Renormalization of One-Boson-Exchange Interactions
in the two-Nucleon system

by

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que la tesis titulada, Renormalization of One-Boson-Exchange Interactions in the two-Nucleon system ha sido realizada por D. Álvaro Calle Cordón bajo su dirección en el Departamento de Física Atómica, Molecular y Nuclear de la Universidad de Granada así como que éste ha disfrutado de estancias en el extranjero, durante un periodo total de ocho meses en la University of Connecticut (USA), en la división teórica del Jefferson Laboratory (USA) y en la University of Liege (Bélgica).

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Published Papers

1. Wigner symmetry, large Nc, and renormalized one-boson exchange potentials.

2. Serber symmetry, Large Nc and Yukawa-like One Boson Exchange Potentials,

3. Renormalization vs Strong Form Factors for One Boson Exchange Potentials,

4. Low Energy Universality and Scaling of Van der Waals Forces,

Proceedings

1. Effective interactions and long distance symmetries in the Nucleon-Nucleon system,
   arXiv:1009.3149v1

2. Renormalization of One Meson Exchange Potentials and Their Currents,
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3. Renormalization and Universality of Van der Waals forces,
   E. Ruiz Arriola, ACC, FB 19, 31 Aug - 5 Sep 2009, Bonn, Germany.
   arXiv:0912.2658

4. Renormalization and Universality of NN interactions in Chiral Quark and Soliton Models,
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   E. Ruiz Arriola, ACC, EFT 09, Valencia, Spain, 2-6 Feb 2009.
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   Renormalization and long distance symmetries for the two-nucleon system (Talk)

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   Universalidad en colisiones atómicas ultrafrías (Talk)

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   Wigner and Serber symmetries for Nucleon-Nucleon Interactions and the large Nc limit (Talk)

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Chapter 1

Introduction

1.1 Nuclear forces

Since the discovery of the neutron by Chadwick in 1932 [1] and the identification of the proton and the neutron\(^1\) as the basic constituents of nuclei, it has been one of the most fundamental goals of theoretical nuclear physics the understanding of the force acting between this constituents, the nucleon-nucleon (NN) interaction, which is responsible for binding the nucleus. The theory of nuclear forces has a long history (see e.g. Ref. [3] for a detailed review). From a fundamental point of view the first idea for the origin of the nuclear force was due to Yukawa in his seminal work [4] where he suggested a massive meson as the responsible for the interaction energy between nucleons. This idea was shortly reconsidered by Yukawa in the framework of quantum field theory and extended by Procca [5] and Kemmer [6] to pseudoscalar and pseudovector particles and even a mixture of this fields with equal mass by Møller and Rosenfeld [7] or with heavier vector boson by Schwinger [8]. However, the discrepancy between these theories with the correct sign of the measured quadrupole and magnetic moments of the deuteron in 1939 by Rabi and co-workers [9–11] led Pauli in 1946 [12] to the hypothesis that an isovector pseudoscalar theory predicted the right sign for the deuteron quadrupole moment. In 1947 the pion was observed experimentally [13] and identified with the particle predicted by Pauli.

Along the 1950’s the pion appeared to be the quantum for the strong interaction, in analogy to the role played by the photon in quantum electrodynamics (QED) and “pion theories” emerged. However, while the one-pion-exchange (OPE) became well established as the long-range part of the nuclear force since it turned out to be very useful in explaining NN scattering data and deuteron properties [14] multi-pion theories ran into serious ambiguities [15–17]. Among them there appeared arbitrariness in the subtraction of the iterated OPE necessary to construct a potential, and a spin-orbit interaction derived from two-pion exchange (TPE) one order of magnitude weaker than experimentally needed. Nonetheless, a crucial contribution from that period was the suggestion by Takehani, Nakamura and Sasaki (TNS) [18] of dividing the nuclear force into three regions. TNS distinguish a long range, \(r \geq 2\) fm, dominated by OPE, an intermediate region, \(1\) fm \(\leq r \leq 2\) fm, where TPE is most important, although heavier mesons may become relevant, and a phenomenological short range or core, \(r \leq 1\) fm, with multi-meson exchange, heavier mesons and more sophisticated effects.

\(^1\)Regarded in the isotopic spin formalism introduced by Cassen and Condon [2] these are two different states of the same particle called nucleon.
Chapter 1. Introduction

This theoretical disaster opened an alternative line of research which took place during the 1950s and the 1960s consisting of a simple phenomenological description of the NN interaction. The basic aim was to provide a phenomenological potential to be used as an input for nuclear calculations. The original phenomenological potentials were improved over the years. Some examples are the Hamada and Johnston [19], the Yale [20] and the Reid [21] hard- and soft-core, the latter being the most applied in nuclear structure physics in the 1970s.

With the discovery of heavy mesons in the early 1960s this disheartening situation in the meson theory of the NN interaction changed when the One-Boson-Exchange (OBE) models came up [24–32]. These models are a natural generalization of the OPE potential proposed by Yukawa [1] and the scalar potential introduced by Johnson and Teller [33] with the additional inclusion of vector mesons as well [28, 34–36]. The traditional point of view within these models is that a group of (correlated) pions can be regarded as a multi-pion resonance, i.e., a single particle with a definite mass and definite intrinsic quantum numbers. The disturbing short distance divergences appearing were first treated by using a hard core boundary condition [28, 34, 35] but it was soon realized that divergences in the potential could be treated by introducing phenomenological form factors mimicking the finite nucleon size [37]. In the mid 1970s a OBE potential with realistic coupling constants based on Regge-pole theory, describing all the available NN scattering data at low energy and deuteron properties, was constructed by the Nijmegen group [38] and Sprung and co-workers [39]. It includes all non-strange mesons of the pseudoscalar, vector and scalar nonet, i.e., \( \pi, \eta, \rho, \omega, \delta, \epsilon \) as well as the \( \eta'(958) \), \( \phi(1020) \) and \( S^*(993) \) and the dominant \( J = 0 \) parts of the Pomeron and tensor \( (f, f', A_2) \) trajectories. At the meson-nucleon vertices a Gaussian form factor was used to kill short distance divergences.

Perhaps the most controversial point in the OBE models is the inclusion of one or even two “fictitious” scalar-isoscalar bosons, the so-called \( \sigma \)-meson (previously denoted as \( \epsilon \)), with a mass ranging 400-800 MeV which is a basic ingredient providing the necessary intermediate-range attraction. Within the OBE models the \( \sigma \)-meson has been always considered a phenomenological ingredient, in part due to the lack of empirical evidence. In recent years, there has been a lot of theoretical effort in the understanding of the nature of the \( \sigma \)-meson. In fact, the scalar meson has been rather problematic with comings and goings in the Particle Data Group (PDG). The situation has gradually changed during the last decade, and the scalar meson has been finally resurrected [40], culminating with the inclusion of the \( 0^{++} \) resonance in the PDG [41] as the \( f_0(600) \) resonance, denoted as the \( \sigma \). The initial wide spread of values ranging from 400 - 1200 MeV for the mass and 600 - 1200 MeV for the width [42] have recently been sharpened by a benchmark prediction based on Roy equations and chiral symmetry [43] in \( \pi \pi \) scattering yielding the unprecedented accurate values \( \sqrt{s} = 441^{+16}_{-8} + i272^{+9}_{-12} \) MeV. Using forward dispersion relations and Roy equations on the real axis the Madrid group [44] has achieved a slightly heavier and narrower \( \sigma \)-resonance determination. While the existence of this broad low lying state is by now out of question, the debate on the nature of the \( \sigma \)-meson is not completely over. Structures of the type tetra-quark or glueball, etc, have been proposed (see e.g. Ref. [45] for a recent review and references therein). Lattice determinations of the lightest scalar mesons [46] have found that the mass of the lightest \( 0^{++} \) meson is suppressed relative to the mass of the \( 0^{++} \) glueball in quenched QCD at an equivalent lattice spacing [47]. In the quenched approximation it has been claimed [48], that \( m_\sigma = 550\text{MeV} \) for pion masses as low as \( m_\pi = 180\text{MeV} \).

2The \( \rho \)-meson was discovered in 1961 at Brookhaven in the \( \pi^- p \rightarrow \pi \pi N \) reaction [22], and the \( \omega \)-meson at Berkeley in \( pp \) annihilation [23]. Both are spin-1 bosons, the \( \rho \) being a \( 2\pi \) resonance and the \( \omega \) a \( 3\pi \) resonance.
Nonetheless, since traditionally this boson has been associated with the correlated TPE, a great theoretical effort was focused in the 1970s on deriving the $2\pi$-exchange contribution to the nuclear force that creates the mid-range attraction. For this, two different ways were explored, namely, dispersion theory and the field theoretical approach. Using dispersion theory one can relate different measured reactions providing a framework for a consistency check of NN from different processes such as $\pi\pi$, $\pi N$ and electron-nucleon scattering. The most important works in this direction were carried out by the Stony Brook [49, 50] and the Paris [51] groups. The Stony Brook group used dispersion results for the $2\pi$-exchange complemented with one-$\pi$ and one-$\omega$ exchange and parameterized the very short distances with strong form factors. In the Paris case, the short-range part of the NN interaction was treated by an energy-dependent repulsive square well and the final version became known as the Paris potential. The field theoretical point of view was first considered by Partovi and Lomon [36, 52] where they invoked a non-relativistic reduction of the Bethe-Salpeter equation to work with a non-static approach to the $2\pi$-exchange. Following this field-theoretical philosophy, one of the most successful OBE potentials was developed in the University of Bonn during the 1970s and the 1980s. They computed all $2\pi$-exchange diagrams including those with virtual $\Delta$-isobar excitations and the relevant $3\pi$- and $4\pi$-exchange diagrams. Apart from these terms, exchanges of heavy mesons were considered as well as correlated $\pi\pi$ and $\pi\rho$ exchanges. The final version was the so-called Bonn full model [53] (see also [54] for a review on relativistic versions).

An important lesson from these developments has been that the non-perturbative nature of the NN force is better handled in terms of quantum mechanical potentials at low energies where relativistic and non-local effects contribute at the few percent level. Although such a framework has remained a useful, appealing and accurate phenomenological model after a suitable ad hoc introduction of phenomenological strong form factors [53, 55] it is far from being a complete description of the intricacies of the nuclear force.

A series of modern potentials based on one-boson-exchanges were constructed by the Nijmegen group in the 1990s. They also carried out the highly successful Partial Wave Analysis (PWA) [56] which provides a spectacular fit with $\chi^2$/DOF = 1.08 comprising a large body of $np$ and $pp$ scattering data.\footnote{The Nijmegen database consists of 1787 $pp$ and 2514 $np$ scattering data in an energy range from 0 to 350 MeV in the lab frame.} A remarkable property of the Nijmegen PWA is that the most important ingredient is the long-range part of the interaction described as charge-dependent OPE. In fact, the PWA checks mainly OPE and some contributions from other mesons, since the interaction below 1.4fm is parameterized by an energy-dependent square well potential. This PWA was used to update the previous Nijmegen potentials to the current high-quality potentials [57]. Shortly a new high-quality version of the Bonn potential came up, the so called (charge-dependent) CD-Bonn potential, based exclusively on OBE but now with charge dependence. The CD-Bonn potential reproduces the world NN data more accurately than by any phase-shift analysis and any other NN potential and this is achieved by the introduction of two effective $\sigma$-mesons, the parameters of which are partial-wave dependent. At the same level of accuracy it is also the more phenomenological Argonne potential [58] which keeps only OPE and represents all the remaining contributions in a rather general operator form.

In spite of the very successful description of the experimental data all the OBE potentials have traditionally been very sensitive to short distances, requiring an unnatural fine-tuning of the vector meson coupling. As a consequence there has been some inconsistency between the couplings required from meson physics, $SU(3)$ or chiral symmetry on the one hand and those from NN scattering fits on the
other hand (see also [59–61]). Part of the disagreement could only be overcome after even shorter scales were explicitly considered [62, 63]. Another common feature of the OBE potentials has been the by-hand introduction of strong form factors. While mimicking finite nucleon size they usually take parameterizations loosely related to the field theoretical meson-baryon Lagrangian from which the meson exchange picture is derived. It is therefore not exaggerated to say that strictly speaking the OBE potentials have not been solved yet.

1.2 Quark models and Effective Field Theories

With the advent of Quantum Chromodynamics (QCD) as part of the Standard Model of particle physics all the meson theories were relegated to just models. Nowadays there is no doubt in admitting QCD as the underlying theory of strong interactions and based on this assertion one would expect that low energy hadron properties and dynamics should be, in principle, derivable from it. In QCD the fundamental degrees of freedom are colored quarks and gluons which have never seen in isolation because they are confined to form strongly interacting hadrons, the colorless bound states built from these fundamental constituents. The resulting nuclear forces responsible for the nuclear binding are then residual color forces, more like the van der Waals forces acting between atoms.

The problem of QCD is its non-perturbative character at nuclear scales, i.e., in the low-energy regime, which makes direct solutions extremely cumbersome. Following the success of the description of single hadron properties in terms of quark models (MIT bag model [64, 65], soliton bag [66, 67], their chiral versions [68, 69], cloudy bag [70–73] and potential models [74–76]) it is not surprising that in the 1980s QCD-inspired quark models were widely used to gain a more fundamental understanding of the NN interaction trying to connect the meson picture with QCD [77–81]. It was soon realized that the colour structure of the one-gluon-exchange (OGE) potential between quarks [75, 82–84] could provide an explanation for the NN short-range repulsion [85]. This led to the attempt of describing the NN interaction from the quark-quark (confinement + OGE) potential using resonating group methods (RGM), generator coordinate methods (GCM) and the Born-Oppenheimer (BO) approximation. From a fundamental point of view the NN interaction should be obtained as a natural solution of the 6-q system. However, in order to describe the NN interaction it is far more convenient to study two 3-q clusters with nucleon quantum numbers. That was the philosophy adopted in the works of Oka et al. [86] in the framework of RGM and of Harvey et al. [87] based on GCM. These methods introduced by Wheeler in 1937 [88] and widely used in nuclear physics [89], allow a treatment of the interaction between two composite systems. Since the perturbative approach of QCD only works at short distances most of the studies of the NN interaction using these quark models concerned the short-range part of the interaction. The origin of the medium- and long-range OPE potential between nucleons was not well understood. In order to avoid this difficulty, the so-called Hybrid Models were introduced [90, 91] where pseudoscalar meson exchanges at the quark level were considered as well as confinement and OGE potentials. More recently, a third category of models was proposed where the quark-quark interaction, besides confinement, is due entirely to meson exchanges between quarks forgetting the OGE contribution. This is the chiral constituent quark model (or Goldstone boson exchange model) proposed by Glozman and Riska in Ref. [92]. Extensions including the scalar-pseudoscalar have been studied as well [93–96].

But an important breakthrough happened in the 1990s when the concept of EFT was applied to low-energy QCD [97] within a Nuclear Physics context. The main idea behind an EFT is the exploitation
of a separation of scales in a given system and the identification of the relevant degrees of freedom such that the physics below a certain energy scale, in which these degrees of freedom live, is not affected by high energy scales, allowing a model-independent description of the system under consideration. For instance, for the ground state and low-energy spectrum of an atomic nucleus as well as conventional nuclear reactions, quarks and gluons are ineffective degrees of freedom while nucleons and pions are the appropriate ones. Moreover, apart from being model independent, the EFT must posses the relevant symmetries of the underlying theory. The basic principle consists on considering the effective or low-energy degrees of freedom and write down the most general Lagrangian consistent with the symmetries of the underlying interaction. With a suitable introduction of counterterms in the effective Lagrangian which encode the unresolved short distance physics together with an organizational scheme or power counting that specifies which terms are required at a desired accuracy, one can distinguish between more and less important contributions and is able to make model independent predictions. The Feynman amplitude can then be written as an expansion in powers of a low-momentum or soft scale \( M_{\text{low}} \sim Q \), with \( Q \) the typical external momentum, over a high-momentum or hard scale \( M_{\text{high}} \sim \Lambda \) where the EFT breaks down. This is usually set by the masses of the heavy particles not considered explicitly, i.e., one can write an expansion in \( Q/\Lambda \) as,

\[
M = \sum \nu \left( \frac{Q}{\Lambda} \right) \nu \mathcal{F} \left( \frac{Q}{\mu}, c_i \right),
\]

where \( \mathcal{F} \) is a function of order one, \( \mu \) is a renormalization scale and \( c_i \) are a collection of low-energy constants (LECs) encoding the high-energy (short-distance) physics that must be fitted to experimental data. The power \( \nu \) at which a given Feynman diagram contributes defines the power counting and in general depends on the number of fields involved in the EFT and the topology of the diagram.

At low energy, the QCD Lagrangean posses a practical symmetry based on the fact that the typical hadronic scale, i.e., the scale of low-mass hadrons which are not Goldstone bosons, e.g., vector mesons, is \( m_\rho = 0.77 \text{GeV} \sim 1 \text{GeV} \), while the masses of the up (u) and down (d) quarks are rather small; \( m_u \approx 2 \pm 1 \text{MeV} \) and \( m_d \approx 5 \pm 2 \text{MeV} \). This symmetry is know as Chiral Symmetry and the corresponding EFT based on it is known as Chiral Perturbation Theory (ChPT). ChPT works therefore in an ideal world of vanishing quark masses and it turns out that QCD is \( SU(2)_L \times SU(2)_R \) symmetric in such a limit. But the chiral symmetry of the QCD lagrangean is spontaneously broken as can be verified from a look to the hadronic spectrum. Therefore, according to Goldstone’s theorem there exist three (massless in the chiral limit) Goldstone bosons that are identified with the (pseudoscalar) pions owing to the fact that they have the same quantum numbers as the broken generators.

The success of ChPT in \( \pi\pi \) scattering \([98, 99]\) its extension to include baryons \([100, 101]\) and the subsequent application to \( \pi N \) scattering \([102–104]\) seemed not to be applicable to the NN sector until Weinberg proposed \([105]\) applying ChPT not to the Feynman amplitudes but to the effective potential itself defined as a sum of connected Feynman diagrams without nucleon intermediate states up to a given chiral order. To obtain the full \( S \)-matrix one iterates this potential to all orders by inserting it into the Lippman-Schwinger (LS) equation in momentum space, or equivalently, in the Schrödinger equation in coordinate space. This scheme complies to the familiar and widely accepted concept of a nuclear potential and was first carried out by Ordóñez, Ray and van Kolck \([106–109]\). Since then, chiral potentials have gained a remarkable popularity and a lot of theoretical effort and intense discussions continue so far. The NN interaction with TPE up to next-to-next-to-leading order (\( N^2\text{LO} \)) has been analyzed in Refs. \([110–113]\). Relativistic \([114]\) and \( 3\pi \) corrections \([115–118]\) have been also analyzed. Relativistic TPE potentials have
been derived in Refs. [119, 120] and the potential up to $N^3LO$ in Refs. [121–123]. TPE potentials with $\Delta$-excitations have been calculated in Ref. [111] and more recently in Ref. [124].

A common feature of these chiral potentials are the divergences appearing when they are naively extrapolated to the origin,

$$U(r) \to \frac{M_N C_n}{r^n},$$

with $M_N$ the nucleon mass, $C_n$ a van der Waals coefficient and $n \geq 2$. In momentum space the divergences are due to divergent integrals appearing in the LS equation. Divergences are a common problem in field theories and methods have been developed to deal with them. One regulates the integrals and then removes the dependence on the regularization parameter (scales or cut-offs) by a renormalization process.

Much of the analysis of the NN interaction using chiral potentials has been done using a finite cut-off in momentum space $\Lambda$, i.e., not doing an exact renormalization which would imply removing the cut-off either in momentum space $\Lambda \to \infty$ or in coordinate space $r_c \to 0$. A stringent constraint follows from the natural requirement of short distance insensitivity: physics not explicitly taken into account should be under control by fixing a sufficient amount of low energy parameters. Such a condition represents the basis of the renormalization procedure developed by the Granada group [125–130] (see also the seminal paper by Case [131] and Frank et al. [132] for a review on singular potentials in which this renormalization procedure is inspired). Renormalization here means the existence of well-defined scattering amplitudes when the cut-off is removed. This condition allows us to identify all the short distance operators needed to remove the cut-off dependence, and once these counterterms are included in the computation approximate cut-off independence is assured and consequently there is no problem in keeping a finite cut-off. In Ref [133] the equivalence between this formulation and the traditional one in momentum space was analyzed and it was shown that renormalization with boundary conditions in coordinate space is equivalent to put counterterms in the LS equation in momentum space. However, the analysis in coordinate space can be carried out much more efficient and transparently.

Coordinate space renormalization has been used for chiral and singular potentials [127, 128] with results converging for practical cut-offs $r_c \sim 0.5 \text{fm} \sim 1/p_{\text{max}}$ with $p_{\text{max}}$ the maximum momentum proven in elastic NN scattering 4. TPE relativistic potentials [134] and TPE with $\Delta$’s [130] have been also studied with good results in spite of the singularities. We show in Fig. 1.1 the results for these analyses where we can see that in all TPE approaches there is an over-binding of $5^0 - 10^0$. This fact can be seen as a defect of the current chiral potentials if the cut-off is removed, i.e., an strict renormalization is carried out. On the other hand since the shortest de Broglie wavelength probed in elastic NN scattering below pion production threshold is $\lambda_{\text{min}} \sim 0.5 \text{fm}$, one expects stable results for similar short distance cut-offs, otherwise the cut-off may become an essential parameter of the theory. This, particularly applies when the cut-off must be fine tuned to physical observables, a situation which actually takes place for specific power counting schemes [121, 135]. In an EFT purist way of thinking one could attribute this to a defect in the power counting but it is clear that other sources of information as well as other manner of dealing with the NN problem consistent with QCD are strongly recommended to gain a deeper understanding on what is going on.

4In fact, the more singular is the potential the better is the convergence. For example, for an attractive singular potential $-1/r^n$ the dependence of the phase shifts for small cut-offs is [129]

$$\frac{d\delta}{dr_c} \sim -k^3 r_c^{n/2}$$

which vanishes quickly in the limit $r_c \to 0$ for $n$ large enough.
1.3 The $1/N_c$ expansion of QCD

As we said in the previous section due to the non-perturbative character of QCD at hadronic scales and the impossibility of using quarks and gluons as degrees of freedom, one seeks for a suitable working approach consistent with the symmetries of QCD to deal with low energy strong interactions. Apart from Chiral Symmetry, QCD posses another limit, namely, the limit of large number of colors $N_c$.

QCD is a non-abelian gauge theory of quarks and gluons with $SU(3)$ symmetry. It was first pointed out by t’Hooft in 1974 [136] that some spectacularly simple properties of QCD emerge if one considers instead of $SU(3)$, the gauge group $SU(N_c)$ in the large-$N_c$ limit fixing $\alpha_S N_c$ with $\alpha_S$ the strong coupling constant. G. t’Hooft’s seminal work was subsequently extended to the baryon context by E. Witten in 1979 [137]. This idea provides a framework where one may deduce properties of large-$N_c$ QCD with general arguments, without even writing down a Lagrangean. Naively, one may think that the analysis of QCD in $SU(N_c)$ must be much more complicated since one is increasing the number of dynamical degrees of freedom, however the analysis can be carried out by considering $1/N_c$ as expansion parameter keeping only the leading terms in this expansion.

It turns out that [137] in the large-$N_c$ limit mesons are stable, free and non-interacting particles with their masses scaling as $m \sim N_c^0$ and widths as $\Gamma \sim 1/N_c$, and baryons are stable heavy particles with their masses scaling as $M \sim N_c$ and with a definite size and shape $\mathcal{O}(1)$. The limit also sets the interaction among them. Meson-baryon interaction is too weak, i.e. $\mathcal{O}(1)$, as to affect the motion of the baryon itself and they propagate freely as if the mesons were not present at all. However baryon-baryon interaction is strong enough $\mathcal{O}(N_c)$. In particular if baryons besides being heavy, are also assumed to be fast, i.e., their momentum scaling as $p \sim N_c$, the interaction between them resembles a time-dependent mean-field picture of elastic scattering of moving solitons.

One of the advantages of taking the large-$N_c$ limit is that nucleons become infinitely heavy, so if their
momentum is taken to be fixed and $N_c$ independent, $p \sim N_c^0$, the non-relativistic potential is a well-defined object and presumably not subjected to the many ambiguities of relativistic potentials. In fact Witten’s $N_c$ counting rules imply non-trivial constraints in meson-baryon couplings which are known as consistency conditions. These consistency conditions generate a contracted spin-flavor $SU(2N_f)_{c}$ algebra with $N_f$ the number of flavors which is exact in the large-$N_c$ limit. Based on this manifest $SU(2N_f)$ spin-flavor symmetry it was shown that if the nucleon momentum scales as $p \sim N_c^0$, the nuclear potentials scale either as $N_c$ or $1/N_c$, depending upon the particular spin-isospin channel, which shows that the NN interaction can be determined with $1/N_c^2 \sim 10\%$ relative accuracy. For two flavors the leading structure $O(N_c)$ of the NN potential turns out to be,

$$V_{NN} = V_C + \tau_1 \cdot \tau_2 [W_S \sigma_1 \cdot \sigma_2 + W_T S_{12}] ,$$

that is, the strongest interaction is the central force terms $V_C$ and $W_S$ as well as the tensor force $W_T$. The remaining contributions, spin-orbit or relativistic corrections, are at least $O(N_c^{-1})$ and therefore relatively suppressed by $O(1/N_c^2)$.

Moreover, one feature of large-$N_c$ that becomes relevant for the NN problem is that it does not specially hold either in a specific energy regime or for long or short distances. This allows us in particular to switch from perturbative quarks and gluons at short distances to the non-perturbative hadrons, the degrees of freedom of interest to nuclear physics. This quark-hadron duality makes possible the applicability of large-$N_c$ counting rules directly to baryon-meson interactions, at distances where explicit quark-gluon effects are not expected to be crucial. Thus, although the large-$N_c$ scaling behavior and spin-flavor structure of the NN potential, Eq. (1.3), is directly established in terms of quarks and gluons, quark-hadron duality at distances larger than the confinement scale requires an identification of the corresponding exchanged mesons, and hence a link to the OBE potentials is provided. At leading order one has exchange of $\pi$, $\sigma$, $\rho$, $\omega$ and $a_1$ and the corresponding OBE potential reads,

$$V_C(r) = \frac{g_{\sigma NN}^2 e^{-m_{\sigma}r}}{4\pi r} + \frac{g_{\omega NN}^2 e^{-m_{\omega}r}}{4\pi r} ,$$

$$W_S(r) = \frac{g_{\rho NN}^2 m_{\rho}^2 e^{-m_{\rho}r}}{48\pi \Lambda_N^2 r} + \frac{f_{\rho NN}^2 m_{\rho}^2 e^{-m_{\rho}r}}{24\pi \Lambda_N^2 r} - \frac{g_{a_1 NN}^2 e^{-m_{a_1}r}}{6\pi r} ,$$

$$W_T(r) = \frac{g_{\rho NN}^2 m_{\rho}^2 e^{-m_{\rho}r}}{48\pi \Lambda_N^2 r} \left[ 1 + \frac{3}{m_{\rho}r} + \frac{3}{(m_{\rho}r)^2} \right] - \frac{f_{\rho NN}^2 m_{\rho}^2 e^{-m_{\rho}r}}{48\pi \Lambda_N^2 r} \left[ 1 + \frac{3}{m_{\rho}r} + \frac{3}{(m_{\rho}r)^2} \right] + \frac{g_{a_1 NN}^2 e^{-m_{a_1}r}}{12\pi r} \left[ 1 + \frac{3}{m_{a_1}r} + \frac{3}{(m_{a_1}r)^2} \right] ,$$

where $\Lambda_N = 3M_N/N_c$, the meson-nucleon coupling constants $g_{m NN} \sim \sqrt{N_c}$ and meson masses $m \sim N_c^0$.

In this Thesis we will carry out a coordinate space renormalization of the above potential using a boundary condition at a short distance cut-off $r_c$ which makes the Hamiltonian self-adjoint for $r > r_c$. Besides being much simpler and efficient, this method allows us to deal with cut-off independent potentials and in practice convergence is achieved for $r_c \sim 0.3$ fm. On the other hand, with this method one can implement
short-distance insensitivity which is very useful to avoid the notorious ambiguities and fine-tuning suffered by traditional OBE potentials.

1.4 Organization and scope of the Thesis

In Chapter 2 we start out with an example to motivate the ideas underlying the concept of renormalization of NN forces as obtained in the Born-Oppenheimer approximation in quark models. Due to the similarity with the atomic physics case \cite{144} we have dubbed the study as the Renormalization of Spin-Flavor Van der Waals Forces. Of course the model is in a sense very simplistic but it embodies many of the features we want to illustrate and enjoys many of the properties characterizing the more popular TPE potentials deduced from the ChPT approach.

A first application in Nuclear Physics of the renormalization approach concerns the interpretation of old nuclear symmetries on the light of the large $N_c$ expansion. In Chapter 3 we show that the pattern of Wigner symmetry coincides with that expected from the large $N_c$ approach with an accuracy of $1/N_c^2$. The symmetry is fulfilled where the $1/N_c$ expansion expects it to be fulfilled and it is violated where the $1/N_c$ expansion allows for a violation. This surprising verification of a fundamental QCD pattern provides much of the motivation underlying the calculations of the present Thesis.

Actually, given that we find that when Wigner symmetry is violated Serber symmetry holds instead, we undertake a study in Chapter 4 buttressing the expectation that this might have a large $N_c$ understanding.

We are lead to consider the role played by $\pi\pi$ resonances within a large $N_c$ framework in the NN potential. Encouraged by these results we face squarely the possibility that large $N_c$ potentials might eventually provide a workable scheme, less directly related to ChPT but closer in spirit to the common wisdom of Nuclear Physics. Actually, large $N_c$ counting rules complemented by quark-hadron duality above the confinement scale support the standard meson exchange picture. Thus, in Chapter 5 we consider the renormalization of large $N_c$ potentials at the OBE level. In passing, we analyze the role played by Strong form factors within a renormalization framework regarding low partial waves as well as the deuteron.

An interesting extension of these ideas pertains the study of Charge Symmetry Breaking as well as the role played by Coulomb forces, a subject discussed in Chapter 6. There, a set of short distance conditions provide a successful connection between two nucleon states in different isospin channels.

Another relevant application adressed in Chapter 7 concerns the interplay between gauge invariance and renormalization. This is of great relevance in the study of meson exchange currents as they occur in electrodisintegration, elastic electron scattering or radiative neutron capture.

Finally in Chapter 7.6 we come to the conclusions and provide some outlook for further work.

Many details and complementary material are collected in the Appendices which are intended to make the Thesis more self-contained.
Chapter 2

Renormalization of Spin-Flavor Van der Waals Forces

2.1 Introduction

In order to introduce the basic ideas that we are going to develop along this thesis it is instructive to consider a model for the NN interaction based in quark-models which shares common features of OBE and Chiral potentials.

As we have discussed in the introduction the meson exchange picture has played a key role in the development of Nuclear Physics [53, 145]. However, the traditional difficulty has been a practical need to rely on short distance information which is hardly accessible directly but becomes relevant when nucleons are placed off-shell. From a theoretical point of view this is unsatisfactory since one must face uncertainties not necessarily linked to our deficient knowledge at long distances and which are difficult to quantify. On the other hand, the purely field theoretical derivation yields potentials which present short distance singularities, thereby generating ambiguities even in the case of the widely used OBE potential. The standard way out to avoid the singularities is to implement vertex functions for the meson-baryon-baryon coupling ($m_{AB}$) in the OBE potentials mimicking nucleon size and shape. Furthermore, due to an extreme fine-tuning of the interaction, mainly in the $^1S_0$ channel, OBE potential models have traditionally needed a too large $g_{\omega NN}$ to overcome the mid range attraction implying one of the largest ($\sim 40\%$) $SU(3)$ violations known to date.

Of course, the extended character of the nucleon as a composite and bound state of three quarks has motivated the use of microscopic models of the nucleon to provide an understanding of the short range interaction besides describing hadronic spectroscopy; quark or soliton models endow the nucleon with its finite size and incorporate basic requirements from the Pauli principle at the quark level or as dictated by the equivalent topology [146–149]. While much effort has been invested into determining the short range interactions, there is a plethora of models and related approximations; it is not obvious what features of the model are being actually tested. In fact, NN studies set the most stringent nucleon size oscillator constant value $b_N = 0.518\text{fm}$ [149] from S-waves and deuteron properties which otherwise could be in a wider range $b_N = 0.4 - 0.6\text{fm}$. This shows that quark models also suffer from a fine-tuning problem.
However, NN scattering in the elastic region corresponds to resolve distances about the minimal de Broglie wavelength associated to the first inelastic pion production threshold, $NN \rightarrow NN\pi$, and corresponds to a center of mass (c.m.) momentum $p_{\text{max}} \sim 350\text{MeV}$ which means $\lambda_{\text{min}} \sim 1/p_{\text{max}} = 0.5\text{fm}$. This scale is smaller than $1\pi$ and $2\pi$ exchange with Compton wavelengths $1.4$ and $0.7\text{fm}$ respectively. Other length scales in the problem are comparable and even shorter, namely 1) nucleon size, 2) correlated meson exchanges and 3) quark exchange effects. All these effects are of similar range and, to some extent, redundant. For example, in quark models, the constituent quark mass is related to the nucleon and vector meson masses through $M_q = M_N/N_c = M_V/2$ which for $N_c = 3$ colors gives the estimate $M_q = 310 - 375\text{MeV}$. Exchange effects due to e.g. One-Gluon-Exchange are $\sim e^{-2M_q r}$ since they correspond to the probability of finding a quark in the opposite baryon. As we will discuss below all these effects are somewhat marginal.

On the other hand, most high precision NN potentials providing $\chi^2/\text{DOF} < 1$ need to incorporate universally the OPE potential (including charge symmetry breaking effects) while the shorter range is described by many and not so similarly looking interactions [57]. This is probably a confirmation that chiral symmetry is spontaneously broken at longer distances than confinement, since hadronization has already taken place. It also suggests that in a quark model aiming at describing NN interactions the pion must be effectively included. Chiral quark models accomplish this explicitly under the assumption that confinement is not crucial for the binding of $\pi, N$ and $\Delta$. Pure quark models including confinement or not have to face in addition the problem of recovering the pion from quark-gluon dynamics. Hybrid models have become practical in dealing with this handicap [146, 147, 149]. But, as it has been mentioned, all these scales around the confinement scale are mixed up. Because these effects are least understood, we will ignore the difficulties by remaining in a regime where confinement is not expected to play a role and stay with standard chiral quark models.

In a series of works [150–152] the derivation of the NN interaction from the Skyrme model [153, 154] have been re-examined. In fact, the Skyrme Lagrangean is related to QCD in the limit of a large number of colors [137]. While both the constituent chiral quark model and the Skyrme soliton model look very disparate, the Chiral Quark Soliton Model embeds both models in the small and the large soliton limit respectively.

In the next sections, we are going to analyze the intuitive non-relativistic chiral quark model (NRCQM) explicitly and comment on the soliton case where similar patterns emerge. The comparison stresses common aspects of the quark soliton model pictures which could be true features of QCD. While the long distance universality between both NRCQM and Skyrme soliton model NN calculations may appear somewhat surprising this is actually so because in a large $N_c$ framework both models are just different realizations of the contracted spin-flavour symmetry [140].

## 2.2 The non-relativistic chiral quark model

As starting point we will consider the chiral quark model which corresponds to the Gell-Mann-Levy sigma model Lagrangean at the quark level [155]. The linear sigma model Lagrangean is a model based on chiral $SU(2)_L \times SU(2)_R$ symmetry and consistent with the symmetries of QCD in the limit of two flavours ($N_f = 2$) massless up and down quarks (see e.g. Refs. [156–159] for a more extended explanation). The Lagrangean can be split into a chiral symmetric part $\mathcal{L}_s$ and a symmetry breaking part $\mathcal{L}_{sb}$ which
generate three Goldstone bosons (spontaneous breaking) and pion masses (explicit breaking) respectively. The invariance of $\mathcal{L}_s$ under isospin rotations $q(x) \to \exp(-ie \cdot \tau/2)q(x)$ leads to the conservation of the vector current,

$$j^\mu = \bar{q} \gamma^\mu \tau^i + \pi \times \partial^\mu \pi.$$  \hspace{1cm} (2.1)

The structure of the linear sigma model Lagrangean comes from imposing conservation of the axial current, 

$$J_5^\mu = \bar{q} \gamma^\mu \sigma \gamma_5 / 2 q + \sigma \partial^\mu \pi - \pi \partial^\mu \sigma.$$ \hspace{1cm} (2.2)

One starts then from a Lagrangean for massless fermions fields $q(x)$ coupled to an isovector-pseudoscalar field $\pi(x)$ and an isoscalar-scalar field $\sigma(x)$ in a chirally invariant fashion,

$$\mathcal{L}_F = \bar{q} \left[ i \partial - g(\sigma + i\gamma_5 \tau \cdot \pi) \right] q.$$ \hspace{1cm} (2.3)

Defining left- and right handed fermion fields

$$q_L = \frac{1}{2} (1 - \gamma_5) q,$$ \hspace{1cm} (2.4)

$$q_R = \frac{1}{2} (1 + \gamma_5) q,$$ \hspace{1cm} (2.5)

the coupling can be re-written as,

$$g\bar{q}(\sigma + i\gamma_5 \tau \cdot \pi)q = g \left[ \bar{q}_L \Sigma q_R + \bar{q}_R \Sigma^\dagger q_L \right].$$ \hspace{1cm} (2.6)

with $\Sigma \equiv \sigma + i\tau \cdot \pi$ which transforms as $\Sigma \to R\Sigma L^\dagger$ being $(R, L)$ an element of the $SU(2)_L \times SU(2)_R$ group. But chiral symmetry must be spontaneously broken since light quarks have a small but finite mass. This effect can be implemented by considering the standard Mexican hat potential $U(\sigma, \pi) = \lambda^2 (\sigma^2 + \pi^2 - \nu^2)^2 / 8$. The $\sigma$ field is decomposed into its vacuum expectation value $\langle \sigma \rangle$ and a fluctuation part. The fermion mass thus becomes $M_q = g\langle \sigma \rangle$ but the pion field still remains massless. In order to give a mass to the pion one has to break the symmetry explicitly. We then add a term proportional to the $\sigma$ field, i.e., $-f_\pi m_\sigma^2 \sigma$, which imply that the axial current is almost conserved (PCAC) apart from a tiny quantity proportional to the pion mass $\partial_\mu j_5^\mu = f_\pi m_\sigma^2 \pi$. All these ingredients together allow us to write the following Lagrangean density,

$$\mathcal{L} = \bar{q} \left[ i \partial - g(\sigma + i\gamma_5 \tau \cdot \pi) \right] q - M_q \bar{q}q + \frac{1}{2} \left[ (\partial^\mu \sigma)^2 + (\partial^\mu \pi)^2 \right] - \frac{1}{2} m_\sigma^2 \sigma^2 - \frac{1}{2} m_\pi^2 \pi^2 - U(\sigma, \pi) - f_\pi m_\sigma^2 \sigma,$$ \hspace{1cm} (2.7)

implementing both spontaneous breaking of chiral symmetry as well as PCAC yielding the Goldberger-Treiman relation at the constituent quark level $M_q = g f_\pi$ with $g = g_{qqq} = g_{qqq}$ and $M_q$ the quark mass.

In this $SU(2)$ model, we have $m_\pi \neq 0$, $m_\sigma^2 = m_\pi^2 + \frac{\lambda^2}{\nu^2} M_q^2$ and $\partial_\mu j_5^\mu = m_\pi^2 \frac{M_q}{\nu} \pi$ although when this model is interpreted from a gradient expansion of the Nambu-Jona-Lasinio (NJL) model [160–162] one gets $m_\pi^2 = m_\sigma^2 + 4M_q^2$, which for $M_q = M_N/3 = M_V/2$ yields $m_\pi = 650 - 770$MeV. Therefore in a
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\[ \frac{R}{2} \times i \eta_i \xi_i \frac{R}{2} \text{c.m.} \]

\[ \text{ClusterB} \]

\[ \text{ClusterA} \]

Figure 2.1: Two-hadron interaction as described by two clusters of \( N_c \) quarks with pairwise interactions.

The model of heavy constituent quarks described by this Lagrangean we obtain the usual \( 1\pi \) and \( 1\sigma \) exchange potentials between quarks \(^1\),

\[
V_{\pi qq}^{1\pi}(r) = -\frac{g_{\pi qq}^2}{4M_q^2} \tau_q \cdot \tau_{q'} \int \frac{d^3 k}{(2\pi)^3} \frac{(\sigma_q \cdot k)(\sigma_{q'} \cdot k)}{k^2 + m_{\pi}^2} e^{ik \cdot r}, \tag{2.8}
\]

\[
V_{\sigma qq}^{1\sigma}(r) = g_{\sigma qq}^2 \int \frac{d^3 k}{(2\pi)^3} \frac{1}{k^2 + m_{\sigma}^2} e^{ik \cdot r}, \tag{2.9}
\]

with \( k \) the three-momentum transfer. Using these meson exchange potentials at the quark level, baryon properties can be obtained by solving the Hamiltonian for \( N_c \) quarks

\[
H = \sum_{i=1}^{N_c} \left[ \frac{p_i^2}{2M_q} + M_q \right] + \sum_{i<j} V(x_i - x_j) = \frac{P^2}{2M} + N_c M_q + H_{\text{int}}, \tag{2.10}
\]

where \( P = \sum_{i=1}^{N_c} p_i \) is the total momentum and \( H_{\text{int}} \) the intrinsic Hamiltonian. Note that the wave function of a moving baryon can be always factorized into a c.m free wave and the intrinsic wave function,

\[
\Psi_B(x_1, \ldots, x_{N_c}) = \phi(\xi_1, \ldots, \xi_{N_c-1}) e^{iP \cdot R}, \tag{2.11}
\]

with \( R = \sum_{i=1}^{N_c} x_i / N_c \) the c.m. of the cluster and \( \xi_i = x_i - R \) the intrinsic coordinates, \( \sum_i \xi_i = 0 \). We would like to find the interaction between two of these clusters. We will proceed by considering hadrons as clusters of \( N_c \) quarks and the interaction between them as sum of pairwise interactions between quarks in each cluster which, for elementary \( \pi qq \) and \( \sigma qq \) vertices, reads

\[
V_{\text{int}}(x_1, \ldots, x_{N_c}, y_1, \ldots, y_{N_c}) = \sum_{i,j} V_{ij}^{\pi \sigma}(x_i - y_j) = \int \frac{d^3 q}{(2\pi)^3} \sum_{i,j} V_{ij}^{\pi \sigma}(q) e^{i q \cdot (x_i - y_j)}. \tag{2.12}
\]

Switching to intrinsic coordinates variables \( x_i = \xi_i + \frac{R}{2} \) and \( y_j = \eta_j - \frac{R}{2} \) with \( \sum_i \xi_i = \sum_j \eta_j = 0 \) where \( R \) is the distance between the c.m. of each cluster (Fig 2.1) and introducing Eqs. (2.8), (2.9) into

\(^1\)The derivation of these potentials is similar to the \( 1\pi \)- and \( 1\sigma \)-exchange potentials explained in Appendix B.
Eq. (2.12), we obtain
\[ V_{1\pi}(R) = -\frac{g_{\pi qq}^2}{M_q^2} \int \frac{d^3q}{(2\pi)^3} e^{iq \cdot R} \frac{g_{ij}}{q^2 + m_i^2} G_A^{ki}(q) G_B^{kj}(q)^*, \] (2.13)
\[ V_{1\sigma}(R) = g_{\pi qq}^2 N_c^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2 + m_i^2} \rho_A(q) \rho_B(q)^*, \] (2.14)
where we have defined the spin-isospin and scalar densities given by (e.g. cluster A)
\[ G_A^{ki}(q) = \frac{1}{N_c} \sum_{a=1}^{N_c} \sigma_a^{i} \xi_a^{k} e^{q \cdot q}, \quad \rho_A(q) = \frac{1}{N_c} \sum_{a=1}^{N_c} e^{q \cdot q}, \] (2.15)
respectively. Thus, the total Hamiltonian is written as
\[ H = H_{A,int} + H_{B,int} + V_{int}(R) + \frac{p^2}{2M_T} + \frac{\mu^2}{2\mu}. \] (2.16)
Galilean invariance implies that inertial masses are \( M_T = 2N_c M_q \) and \( \mu = N_c M_q/2 \). Introducing the two independent cluster complete states \( H_{A,int} \phi_{A,n} = M_{A,n} \phi_{A,n} \) and \( H_{B,int} \phi_{B,m} = M_{B,m} \phi_{B,m} \) the two-clusters c.m. frame unperturbed wave function is just a product
\[ \Psi_{A_n,B_m}^{(0)}(\xi_1, \ldots, \xi_{N_b}; \eta_1, \ldots, \eta_{N_b}) = \phi_{A,n}(\xi_1, \ldots, \xi_{N_b}; R/2) \phi_{B,m}(\eta_1, \ldots, \eta_{N_b}; -R/2) e^{iQ \cdot R}, \]
where \( Q \) is the relative momentum between the two clusters. The above problem is usually handled by Resonating Group Methods [93, 146, 147, 149]. We will analyze this coupled channel scattering problem perturbatively where the transition potentials, defined as \( V^{1\pi}_{A_n,B_m:A_k,B_l}(R) = \langle \phi_{A,n} \phi_{B,m} | V_{int} | \phi_{A,k} \phi_{B,l} \rangle \), have a familiar structure
\[ V^{1\pi}_{A_n,B_m:A_k,B_l}(R) = -\frac{g_{\pi qq}^2}{M_q^2} \int \frac{d^3q}{(2\pi)^3} \frac{g_{ij}}{q^2 + m_i^2} e^{iq \cdot R} \langle A_n | G_A^{ij}(q) | A_k \rangle \langle B_m | G_B^{ij}(q) | B_l \rangle, \] (2.17)
\[ V^{1\sigma}_{A_n,B_m:A_k,B_l}(R) = g_{\pi qq}^2 N_c^2 \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2 + m_i^2} e^{iq \cdot R} \langle A_n | \rho_A(q) | A_k \rangle \langle B_m | \rho_B(q) | B_l \rangle. \] (2.18)
Note that the matrix elements at the baryon vertices contain all the baryon structure information and express the finite size of the baryon.

At long distances the leading singularities \( q = im_\pi \) and \( q = im_\sigma \) dominate and with a suitable definition of coupling constant at the hadronic level [163] one can obtain the usual \( 1\pi \) and \( 1\sigma \) exchange potentials between nucleons. Actually, using that \( |\langle N|\rho(q)|N\rangle|^2 \) is an even function of \( q \) we get the structure for the \( NN \to NN \) potentials
\[ V_\sigma(R) = g_{\pi qq}^2 N_c^2 \int \frac{d^3q}{(2\pi)^3} e^{iq \cdot R} \frac{|\langle N|\rho(0)|N\rangle|^2}{q^2 + m_i^2} + C_0 \delta^{(3)}(R) + C_2 (-\nabla^2 + m_i^2) \delta^{(3)}(R) + \ldots \]
\[ = -\frac{g_{\pi NN}^2}{4\pi} \frac{e^{-m_\sigma r}}{r} + \text{distributions}, \] (2.19)
for the sigma and
\[ V_{\sigma}(R) = \frac{g_{\sigma\pi}^2}{M_{\pi}^2} \int \frac{d^3q}{(2\pi)^3} e^{iq \cdot R} \langle \sigma | \rho(\text{im}\pi) | N \rangle^2 \langle (\tau^a \sigma^i)_{\pi NN} (\tau^a \sigma^i)_{NN} \rangle 
+ C_0 \delta^{(3)}(R) + C_2 (-\nabla^2 + m_\pi^2) \delta^{(3)}(R) + \ldots 
= \frac{g_{\sigma NN}^2 m_\pi^2}{48\pi M_N^2} \left[ Y_0(m_\pi r) \sigma_A \cdot \sigma_B + Y_2(m_\pi r) S_{12}(\hat{r}) \right] \tau_A \cdot \tau_B + \text{distributions}, \]

(2.20)

for the pion, with \( Y_0(x) = e^{-x}/x \) and \( Y_2(x) = Y_0(x)(1 + 3/x + 3/x^2) \). Here, the couplings are given by \( g_{\sigma NN} = N_c g_{\sigma\pi q} |\rho(\text{im}\pi)| \) and \( g_{\pi NN} = N_c g_{\pi\pi q q} |\rho(\text{im}\pi)| \) where \( g_A = (N_c - 2)/3 \) [164] and \( \langle N | G^{\alpha\alpha}(q) | N \rangle = \rho(q^2) \chi_{\sigma, \alpha}^{\beta, \gamma} \sigma^\alpha \sigma^\gamma \chi_N \). Assuming \( |\rho(\text{im}\pi)| \sim |\rho(0)| = 1 \) one has the Goldberger-Treiman relation \( g_A M_N = g_{\pi NN} f_\pi \) at the nucleon level. As we see, in addition to the long range 1\( \pi \) and 1\( \sigma \) potentials at the nucleon level, an infinite sum of delta functions and derivatives thereof representing size effects appears. Therefore, we can separate the potential into a long-range and a contact interaction,
\[ V(R) = V_L(R) + V_C(R), \]

with \( V_L(R) = V_{1\pi} + V_{1\sigma} \) and,
\[ V_C(R) = C_0 \delta^{(3)}(R) + C_2 \nabla^2 \delta^{(3)}(R) + \ldots, \]

(2.22)

the contact interaction implementing finite range effects. Only an infinite number of terms may yield a finite size effect but any finite truncation will produce a negligible contribution at any non-vanishing distance. In a sense, this result is reminiscent of the Gauss theorem for charged objects with a sharp non-overlapping boundary; the interaction is mainly due to the total charge and regardless on the density profiles of the system. Note that the coefficients of the contact interactions are fixed numbers having a meaning perturbatively. However, if one tries to play with them to characterize finite resolution effects (nucleon size and potential range) in a model independent way non-perturbatively (solving e.g. the Schrödinger equation) important restrictions arise. Unlike the \( \delta \)'s, the OPE short distance \( 1/r^3 \) singularity is not located in a compact region and contributes to all arbitrarily small distances. Thus, one can effectively drop the derivatives of distributions. This simple-minded argument was advanced in Ref. [127] and explicitly verified in momentum space by taking \( C_0 \) and \( C_2 \) as real counterterms in the LS equation in Ref. [133]; either \( C_2 \) becomes irrelevant or the scattering amplitude does not converge. Therefore, we represent \( C_0 \) as an energy independent boundary condition in our coordinate space renormalization method which will basically consist in the following:

1. Fix some low energy constants such as e.g. the scattering length for S-waves, \( a_0 \), at zero energy as an independent variable of the potential,
2. Integrate in down to an arbitrarily small cut-off radius \( r_c \),
3. Construct an orthogonal finite energy state by matching log-derivatives at \( r_c \) and
4. Integrate out generating a phase-shift \( \delta_0(p) \) with a prescribed scattering length \( a_0 \).

The crucial aspect is that short distance insensitivity is implemented. In the next section we will apply this method to an interesting example describing the NN interaction with multi-pion effects included in an effective way.
2.3 Renormalization of spin-flavour Van der Waals forces

The non-linear chiral quark model [165] corresponds to take \( m_\pi \to \infty \), reducing to just OPE. An improvement of the simple OPE may be carried out if we introduce the \( \Delta \)-isobar as a dynamical degree of freedom. In this case the long distance OPE transition potentials (see Sec. B.6 in Appendix B for explicit expressions) have the form

\[
V_{AB,CD}(r) = (\tau_{AB} \cdot \tau_{CD}) \left\{ \sigma_{AB} \cdot \sigma_{CD}[W^{12\pi}_B(r)]_{AB,CD} + |S_{12}]_{AB,CD}[W^{12\pi}_B(r)]_{AB,CD} \right\},
\]

where the tensor term and radial functions are defined by

\[
|S_{12}]_{AB,CD} = 3(\sigma_{AB} \cdot \hat{r})(\sigma_{CD} \cdot \hat{r}) - \sigma_{AB} \cdot \sigma_{CD},
\]

\[
[W^{12\pi}_B(r)]_{AB,CD} = \frac{m_\pi f_{\pi AC} f_{\pi BD}}{4\pi} Y_{0,2}(m_\pi r).
\]

In this particular form the resulting potential is model independent as it does not include e.g. vertex form factors. In fact, the corresponding couplings are \( f_{\pi AB} = |F_{\pi AB}(im_\pi)| \) where the transition form factors are defined as \( F_{\pi AB}(q^2) = g^{AB} S^A \chi_B = (A|G(q)|B) \). In the SU(4) \( \otimes \) SU(\( N_c \)) quark model [164] and in the chiral limit the couplings fulfill \( f_{\pi \Delta \Delta}/f_{\pi NN} = 1/5 \) and \( f_{\pi N\Delta}/f_{\pi NN} = 3/2(\pi c - 1)(\pi c + 5)/\pi c \). This gives the familiar result \( f_{\pi N\Delta}/f_{\pi NN} = 6/5 \) for \( \pi c = 3 \) and \( f_{\pi N\Delta}/f_{\pi NN} = 3/2 \) for \( \pi c \to \infty \). The \( \Delta \to N\pi \) width in the Born approximation with relativistic kinematics yields [158] (see also Sec. B.7 in Appendix B) \( f_{\pi N\Delta}^2/(4\pi) = 0.37 \) while the Chew-Low value is \( f_{\pi N\Delta} = 2 f_{\pi NN} \) (see e.g. Ref. [166]).

In general, this requires solving a coupled channel problem [167, 168] but if we only care about the elastic NN channel with \( T_{c.m.} = m_\pi < \Delta \equiv M_\Delta - M_N = 293\text{MeV} \) we may take into account the effect of the closed channels as sub-threshold effects in perturbation theory. To do so we start from the LS equation

\[
T = V + VG_0T,
\]

with \( G_0 = (E - H_0)^{-1} \). Let be \( n, k \) the in-going channel with c.m. momentum \( k \) and \( n', k' \) the out-going channel with c.m. momentum \( k' \) being \( n, n' = N, N\Delta, \Delta \Delta \). Restricting to the two particle ground \( |0\rangle = |N N\rangle \) and excited \( |n\rangle = |N\Delta, \Delta N\rangle, |\Delta\Delta\rangle \) in-going and out-going channels with resolvent \( G_{0,m}(E) = (E - H_{0,m})^{-1} \) where \( H_{0,m} = q^2/(2\mu_m) + E_m \), we can write the LS equation as

\[
\langle k', n'|T|k, n \rangle = \langle k', n'|V|k, n \rangle + \sum_m \int \frac{d^3q}{(2\pi)^3} \langle k', n'|V|q, m \rangle \frac{1}{E - q^2/(2\mu_m) - E_m} \langle q, m|T|k, n \rangle,
\]

with \( E_0 = 2M_N, E_{1,2} = M_N + M_\Delta \) and \( E_3 = 2M_\Delta \) the corresponding threshold energies. For the elastic channel \( (NN \to NN) \) this equation reads

\[
\langle k', 0|T|k, 0 \rangle = \langle k', 0|V|k, 0 \rangle + \sum_m \int \frac{d^3q}{(2\pi)^3} \langle k', 0|V|q, m \rangle \frac{1}{E - q^2/(2\mu_m) - E_m} \langle q, m|T|k, 0 \rangle.
\]

Thus, separating the elastic term \( m = 0 \) explicitly from the sum we get the effective potential in the elastic scattering channel corresponding to higher pion exchanges, which, when iterated to second order
yields the elastic scattering amplitude \( T_{00} \).

\[
\langle k', 0 | \hat{V} | k, 0 \rangle = \langle k', 0 | V | k, 0 \rangle \\
+ \sum_{m \neq 0} \int \frac{d^3q}{(2\pi)^3} \frac{1}{E-q^2/(2\mu_m)-E_m} \langle q, m | V | k, 0 \rangle \\
+ \mathcal{O}(V^3).
\]

Specifically, defining the momentum space potential \( V_{nm}(k' - k) \equiv \langle k', n | V | k, m \rangle = \int d^3r V_{nm}(r) e^{i(k-k') \cdot r} \) we get

\[
\hat{V}_{00}(k' - k, E) = V_{00}(k' - k) + \sum_{m \neq 0} \int \frac{d^3q}{(2\pi)^3} \frac{V_0(k' - q)V_{m0}(q - k)}{E-q^2/2\mu_m-E_m} + \mathcal{O}(V^3),
\]

which, expectedly, depends on the energy. Evaluating on-shell at \( E = E_0 + p^2/2\mu_0 \), assuming a large splitting \( p \ll \sqrt{M_\Delta} = 600\text{MeV} \) and neglecting the kinetic energy piece in the \( N\Delta \) channel, we get the perturbative and local optical potential in coordinate space

\[
\hat{V}_{N\pi N}^{1\pi+2\pi+\cdots}(r) = V_{N\pi N}^{1\pi}(r) + 2 \frac{|V_{N\pi N\Delta}^{1\pi}(r)|^2}{M_N - M_\Delta} + \frac{1}{2} \frac{|V_{N\pi N\Delta}^{1\pi}(r)|^2}{M_N - M_\Delta} + \mathcal{O}(V^3),
\]

that is the Born-Oppenheimer (BO) approximation to second order generating more complicated spin-isospin structures than just OPE including a central force, all of them \( \sim e^{-2m_\pi r} \) like TPE. This similarity with TPE is also clear at short distances where \( V_{N\pi N\Delta}^{1\pi}(r) \sim V_{N\pi N\Delta}^{1\pi}(r) \sim g_A^2/(f_\pi^2 m_\pi) \) and hence the effective potential becomes an attractive singular van der Waals potential \( \hat{V}_{N\pi N} \sim -g_A^2/(\Delta f_\pi^2 m_\pi) \).

The explicit form of this optical potential can be derived using the transition potentials of Appendix B given, in coordinate space, by \(^2\)

\[
V_{N\pi N}(r) = (\tau_1 \cdot \tau_2) \left\{ \sigma_1 \cdot \sigma_2 [W_{S}^{1\pi}(r)]_{N\pi N} + S_{12}^I [W_{T}^{1\pi}(r)]_{N\pi N} \right\},
\]

\[
V_{N\pi N\Delta}(r) = (\tau_1 \cdot T_2) \left\{ \sigma_1 \cdot S_2 [W_{S}^{1\pi}(r)]_{N\pi N\Delta} + S_{12}^{II} [W_{T}^{1\pi}(r)]_{N\pi N\Delta} \right\},
\]

\[
V_{N\pi N\Delta}(r) = (T_1 \cdot T_2) \left\{ S_1 \cdot S_2 [W_{S}^{1\pi}(r)]_{N\pi N\Delta} + S_{12}^{III} [W_{T}^{1\pi}(r)]_{N\pi N\Delta} \right\},
\]

where \( S(T) \) are the transition operators connecting spin(isospin) 1/2 and 3/2. The explicit form of the tensor operators is

\[
S_{12}^I = 3(\sigma_1 \cdot \hat{r})(\sigma_2 \cdot \hat{r}) - \sigma_1 \cdot \sigma_2, \quad \text{(2.27)}
\]

\[
S_{12}^{II} = 3(\sigma_1 \cdot \hat{r})(S_2 \cdot \hat{r}) - \sigma_1 \cdot S_2, \quad \text{(2.28)}
\]

\[
S_{12}^{III} = 3(S_1 \cdot \hat{r})(S_2 \cdot \hat{r}) - S_1 \cdot S_2. \quad \text{(2.29)}
\]

\(^2\)Henceforward we will use the following notation for the tensor operators

<table>
<thead>
<tr>
<th></th>
<th>( N\pi \rightarrow N\pi )</th>
<th>( N\pi \rightarrow N\Delta )</th>
<th>( N\pi \rightarrow \Delta\Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I)</td>
<td>( \sigma )</td>
<td>( \tau )</td>
<td>( \sigma \cdot \sigma )</td>
</tr>
<tr>
<td>(II)</td>
<td>( \sigma \cdot \tau )</td>
<td>( \sigma \cdot \sigma )</td>
<td>( \tau \cdot \tau )</td>
</tr>
<tr>
<td>(III)</td>
<td>( \sigma \cdot \sigma )</td>
<td>( \sigma \cdot \sigma )</td>
<td>( \tau \cdot \tau )</td>
</tr>
</tbody>
</table>
When calculating the squared transition potentials appearing in Eq. (2.26) we have to deal with the following products in each channel,

\[(T_1 \cdot \tau_2)(\tau_2 \cdot T_1^\dagger), \quad S_{12}^{II}[S_{12}^{II}]^\dagger(\sigma_1 \cdot S_2)(\sigma_1 \cdot S_2)^\dagger, \quad S_{12}^{II}(\sigma_1 \cdot S_2)^\dagger + (\sigma_1 \cdot S_2)[S_{12}^{II}]^\dagger, \]

\[(T_1 \cdot T_2)(T_1^\dagger \cdot T_1^\dagger), \quad S_{12}^{II}[S_{12}^{II}]^\dagger(\sigma_1 \cdot S_2)(\sigma_1 \cdot S_2)^\dagger, \quad S_{12}^{II}(\sigma_1 \cdot S_2)^\dagger + (\sigma_1 \cdot S_2)[S_{12}^{II}]^\dagger . \]

We list the results for these products in Table 2.1 but the calculation details can be found in Appendix D. In this table we have written, in addition to the isospin operator products, the pieces we have to multiply by the squared Yukawa functions appearing in the first column.

<table>
<thead>
<tr>
<th>Isospin factors</th>
<th>Products in channel $NN \rightarrow N\Delta$</th>
<th>Products in channel $NN \rightarrow \Delta\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(T_1 \cdot \tau_2)(\tau_2 \cdot T_1^\dagger)$</td>
<td>$2 + \frac{2}{3}(\tau_1 \cdot \tau_2)$</td>
<td>$2 \frac{2}{3} (\sigma_1 \cdot \sigma_2)$</td>
</tr>
<tr>
<td>$(T_1 \cdot T_2)(T_1^\dagger \cdot T_1^\dagger)$</td>
<td>$\frac{1}{3} - \frac{4}{3}(\tau_1 \cdot \tau_2)$</td>
<td>$4 \frac{2}{3} S_{12}^\dagger$</td>
</tr>
<tr>
<td>$[W_{1S}^{12}(r)]^2$</td>
<td>$2 + \frac{2}{3} (\sigma_1 \cdot \sigma_2)$</td>
<td>$\frac{2}{3} S_{12}^\dagger$</td>
</tr>
<tr>
<td>$[W_{2S}^{12}(r)]^2 [W_{2T}^{12}(r)]^2$</td>
<td>$\frac{2}{3} - \frac{4}{3} (\sigma_1 \cdot \sigma_2)$</td>
<td>$\frac{4}{3} S_{12}^\dagger + \frac{2}{3} (\sigma_1 \cdot \sigma_2)$</td>
</tr>
</tbody>
</table>

Table 2.1: Products involved in the Born-Oppenheimer potential up to NLO where the $\Delta$-isobar is integrated out. In this table we show both the isospin factors and the pieces which appear multiplying the squared Yukawa functions in the first column. $S_{12}^\dagger$ is the tensor operator in the $NN \rightarrow NN$ channel.

The potential can be then written as

\[
V_{NN,NN}^{1\pi+2\pi+\cdots}(r) = \left[ V_C(r) + V_S(r) (\sigma_1 \cdot \sigma_2) + V_T(r) S_{12}^\dagger \right] \left( \tau_1 \cdot \tau_2 \right),
\]

with components,

\[
V_C(r) = \frac{\mathcal{F}_{2N\Delta}^2 (\mathcal{F}_{2N\Delta}^2 + 9 \mathcal{F}_{2NN}^2) m_\pi^2}{16\pi^2 \Delta} \left[ 2Y_2(m_\pi r)^2 + Y_0(m_\pi r)^2 \right],
\]

\[
V_S(r) = \frac{\mathcal{F}_{2N\Delta}^2 (\mathcal{F}_{2N\Delta}^2 - 18 \mathcal{F}_{2NN}^2) m_\pi^2}{972\pi^2 \Delta} \left[ Y_2(m_\pi r)^2 - Y_0(m_\pi r)^2 \right],
\]

\[
V_T(r) = \frac{\mathcal{F}_{2N\Delta}^2 (\mathcal{F}_{2N\Delta}^2 - 18 \mathcal{F}_{2NN}^2) m_\pi^2}{972\pi^2 \Delta} \left[ Y_2(m_\pi r)^2 - Y_0(m_\pi r)Y_2(m_\pi r) \right],
\]

\[
W_C(r) = \frac{\mathcal{F}_{2\pi\pi}^2 m_\pi}{12\pi} Y_0(m_\pi r)
\]

\[
W_S(r) = \frac{\mathcal{F}_{2\pi\pi}^2 m_\pi}{12\pi} Y_2(m_\pi r)
\]

\[
W_T(r) = \frac{\mathcal{F}_{2\pi\pi}^2 m_\pi}{5832\pi^2 \Delta} \left[ Y_2(m_\pi r)^2 - Y_0(m_\pi r)Y_2(m_\pi r) \right].
\]
Note that this potential is local and there are neither relativistic correction nor spin-orbit interaction. Therefore, this potential can be directly inserted into the Schrödinger equation to study the elastic NN channel and this is what we are going to do in the next sections.

2.4 Schrödinger equation in the elastic NN channel

2.4.1 Short distance behaviour

Specific knowledge of the short distance behaviour is needed to carry on with the renormalization program. Generally, on purely dimensional grounds, we have for \( r \to 0 \)

\[
V_i(r) \to \frac{C^{V,i}}{r^k}, \quad W_i(r) \to \frac{C^{W,i}}{r^k},
\]  

(2.37)

where

\[
C^{i}_{k=2n+m+r+1} \sim \frac{1}{f_{\pi}^n M_N^m \Delta^r},
\]

(2.38)

with \( \Delta \) the \( N\Delta \) splitting, and \( n, m \) and \( r \) non-negative integers. At short distances, the angular momentum dependence may be neglected when the index \( k > 2 \). The relevant issue to carry out the renormalization procedure and to generate finite results is to know whether the interaction is attractive or repulsive as we will see later on. As noted in the last section it turns out that the potential Eq. (2.30) has a leading singularity behaviour of the form \( \tilde{V}_{NN,NN} \sim -g_A^4/(\Delta f_{\pi}^4 r^6) \). In fact, the short-distance expansion of this potential can be written as

\[
V_C(r) = \frac{C^V_C}{r^6} + \ldots, \quad W_C(r) = \frac{C^W_C}{r^6} + \ldots,
\]

\[
V_S(r) = \frac{C^V_S}{r^6} + \ldots, \quad W_S(r) = \frac{C^W_S}{r^6} + \ldots,
\]

\[
V_T(r) = \frac{C^V_T}{r^6} + \ldots, \quad W_T(r) = \frac{C^W_T}{r^6} + \ldots,
\]

where the van der Waals coefficients are given by

\[
C^V_C = -\frac{f_{\pi N\Delta}^2 (9f_{\pi NN}^2 - f_{\pi N\Delta}^2)}{9m_{\pi}^4 \pi^2 \Delta}, \quad C^W_C = -\frac{f_{\pi N\Delta}^2 (18f_{\pi NN}^2 - f_{\pi N\Delta}^2)}{54m_{\pi}^4 \pi^2 \Delta},
\]

(2.39)

\[
C^V_S = \frac{f_{2\pi N\Delta}^2 (18f_{2\pi NN}^2 - f_{2\pi N\Delta}^2)}{108m_{\pi}^4 \pi^2 \Delta}, \quad C^W_S = \frac{f_{2\pi N\Delta}^2 (36f_{2\pi NN}^2 + f_{2\pi N\Delta}^2)}{648m_{\pi}^4 \pi^2 \Delta},
\]

(2.40)

\[
C^V_T = -\frac{f_{2\pi N\Delta}^2 (18f_{2\pi NN}^2 - f_{2\pi N\Delta}^2)}{108m_{\pi}^4 \pi^2 \Delta}, \quad C^W_T = -\frac{f_{2\pi N\Delta}^2 (36f_{2\pi NN}^2 + f_{2\pi N\Delta}^2)}{648m_{\pi}^4 \pi^2 \Delta}.
\]

(2.41)

An interesting feature of this potential arises with the identification \( h_A/g_A = f_{\pi N\Delta}/f_{\pi NN} \) which gives an identical short-distance behaviour as the one of the NLO-\( \Delta \) potential of ref. [111] (see also Appendix of ref. [130]).

The numerical values of the van der Waals coefficients \( MC_6 \) are summarized in Table 2.2 for the different components of the Born-Oppenheimer potential and two different choices for \( f_{\pi N\Delta} \). The van der Waals coefficients are additive: therefore one can obtain the coefficient corresponding to some given partial
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<table>
<thead>
<tr>
<th>$f_{\pi N\Delta}$ = $\frac{2}{3} f_{\pi NN}$</th>
<th>$MC_{6}^{V,C}$</th>
<th>$MC_{6}^{W,C}$</th>
<th>$MC_{6}^{V,S}$</th>
<th>$MC_{6}^{W,S}$</th>
<th>$MC_{6}^{V,T}$</th>
<th>$MC_{6}^{W,T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{\pi N\Delta}$ = $\frac{3}{2} f_{\pi NN}$</td>
<td>-1.952</td>
<td>-0.516</td>
<td>0.258</td>
<td>0.097</td>
<td>-0.258</td>
<td>-0.097</td>
</tr>
<tr>
<td>$f_{\pi N\Delta}$ = $\frac{3}{2} f_{\pi NN}$</td>
<td>-3.287</td>
<td>-0.767</td>
<td>0.383</td>
<td>0.155</td>
<td>-0.383</td>
<td>-0.155</td>
</tr>
</tbody>
</table>

Table 2.2: van der Waals $MC_{6}$ coefficients (in fm$^4$) for the different spin-isospin components of the Born-Oppenheimer potential. The number for two different choices of $f_{\pi N\Delta}$ corresponding to the quark-model formula with $N_c = 3$ and $N_c \to \infty$ are shown.

<table>
<thead>
<tr>
<th>$f_{\pi N\Delta}$ = $\frac{6}{5} f_{\pi NN}$</th>
<th>$MC_{6,1}^{1S_0}$</th>
<th>$MC_{6,3}^{3S_1}$</th>
<th>$MC_{6,6}^{E_1}$</th>
<th>$MC_{6,3}^{3D_1,-}$</th>
<th>$-R_{1}^{\pi}$</th>
<th>$-R_{-}^{\pi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{\pi N\Delta}$ = $\frac{3}{2} f_{\pi NN}$</td>
<td>-3.534</td>
<td>-0.438</td>
<td>0.095</td>
<td>-0.505</td>
<td>-0.572</td>
<td>-0.370</td>
</tr>
<tr>
<td>$f_{\pi N\Delta}$ = $\frac{3}{2} f_{\pi NN}$</td>
<td>-5.669</td>
<td>-1.068</td>
<td>0.232</td>
<td>-1.232</td>
<td>-1.397</td>
<td>-0.904</td>
</tr>
</tbody>
</table>

Table 2.3: van der Waals $MC_{6}$ coefficients (in fm$^4$) in the $^1S_0$ and $^3S_1 - ^3D_1$ (deuteron) channels of the Born-Oppenheimer potential. We also present the corresponding negative eigenvalues $-R_{1}^{\pi}$ and $-R_{-}^{\pi}$ in the deuteron channel case. The number for two different choices of $f_{\pi N\Delta}$ corresponding to the quark-model formula with $N_c = 3$ and $N_c \to \infty$ are shown.

wave by adding the individual contributions, i.e.

$$MC_{6} = MC_{6}^{V,C}(r) + \tau MC_{6}^{W,C} + \sigma \left( MC_{6}^{V,S} + \tau MC_{6}^{W,S} \right) + S_{12} \left( MC_{6}^{V,T} + \tau MC_{6}^{W,T} \right),$$

with $\tau = \tau_1 \cdot \tau_2$ and $\sigma = \sigma_1 \cdot \sigma_2$. This is done for the singlet $^1S_0$ and the triplet $^3S_1 - ^3D_1$ channels in Table 2.3. For our numerical calculations we take $m_\pi = 138.03$ MeV, $2\mu_{np} = M_N = 938.918$ MeV, $M_\Delta = 1232$ MeV, $g_A = 1.29$, $f_\pi = 92.4$, $f_{\pi NN} = g_A m_\pi / 2 f_\pi$. For the $\pi N\Delta$ coupling constant we will take $f_{\pi N\Delta} = 6/5 f_{\pi NN}$ which corresponds to $N_c = 3$ a value in between $f_{\pi N\Delta} = 3/2 f_{\pi NN}$ for $N_c \to \infty$ and the Chew-Low value $f_{\pi N\Delta} = 2 f_{\pi NN}$.

2.4.2 The singlet channel

As already noted the improved OPE potential diverges when extrapolating to the origin just like a vdW potential. The use of renormalization is then mandatory [127, 130]. To carry out the renormalization program, we will use the superposition principle of boundary conditions since the potential and the renormalization conditions can be explicitly disentangled. The $^1S_0$ wave function in the $np$ c.m. system can be written as

$$\Psi(x) = \frac{1}{\sqrt{4\pi r}} u(r) \chi_{p,n}^{m_s},$$

with the total spin $s = 0$ and $m_s = 0$. The function $u(r)$ is the reduced S- wave function and satisfies the following reduced Schrödinger equation

$$-u''_k(r) + U_{1S_0}(r) u_k(r) = k^2 u_k(r),$$
where $k$ is the c.m. and $U_{1,S_0}$ the reduced potential defined as

$$U_{1,S_0}(r) = M \left( V_C(r) + W_C(r) - 3W_S(r) - 3W_S(r) \right). \quad (2.45)$$

For a finite energy scattering state we solve for the BO potential Eq. (2.30) with the asymptotic normalization

$$u_k(r) \to \frac{\sin(kr + \delta_0(k))}{\sin \delta_0(k)}, \quad (2.46)$$

with $\delta_0(k)$ the phase shift. For a potential falling off exponentially $\sim e^{-mx\tau}$ at large distances, we have the effective range expansion, valid for momenta $|k| < m_x/2$,

$$k \cot \delta_0(k) = -\frac{1}{\alpha_0} + \frac{1}{2} r_0 k^2 + v_2 k^4 + \ldots,$$  
(2.47)

with $\alpha_0$ the scattering length and $r_0$ the effective range. At short distances the BO potential behaves as

$$U_{1,S_0}(r) \to \frac{MC_0 S_0}{r^6} = -\frac{R^4}{r^6}, \quad (2.48)$$

which is a van der Waals type interaction. The numerical value for $MC_0 S_0$ is listed in Tab. 2.3. Since the value of the coefficient is negative, with the typical length scale $R = (-MC_0)^{1/4}$, the solution at short distances is of oscillatory type 3

$$u_k(r) \to A \left( \frac{r}{R} \right)^{3/2} \sin \left[ \frac{1}{2} \left( \frac{R}{r} \right)^2 + \varphi(k) \right], \quad (2.51)$$

where $A$ is a normalization constant and $\varphi$ an undetermined phase, i.e., a free parameter which may in principle depend on energy. If we impose orthogonality for $r > r_c$ between two different energy states we get

$$u_k'(r_c)u_k(r_c) - u_k'(r_c)u_p(r_c) = (k^2 - p^2) \int_{r_c}^{\infty} u_k(r)u_p(r) dr = 0. \quad (2.52)$$

In particular we fix the undetermined phase $\varphi$ by imposing orthogonality between the zero energy state and the state with momentum $p$. Renormalization implies taking the limit $r_c \to 0$ and the short distance

\[3\]Indeed for small distances one can undertake a WKB approximation. At this point it is interesting to discriminate the crucial difference between repulsive and attractive singular potentials as compared to the customary case of non-singular ones (i.e. $\lim_{r \to 0} r^2 |V(r)| < \infty$). For the usual regular potentials there are a regular and irregular solution at the origin, and the regularity condition $u(0) = 0$ fixes uniquely the solution. However, since the potential is singular and attractive there are two linearly independent solutions, so regularity at the origin does not select a unique solution. Indeed, at short distances the De Broglie wavelength is slowly varying, $d[U(r)] \sim \frac{\varpi}{dr} \ll 1$ and hence a WKB approximation holds [131, 132], yielding for $r \to 0$

$$u_k(r) \to C \left( \frac{r}{R_n} \right)^{n/4} \sin \left[ \frac{2}{n-2} \left( \frac{R_n}{r} \right)^{n-1} + \varphi_k \right], \quad (2.49)$$

where $R_n = (2\mu C_n)^{1/(n-2)}$ corresponds to the highest vdW scale. The phase $\varphi_k$ is arbitrary and could, in principle, be energy dependent. In the case of a singular repulsive potential i.e. $R^2U(r) = +(R/r)^n$ one can likewise apply a WKB approximation where one has the general solution

$$u_k(r) \to A \left( \frac{r}{R_n} \right)^{n/4} \exp \left[ -\frac{2}{n-2} \left( \frac{R_n}{r} \right)^{n-1} \right] + B \left( \frac{r}{R_n} \right)^{n/4} \exp \left[ +\frac{2}{n-2} \left( \frac{R_n}{r} \right)^{n-1} \right]. \quad (2.50)$$

In this case the regular solution fixes $B = 0$. 

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phase $\varphi$ becomes energy independent when orthogonality is applied to the short distance solution,

$$\sin[\varphi(k) - \varphi(p)] = 0. \tag{2.53}$$

As a consequence we get the following energy independent condition

$$\frac{u_p'(r_c)}{u_p(r_c)} = \frac{u_k'(r_c)}{u_k(r_c)}. \tag{2.54}$$

Thus, for the zero energy state we solve

$$- u_0''(r) + U_{1S_0}(r) u_0(r) = 0, \tag{2.55}$$

with the asymptotic normalization at large distances

$$u_0(r) \to 1 - \frac{r}{\alpha_0}, \tag{2.56}$$

where $\alpha_0$ is the scattering length. In this equation $\alpha_0$ is an input, so one needs to integrate Eq. (2.55) from infinity to the origin (in contrast with the usual procedure of integrating from the origin to infinity).

Now, we can use the superposition principle of boundary conditions to write

$$u_0(r) = u_{0,c}(r) - \frac{1}{\alpha_0} u_{0,s}(r), \tag{2.57}$$

where $u_{0,c}(r) \to 1$ and $u_{0,s}(r) \to r$ when $r \to \infty$. On the other hand the finite energy state can be written via superposition principle as

$$u_k(r) = u_{k,c}(r) + k \cot \delta_0(k) u_{k,s}(r), \tag{2.58}$$

where $u_{k,c}(r) \to \cos kr$ and $u_{k,s}(r) \to \sin kr/r$ when $r \to \infty$. One can use orthogonality between the zero- and finite-energy states to deduce the phase shifts fixing the $^1S_0$ scattering length as an input parameter. The result is shown in Fig. 2.3 (left-upper panel) in comparison with OPE and an average of the Nijmegen potentials. The superposition principle allows further simplifications which will be analyzed in the next chapter.

### 2.4.3 Triplet channel with three counterterms

The $^3S_1 - ^3D_1$ wave function in the $np$ c.m. system can be written as

$$\Psi(x) = \frac{1}{\sqrt{4\pi r}} \left[ u(r) \sigma_p \cdot \sigma_n + \frac{w(r)}{\sqrt{8}} \left( 3\sigma_p \cdot \hat{x} \sigma_n \cdot \hat{x} - \sigma_p \cdot \sigma_n \right) \right] \chi_{^3S_0}^{m_s}, \tag{2.59}$$

with total spin $s = 1$ and $m_s = 0, \pm 1$; $\sigma_p$ and $\sigma_n$ are the Pauli matrices for the proton and the neutron respectively. The functions $u(r)$ and $w(r)$ are the reduced S- and D-wave components of the relative
wave function respectively. They satisfy the coupled set of equations in the $^3S_1 - ^3D_1$ channel

\[ -u''(r) + U_{S_1}(r)u(r) + U_{E_1}(r)w(r) = k^2 u(r), \]
\[ -w''(r) + U_{E_1}(r)u(r) + \left[U_{D_1}(r) + \frac{6}{r^2}\right]w(r) = k^2 w(r), \]

with $U_{S_1}(r)$, $U_{E_1}(r)$ and $U_{D_1}(r)$ the corresponding matrix elements of the coupled channel potential, which are

\[ U_{S_1} = M \left[V_C - 3W_C + V_S - 3W_S \right], \]
\[ U_{E_1} = M \left[2\sqrt{2}(V_T - 3W_T) \right], \]
\[ U_{D_1} = M \left[V_C - 3W_C + V_S - 3W_S - 2V_T + 6W_T \right]. \]

In fact, our potential shares a remarkable feature with the chiral potentials at LO, NLO-Δ and N^2LO-Δ of Refs. [109–111, 124], namely, the spin-orbit coupling vanishes as well as relativistic corrections. As a consequence, for a given isospin the total potential can be diagonalized. The corresponding eigen-potentials depend just on the isospin of the channel and not on the total angular momentum j. All the j dependence goes into the matrix which diagonalizes the potential. Therefore, in the triplet channel $j = 1$ one can diagonalize the corresponding potential matrix at short distances

\[
\begin{pmatrix}
MC_{6,^3S_1} & MC_{6,E_1} \\
MC_{6,E_1} & MC_{6,^3D_1}
\end{pmatrix} = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
-R^4_+ & 0 \\
0 & -R^4_-
\end{pmatrix} \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix},
\]

with $\cos \theta = 1/\sqrt{3}$ the mixing angle in this channel. In the diagonal basis $(v_+, v_-)$ the coupled channel Eqs. (2.60), (2.61) decouple with the transformation

\[ \begin{pmatrix} v_+ \\ v_-
\end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix} u \\ w
\end{pmatrix}, \]

where the eigen functions $v_{\pm}$ satisfy the Schrödinger equations

\[ -v''_{\pm} - \frac{R^4_{\pm}}{r^6} v_{\pm} = k^2 v_{\pm}. \]

At this point it is important to discuss how the renormalization program works which depends on the sign of this eigen-values:

1. If both eigen-values are negative, i.e., attractive-attractive eigen-potentials, the general solutions are of oscillatory type

\[ v_+(r) = \left(\frac{r}{R_+}\right)^{\frac{3}{2}} C_+ \sin \left[\frac{1}{2} \frac{R^2_+}{r^2} + \varphi_+(k)\right], \]
\[ v_-(r) = \left(\frac{r}{R_-}\right)^{\frac{3}{2}} C_- \sin \left[\frac{1}{2} \frac{R^2_-}{r^2} + \varphi_-(k)\right], \]

with $\varphi_{\pm}$ two short distance phases which must be fixed independently and $C_{\pm}$ suitable normalization constants. This is for example the case of chiral NNLO [110], NLO-Δ [110, 111, 124] and our BO potential Eq. (2.30). However, orthogonality of different energy solutions of Eqs. (2.60), (2.61)
requires these phases to be energy independent,

\[
u'_p(r_c)u_k(r_c) - u_p(r_c)u'_k(r_c) + w'_p(r_c)w_k(r_c) - w_p(r_c)w'_k(r_c) = \frac{1}{R_+} \sin (\phi_+(k) - \phi_+(p)) + \frac{1}{R_-} \sin (\phi_-(k) - \phi_-(p)) = 0. \tag{2.70}
\]

We fix them from deuteron physical properties, namely, the binding energy and asymptotic D/S ratio (see below). Once these two quantities are fixed, scattering states can be completely determined by fixing in addition the scattering length of the \(3\)\(S_1\) phase, and then imposing orthogonality to the deuteron state. This is equivalent to renormalizing with three counter-terms in momentum space.

2. If one of the eigen-values is positive and other is negative, i.e., attractive-repulsive eigen-potentials, then one of the solutions is regular and exponentially growing function and the other is of oscillatory type with a short distance phase \(\phi\). The orthogonality condition between different energy solutions fixes this phase to be energy independent as in the singlet channel case. This is e.g. the case of the OPE and OBE potential that will be analyzed in detail latter.

3. Finally, if both eigen-values are positive, both solutions are regular and exponentially growing functions without any short distance phase. The orthogonality conditions are immediately satisfied.

In the case of the deuteron one solves Eqs. (2.60), (2.61) for negative energy

\[
k^2 = -\gamma^2 = -MB_d, \tag{2.71}
\]

with \(\gamma\) the deuteron wave number and \(B_d\) the deuteron binding energy together with the asymptotic condition at infinity

\[
u(r) \rightarrow A_S e^{-\gamma r}, \tag{2.72}
\]
\[
w(r) \rightarrow A_D e^{-\gamma r} \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2}\right), \tag{2.73}
\]

where \(A_S\) and \(A_D\) are the S- and D-wave normalization factors. The asymptotic D/S ratio parameter \(\eta\) is defined as \(\eta = A_D/A_S\). Further, we can use the superposition principle of boundary conditions to write

\[
u(r) = u_S(r) + \eta u_D(r), \tag{2.74}
\]
\[
w(r) = w_S(r) + \eta w_D(r), \tag{2.75}
\]

with the auxiliary problems

\[
\begin{pmatrix} u_S \\ w_S \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\gamma r}, \tag{2.76}
\]
\[
\begin{pmatrix} u_D \\ w_D \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\gamma r} \left(1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2}\right), \tag{2.77}
\]

which solutions depend on the deuteron binding energy through \(\gamma\) and the potential.

In order to obtain the regularized wave functions we need to fix two free parameters and we choose to fix \(\gamma\) and \(\eta\) to their experimental values (see table 2.4). This way we obtain \(u(r)\) and \(w(r)\) by integrating
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Table 2.4: Deuteron properties for the Born-Oppenheimer potentials without (BO) and with (BO-Regular) vertex form factors. The computation is made by fixing $\gamma$ and $\eta$ to their experimental values and renormalizing in the case of BO with no form factors and by fixing $\gamma$ and a regular boundary condition in the case of BO with form factors (see Sec. (2.4.4)). We compare with OPE, NijmII, Reid93 and the experimental values of Ref. [169] and references therein.

<table>
<thead>
<tr>
<th>Set</th>
<th>$\gamma$ (fm$^{-1}$)</th>
<th>$\eta$</th>
<th>$A_S$ (fm$^{-1/2}$)</th>
<th>$r_m$ (fm)</th>
<th>$Q_d$(fm$^2$)</th>
<th>$P_D$ (%)</th>
<th>$\langle r^{-1}\rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>OPE</td>
<td>Input</td>
<td>0.02633</td>
<td>0.8681</td>
<td>1.9351</td>
<td>0.2762</td>
<td>7.88</td>
<td>0.476</td>
</tr>
<tr>
<td>BO ($N_c = 3$)</td>
<td>Input</td>
<td>0.0259</td>
<td>0.8659</td>
<td>1.9281</td>
<td>0.2634</td>
<td>8.08</td>
<td>0.506</td>
</tr>
<tr>
<td>BO (Regular)</td>
<td>Input</td>
<td>0.0259</td>
<td>0.8847</td>
<td>1.9680</td>
<td>0.2797</td>
<td>6.43</td>
<td>0.489</td>
</tr>
<tr>
<td>NijmII</td>
<td>0.231605</td>
<td>0.02521</td>
<td>0.8845</td>
<td>1.9675</td>
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<td>0.4502</td>
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<tr>
<td>Reid93</td>
<td>0.231605</td>
<td>0.02514</td>
<td>0.8845</td>
<td>1.9686</td>
<td>0.2703</td>
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</tr>
<tr>
<td>Exp.</td>
<td>0.231605</td>
<td>0.0256(4)</td>
<td>0.8846(9)</td>
<td>1.9754(9)</td>
<td>0.2859(3)</td>
<td>5.67(7)</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2.4: Deuteron properties for the Born-Oppenheimer potentials without (BO) and with (BO-Regular) vertex form factors. The computation is made by fixing $\gamma$ and $\eta$ to their experimental values and renormalizing in the case of BO with no form factors and by fixing $\gamma$ and a regular boundary condition in the case of BO with form factors (see Sec. (2.4.4)). We compare with OPE, NijmII, Reid93 and the experimental values of Ref. [169] and references therein.

Figure 2.2: Deuteron wave functions, $u$ (left panel) and $w$ (right panel), as a function of the radius (in fm) for the BO potential and for the regularized-BO potential with vertex form factors, compared to the OPE potential and the Nijmegen II wave functions [57]. The value $\eta = 0.0256(4)$ is taken for the asymptotic D/S ratio.

Eqs. (2.60), (2.61) from $r \to \infty$ to $r = 0$ with Eqs. (2.74)- (2.77) as boundary conditions. $A_S$ can be later determined from

$$\int_0^{\infty} dr \left[ u(r)^2 + w(r)^2 \right] = 1,$$

i.e., from demanding the deuteron normalization to be equal to unity. The renormalized deuteron wave functions are depicted in Fig. 2.2 and compared to the Nijmegen II wave functions [57]. The asymptotic D/S ratio is taken to be $\eta = 0.0256$. One can see in Fig. 2.2 the appearance of an increasing number of oscillations in the wave function when the radius approaches zero. They have no appreciable effect on the physics of the deuteron, as they happen at very short distances. Therefore they have a very small effect on the computation of deuteron observables.

Here, we also compute the matter radius, which reads,

$$r^2_m = \left< r^2 \right> = \frac{1}{4} \int_0^{\infty} r^2 (u(r)^2 + w(r)^2) dr,$$
the quadrupole moment (without meson exchange currents)

\[ Q_d = \frac{1}{20} \int_0^\infty r^2 w(r)(2\sqrt{2}u(r) - w(r))dr, \]  

(2.80)

the D-state probability

\[ P_D = \int_0^\infty w(r)^2 dr, \]  

(2.81)

and the inverse moment of the radius

\[ \langle r^{-1} \rangle = \int_0^\infty r^{-n}(u(r)^2 + w(r)^2)dr, \]  

(2.82)

which, as is well known, appear in the multiple scattering expansion of the \( \pi \)-deuteron scattering length.

In Tab. 2.4 we show our renormalized results for \( f_{\pi N\Delta} = \frac{6}{5}f_{\pi NN} \).

For scattering states in the \( ^3S_1 - ^3D_1 \) channel the Schrödinger equation has two linearly independent solutions that we denote, as usually, the \( \alpha \) and \( \beta \) solutions. These solutions are defined by its asymptotic normalization, which read,

\[ u_{k,\alpha}(r) \rightarrow \hat{j}_0(kr) \cot \delta_1 - \hat{y}_0(kr), \]  

(2.83)

\[ w_{k,\alpha}(r) \rightarrow \tan \epsilon \left( \hat{j}_2(kr) \cot \delta_1 - \hat{y}_2(kr) \right), \]  

(2.84)

\[ u_{k,\beta}(r) \rightarrow -\tan \epsilon \left( \hat{j}_0(kr) \cot \delta_2 - \hat{y}_0(kr) \right), \]  

(2.85)

\[ w_{k,\beta}(r) \rightarrow \hat{j}_2(kr) \cot \delta_2 - \hat{y}_2(kr), \]  

(2.86)

where \( \hat{j}_l(x) = x j_l(x) \) and \( \hat{y}_l(x) = x y_l(x) \) are the reduced spherical Bessel functions and \( \delta_1 \) and \( \delta_2 \) are the eigen-phases \(^4\) in the \( ^3S_1 \) and \( ^3D_1 \) channels; \( \epsilon \) is the mixing angle \( E_1 \).

In order to determine the eigen phase shifts (\( \delta_1, \delta_2 \) and \( \epsilon \)), we define the four auxiliary problems

\[ u_1(r) \rightarrow \hat{y}_0(kr), \quad w_1(r) \rightarrow 0, \]  

\[ u_2(r) \rightarrow \hat{j}_0(kr), \quad w_2(r) \rightarrow 0, \]  

\[ u_3(r) \rightarrow 0, \quad w_3(r) \rightarrow \hat{y}_2(kr), \]  

\[ u_4(r) \rightarrow 0, \quad w_4(r) \rightarrow \hat{j}_2(kr), \]

which depend on the potential and can be obtained by integrating in Eqs. (2.60), (2.61) for positive energy \( k > 0 \). Thus, the general solution satisfying the \( \alpha \) and \( \beta \) asymptotic conditions can be written

\(^4\)There are two commonly used parameterizations, which are the same for un-coupled waves, but different for coupled waves. These are the bar-phases (Stapp, Ypsilantis and Metropolis) and the eigen-phases (Blatt and Biedenharn). We refer to Appendix E for more details.
via superposition principle as

\begin{align*}
    u_{k,\alpha}(r) &= -u_1(r) + \cot \delta_1 u_2(r) - \tan \epsilon u_3(r) + \tan \epsilon \cot \delta_1 u_4(r), \\
    w_{k,\alpha}(r) &= -w_1(r) + \cot \delta_1 w_2(r) - \tan \epsilon w_3(r) + \tan \epsilon \cot \delta_1 w_4(r),
\end{align*}

\begin{align*}
    u_{k,\beta}(r) &= \tan \epsilon u_1(r) - \tan \epsilon \cot \delta_2 u_2(r) - u_3(r) + \cot \delta_2 u_4(r), \\
    w_{k,\beta}(r) &= \tan \epsilon w_1(r) - \tan \epsilon \cot \delta_2 w_2(r) - w_3(r) + \cot \delta_2 w_4(r).
\end{align*}

The orthogonality constraints between the deuteron and scattering states generate the following boundary conditions

\begin{align*}
    u_{r=rc} u_{k,\alpha}' - u_{r=rc} u_{k,\alpha} + w_{r=rc} w_{k,\alpha}' - w_{r=rc} w_{k,\alpha} &= 0, \\
    u_{r=rc} u_{k,\beta}' - u_{r=rc} u_{k,\beta} + w_{r=rc} w_{k,\beta}' - w_{r=rc} w_{k,\beta} &= 0.
\end{align*}

Another equation arises from orthogonality between the \( \alpha \) and \( \beta \) solutions, i.e.,

\begin{align*}
    u_{k,\alpha} u_{k,\beta}' - u_{k,\alpha}' u_{k,\beta} + w_{k,\alpha} w_{k,\beta}' - w_{k,\alpha}' w_{k,\beta} &= 0,
\end{align*}

Therefore, once the deuteron is known, the renormalized eigen phase shifts can be calculated solving Eqs. (2.91), (2.92) and (2.93) by integrating in the four auxiliary problems defined above. Our results are shown in Fig. 2.3 in comparison with OPE and an average of the Nijmegen potentials. As we can see from Fig. 2.3, although a clear improvement of OPE is achieved the singlet \( ^1S_0 \) channel suffers an over-binding of \( 5^0 - 10^0 \). This result is a common problem in all TPE approaches [130] and as we see it persists in our approach. In the next section we will try to improve on this description by introducing vertex form factors implementing nucleon finite effects.

### 2.4.4 Regularization of the potential with vertex form factors

One of the ways of getting the needed repulsion in the \( ^1S_0 \)-channel is by taking into account the finite size of baryons. For example, the use of harmonic oscillator baryon wave functions at a quark-model level may justify the introduction of a Gaussian vertex form factor for nucleons. In fact, this corresponds to the replacement of the potential in momentum space \( V_{AB,CD}(q) \rightarrow V_{AB,CD}(q) [\Gamma_{mAB}(q)\Gamma_{mCD}(q)] \), with an exponential parameterization for \( \Gamma_{mAB} \), i.e.,

\begin{equation}
    \Gamma_{mAB}^{\exp}(q^2) = \exp \left[ \frac{q^2 - m^2}{\Lambda^2_{mAB}} \right],
\end{equation}

being in our case the meson \( m \) the pion. To obtain the corresponding regularized potential in coordinate space when one introduces vertex form factors the following Fourier transform appears,

\begin{equation}
    \int \frac{d^3k}{(2\pi)^3} \frac{(s_1 \cdot k)(s_2 \cdot k)}{k^2 + m^2} e^{-2 \left( \frac{s_1^2 + s_2^2}{k^2} \right)} e^{i k \cdot r},
\end{equation}
Chapter 2. Renormalization of Spin-Flavor Van der Waals Forces

Figure 2.3: Renormalized (eigen) phase shifts for the OPE and ∆-Born-Oppenheimer potentials as a function of the c.m. np momentum $p$ in the spin singlet $^1S_0$ (one counterterm) and triplet $^3S_1-^3D_1$ (three counterterms) channels, compared to an average of the Nijmegen potentials [57]. We take $f_{NN}^2/4\pi = 0.07388[57]$ and $f_{N\Delta}/f_{NN} = 5/6$.

where $s$ denotes both, the spin-$\frac{1}{2}$ operator $\sigma$ or the spin-$\frac{3}{2}$ transition operator $S$ depending on the channel under consideration, and

$$1/\Lambda^2 = \begin{cases} 
1/\Lambda_{sNN}^2, & NN \rightarrow NN \text{ channel} \\
1/\Lambda_{sN\Delta}^2, & NN \rightarrow N\Delta \text{ channel} \\
\frac{1}{7} (1/\Lambda_{sNN}^2 + 1/\Lambda_{sN\Delta}^2), & NN \rightarrow \Delta\Delta \text{ channel}
\end{cases}$$

These integrals can be easily done as follows. Using that,

$$((s_1 \cdot k)(s_2 \cdot k)) e^{+ik \cdot r} = -((s_1 \cdot \nabla)(s_2 \cdot \nabla)) e^{+ik \cdot r}, \quad (2.96)$$

we can write the integral in the form

$$\int \frac{d^3k}{(2\pi)^3} \frac{(s_1 \cdot k)(s_2 \cdot k)}{k^2 + m^2} e^{-2\left(\frac{k^2 + m^2}{\Lambda^2}\right)} e^{+ik \cdot r}$$

$$= -e^{-2m^2/\Lambda^2} ((s_1 \cdot \nabla)(s_2 \cdot \nabla)) F(r)$$

$$= -\frac{1}{3} e^{-2m^2/\Lambda^2} \left[ \left(\frac{2}{r} F'(r) + F''(r)\right) (s_1 \cdot s_2) - \left(\frac{2}{r} F'(r) + F''(r)\right) \hat{S}_{12} \right], \quad (2.97)$$
being $\mathbf{S}_{12} = 3(\mathbf{s}_1 \cdot \hat{r})(\mathbf{s}_2 \cdot \hat{r}) - \mathbf{s}_1 \cdot \mathbf{s}_2$ a generalized tensor operator and $F(r)$ a radial function which depends on $\Lambda$,

$$F(r) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + m^2_\pi} e^{-2k^2/\Lambda^2} e^{ik \cdot r}, \quad (2.98)$$

satisfying the differential equation

$$(-\nabla^2 + m^2_\pi) F(r) = \int \frac{d^3k}{(2\pi)^3} e^{-2k^2/\Lambda^2} e^{ik \cdot r} = \frac{\Lambda^3}{16\sqrt{2}\pi^3} e^{-r^2(\Lambda^2/8)}, \quad (2.99)$$

that is

$$- F''(r) - \frac{2}{r} F'(r) + m^2_\pi F(r) = \frac{\Lambda^3}{16\sqrt{2}\pi^3} e^{-r^2(\Lambda^2/8)}. \quad (2.100)$$

To solve this ordinary differential equation we need two boundary conditions which are obtained from a direct evaluation of the integrals,

$$F(0) = \sqrt{\frac{\pi}{2}} \frac{\Lambda}{4\pi^2} \frac{m_\pi}{\sqrt{m_\pi}} \text{Erf} \left( \frac{\sqrt{2}m_\pi}{\Lambda} \right), \quad (2.101)$$

$$F'(0) = 0. \quad (2.102)$$

Finally, we find

$$F(r) = \frac{1}{8\pi r} e^{-m_\pi r} \left\{ 1 + \text{Erf} \left( \frac{r\Lambda^2 - 4m_\pi^2}{2\sqrt{2}\Lambda} \right) - e^{2m_\pi r} \text{Erf} \left( \frac{r\Lambda^2 + 4m_\pi}{2\sqrt{2}\Lambda} \right) \right\} e^{2m_\pi^2/\Lambda^2}, \quad (2.103)$$

where Erf and Erfc are the error function and complementary error function respectively. Making use of this $F(r)$ and Eq. (2.98) the transition potentials after introducing vertex form factors read

$$V_{NN,NN}^{Reg}(r) = (\pi_1 \cdot \pi_2) \left\{ \sigma_1 \cdot \sigma_2 [W_{1S}^{1\pi}(r)]_{NN,NN} + S_{12}^{I} [W_{1T}^{1\pi}(r)]_{NN,NN} \right\}, \quad (2.104)$$

$$V_{NN,NN}^{Reg}(r) = (\pi_1 \cdot \pi_2) \left\{ \sigma_1 \cdot \sigma_2 [W_{1S}^{1\pi}(r)]_{NN,NN} + S_{12}^{II} [W_{1T}^{1\pi}(r)]_{NN,NN} \right\}, \quad (2.105)$$

$$V_{NN,NN}^{Reg}(r) = (\pi_1 \cdot \pi_2) \left\{ \sigma_1 \cdot \sigma_2 [W_{1S}^{1\pi}(r)]_{NN,NN} + S_{12}^{III} [W_{1T}^{1\pi}(r)]_{NN,NN} \right\}, \quad (2.106)$$

where we have defined the regularized Yukawa functions

$$[W_{1S}^{1\pi}(r)]_{AB,CD} = \frac{f_{AC}f_{BD}}{3m_\pi^2} e^{-2m_\pi^2/\Lambda_{AB,CD}^2} \left( \frac{2}{r} F'(r) + F''(r) \right), \quad (2.107)$$

$$[W_{1T}^{1\pi}(r)]_{AB,CD} = -\frac{f_{AC}f_{BD}}{3m_\pi^2} e^{-2m_\pi^2/\Lambda_{AB,CD}^2} \left( \frac{F'(r)}{r} - F''(r) \right). \quad (2.108)$$

In the next section we will use this regularized BO potential incorporating vertex form factors to see if we can improve the description of the deuteron as well as the phase shifts.

---

5They are defined as

$$\text{Erf}(z) = 1 - \text{Erfc}(z) = \frac{2}{\sqrt{\pi}} \int_0^z dt e^{-t^2} = 1 - \frac{e^{-z^2}}{\sqrt{\pi}z} \left[ 1 + O(z^{-1}) \right]$$
2.4.5 Triplet channel with one counterterm and higher partial waves

One of the advantages of regularizing the potential is that singularities at short distances are removed. This means that one can proceed as usual integrating out with a regular boundary condition, \( u(0) = 0 \). However, if so, one pays the price of fine-tuning, especially in S-waves. To avoid this fine-tuning one can renormalize the S-waves with a regular boundary condition. In the singlet channel this is done just as explained in Sec. 2.4.2. In the case of the triplet channel at short distances one has a coupled channel Coulomb problem where the short distance behaviour can generally be written as a linear admixture of regular and irregular solutions,

\[
\begin{align*}
  u(r) &\sim a_1 r + a_2, \\
  w(r) &\sim b_1 r^3 + b_2 r^{-2}. 
\end{align*}
\]

In order to get a normalizable wave function we must impose the regular solution for the D-wave, meaning \( b_2 = 0 \). The renormalized deuteron then corresponds to integrate in the \( u_S, u_D \) and \( w_S, w_D \) functions, Eqs. (2.76) and (2.77), fixing the bound state energy \( \gamma \) to its experimental value and imposing the regularity condition of the D-wave, which for \( b_2 = 0 \), can be re-written as

\[
 r_c \frac{w'(r_c)}{w(r_c)} = 3, 
\]

enabling to extract the D/S-wave ratio as

\[
\eta = \frac{-3w_S(r_c) + r_c w'_S(r_c)}{3w_D(r_c) - r_c w'_D(r_c)}. 
\]

Hence we know \( u(r) \) and \( w(r) \) up to the normalization constant \( A_S \) that we can fixed \textit{a posteriori}. This procedure provides the deuteron wave functions unambiguously. Note that in our regularized BO potential we have three free parameters to be fitted, namely, \( \Lambda_{\pi NN}, \Lambda_{\pi N\Delta} \) and \( f_{\pi N\Delta} \). Actually the variation of \( f_{\pi N\Delta} \) is associated with the usual fit of \( h_A \) to the experimental data carried out in chiral potentials. Now, one can try to fit these free parameters to an average of the Nijmegen \( ^1S_0 \) phase shifts plus the Nijmegen deuteron, i.e., an average of \( \eta \) and \( A_S \) as obtained from the Nijmegen potentials. Doing so one gets,\(^6\) \( \Lambda_{\pi NN} = 2401.11 \text{ MeV}, \Lambda_{\pi N\Delta} = 1084.77 \text{ MeV} \) and \( f_{\pi N\Delta} = 2.045 \) with \( \chi^2/\text{DOF} = 0.542 \). The deuteron properties for this set of constants are shown in Tab. 2.4 which, with the exception of \( P_D \), are in good agreement to the Nijmegen values as should be expected. Therefore once the potential has been determined one can proceed to predict the rest of partial waves by integrating out with a regular boundary condition.\(^7\) All the partial waves phase shifts with total angular momentum \( j \leq 5 \) when a fit to the \( ^1S_0 + \) deuteron together is done, are plotted in Figs. 2.4 to 2.9.

As one can note by looking at these higher partial waves, the \( ^1S_0 \) wave together with the deuteron does not give enough information about the form factor in order to extract the correct P- and D- phase shifts. Actually waves with \( L \geq 3 \) are not as sensible to the form factor as can be the P- and D- waves and this is in part because for waves with high angular momentum what is most important is the tail of the potential being perturbation theory applicable in this case. The fact is that, since our potential has not

\(^6\)For the \( ^1S_0 \) phase shifts we have taken an average of the Nijmegen potentials at 11 different energies \([57, 170, 171]\) which correspond to the PWA energies. A fit to the PWA \([56]\) is much more limited due to the small phase errors and as a consequence bigger \( \chi^2/\text{DOF} \) arise.

\(^7\)In Appendix E we explain how to extract the correct partial wave phase shift in the case of a regular potential.
spin-orbit interaction we should not expect to have a good agreement by looking partial waves separately. This is a fundamental problem of this regularized BO potential which would need further study. We note that recently [172] a renormalization of the chiral TPE potential with explicit delta excitations for NN scattering has been carried out. Due to the singular nature of the chiral potentials correlations between different partial waves are generated with a better general description. In view of the identical short distance behaviour of our BO potential it would be interesting to reanalyze the present calculations incorporating the above mentioned correlations.
Figure 2.5: Predicted phase shifts with $J = 1$. 
Figure 2.6: Predicted phase shifts with $J = 2$. 
Figure 2.7: Predicted phase shifts with $J = 3$. 
Figure 2.8: Predicted phase shifts with $J = 4$. 
Figure 2.9: Predicted phase shifts with $J = 5$. 
Chapter 3

Wigner $SU(4)$ symmetry as a long distance symmetry

3.1 Introduction

Symmetries have traditionally been very useful in Nuclear Physics partly because the force at the hadronic level is not well known at short distances [173–175]. In some cases, like isospin, chiral or heavy quark symmetry, the relation with the underlying theory QCD is clear and can be formulated in terms of quark and gluonic degrees of freedom. In some other cases the connection is less straightforward and for that reason are called accidental symmetries.

Many years ago Wigner and Hund proposed [176, 177] extending the spin and isospin $SU_S(2) \otimes SU_I(2)$ symmetry into the larger $SU(4)$ group where the nucleon-spin states $p \uparrow, p \downarrow, n \uparrow, n \downarrow$ correspond to the fundamental representation, and hence providing a super-multiplet structure of nuclear energy levels as well as new selection rules for nuclear transitions and response functions [178]. The corresponding $SU(4)$ mass formula was found to be at least as good as the well known Weizsäcker one [179, 180]. Spin-orbit interaction of the shell model obviously violate the symmetry, and indeed a breakdown of $SU(4)$ has been reported for heavier nuclei [181] while nuclear matter has been addressed in [182]. Double binding energy differences have been shown to be a useful test of the symmetry [183]. Recently, inequalities for light nuclei based on $SU(4)$ and Euclidean path integrals have been derived by neglecting all but S-wave interactions [184].

Despite its relative success along the years, $SU(4)$ symmetry has been treated as an accidental one within the traditional approach to Nuclear Physics and its origin from QCD has been a subject of some interest in the last decade. Indeed, attempts to justify $SU(4)$ spin-flavour symmetry from a more fundamental level have been carried out along several lines. Based on the limit of large number of colors $N_c$ of QCD [136, 137, 185], it was shown [140, 141] that if the nucleon momentum scales as $p \sim N_c^0$, the nuclear potentials scale either as $N_c$ or $1/N_c$, depending upon the particular spin-isospin channel$^1$, which shows that the NN force could be determined with $1/N_c^2$ relative accuracy. It was found that the leading potential would be $SU(4)$ symmetric if the tensor force was neglected in addition, a plausible assumption for light nuclei where S-waves dominate.

$^1$The unfamiliar reader can find a concise review of the large-$N_c$ limit of QCD and the NN interaction in Appendix A.
3.2 Wigner symmetry

It is worth to quickly review the consequences of the symmetry in the NN interaction. Wigner SU(4) spin-isospin symmetry consists of the following 15-generators \[173-175\]

\[
T^a = \frac{1}{2} \sum_A \tau_A^a, \quad (3.1)
\]
\[
S^i = \frac{1}{2} \sum_A \sigma^i_A, \quad (3.2)
\]
\[
G^{ia} = \frac{1}{2} \sum_A \sigma^i_A \tau^a_A, \quad (3.3)
\]

where \(\tau_A^a\) and \(\sigma^i_A\) are isospin and spin Pauli matrices for nucleon \(A\) respectively, and \(T^a\) is the total isospin, \(S^i\) the total spin and \(G^{ia}\) the Gamow-Teller transition operator. The quadratic Casimir operator reads

\[
C_{SU(4)} = T^a T_a + S^i S_i + G^{ia} G_{ia}, \quad (3.4)
\]

and a complete set of commuting operators can be taken to be \(C_{SU(4)}, T_3\) and \(S_z, G_{z3}\). The fundamental representation has \(C_{SU(4)} = 4\) and corresponds to a single nucleon state with a quartet of states \(p \uparrow, p \downarrow, n \uparrow, n \downarrow\), with total spin \(S = 1/2\) and isospin \(T = 1/2\) represented \(4 = (S, T) = (1/2, 1/2)\). For two nucleon states with good spin \(S\) and good isospin \(T\) Pauli principle requires \((-)^{S+T+L} = -1\) with \(L\) the angular momentum, thus

\[
C_{ST}^{SU(4)} = \frac{1}{2} (\sigma + \tau + \sigma \tau) + \frac{15}{2}, \quad (3.5)
\]

where \(\tau = \tau_1 \cdot \tau_2 = 2T(T+1) - 3\) and \(\sigma = \sigma_1 \cdot \sigma_2 = 2S(S+1) - 3\) and the corresponding wave function is of the form

\[
\Psi(\vec{x}) = \frac{u_{LS}^T(r)}{r} Y_{LM} (\hat{x}) \chi^{SM} \chi^{TM}. \quad (3.6)
\]

One has two super-multiplets, which Casimir values are

\[
C_{SU(4)}^{00} = C_{SU(4)}^{11} = 9, \quad (3.7)
\]
\[
C_{SU(4)}^{01} = C_{SU(4)}^{10} = 5, \quad (3.8)
\]

corresponding to an antisymmetric sextet \(6_A = (0, 1) \oplus (1, 0)\) when \(L = \text{even}\) and a symmetric decuplet \(10_S = (0, 0) \oplus (1, 1)\) when \(L = \text{odd}\). The radial wave functions fulfill \(u_{L}^{01}(r) = u_{L}^{10}(r)\) and \(u_{L}^{00}(r) = u_{L}^{11}(r)\) respectively. This means that one has the following super-multiplets;

- Sextet: \(C_{SU(4)}^{01} = C_{SU(4)}^{10} = 5\)

\[
6_A = (0, 1) \oplus (1, 0), \quad L = 0, 2, \ldots \quad (^1S_0, ^3S_1), (^1D_2, ^3D_{1,2,3}) \ldots
\]

- Decuplet: \(C_{SU(4)}^{00} = C_{SU(4)}^{11} = 9\)

\[
10_S = (0, 0) \oplus (1, 1), \quad L = 1, 3, \ldots \quad (^1P_1, ^3P_{0,1,2}), (^1F_3, ^3F_{2,3,4}) \ldots
\]
When applied to the NN potential, the requirement of Wigner symmetry for all states, implies,

\[
V_T = W_T = V_{LS} = W_{LS} = 0, \quad (3.11)
\]

\[
W_S = V_S = W_C, \quad (3.12)
\]

so that the potential may be written as

\[
V = V_C + (2C^{ST}_{SU(4)} - 15)W_S.
\]

(3.13)

Note that the particular choice \(W_S = 0\) corresponds to a spin-isospin independent potential, but in this case no distinction between the \(6_A\) and \(10_S\) super-multiplets arises. The Wigner symmetry does not distinguish between different total angular momentum values, so admitting that the potentials are different we may define a center of the multiplet potential

\[
V_{LST}(r) \equiv \frac{\sum_{J=L-S}^{L+S}(2J + 1)V_{JST}(r)}{(2S + 1)(2L + 1)},
\]

(3.14)

where similarly to the perturbation theory for energy levels where the center of a multiplet of states is predicted, the appropriate statistical weights related to the angular momentum have been used. The previous expression makes sense if the symmetry is broken linearly by spin-orbit coupling. In terms of these mean potentials the symmetry would be \(V_{1_L}(r) = V_{3_L}(r)\), or equivalently,

\[
V_{1_J, J}(r) = \frac{\sum_{J=L-1}^{L+1}(2J + 1)V_{1_J, L}(r)}{3(2L + 1)}.
\]

(3.15)

If the symmetry is taken literally at all distances one should have for the phase shifts a similar result \(\delta_{1_L} = \delta_{3_L}\).

Now, low energy NN scattering is dominated by S-waves and Wigner SU(4) symmetry predicts identical interactions in both \(^1S_0\) and \(^3S_1\) channels. The above mentioned identity of the \(^1S_0\) and \(^3S_1\) potentials holds also in the large \(N_c\)-expansion [140, 141]. In recent quenched lattice QCD evaluations of the NN potential [186] the \(^1S_0\) and \(^3S_1\) potentials appear to be very similar Fig. 3.1 as well although indeed unphysical pion masses are probed.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.1.png}
\caption{Lattice QCD result of Ref. [186] for the central part of the NN potential \(V_C(r)\) in the \(^1S_0\) and \(^3S_1\) channels.}
\end{figure}
In contrast, the corresponding phase shifts from Partial Wave Analyses [56] are very different at all energies as is shown in Fig 3.2.

We are thus confronted with an intriguing puzzle since it is not obvious at all in what sense should the symmetry be interpreted for the NN system; it would be difficult to understand otherwise the successes of SU(4) for light nuclei. A second puzzle arises from an embarrassing cohabitation of conflicts and agreements between large-$N_c$ studies and Wigner symmetry. Despite the initial claim [141] a more complete analysis [140] could only justify the Wigner symmetry in even-L partial waves while for odd-L a violation of the symmetry was expected. However, doing so required neglecting the tensor force, which according to the Wigner symmetry should vanish, but it is a leading contribution to the potential in the large $N_c$ limit. Thus, while some pieces of the NN potential (such as e.g. spin-orbit) are suppressed in both schemes, some others are not simultaneously small. In one side these conflicts between the SU(4) Wigner symmetry and the QCD based large-$N_c$ expansion for odd-L channels lead to the question of the validity of either frameworks and on the other side it is clear that they require an explanation.

As we will see both puzzles can be understood by introducing the concept of long distance symmetry. The discussion is again done in the context of coordinate space renormalization already used in Chapter 2. We find that for S-waves the Wigner symmetry holds in a much wider range than the applicability of a contact interaction suggests if the finite range of the interaction is incorporated. As a byproduct we provide quantitative predictions; the seemingly independent triplet and singlet S-waves phase shifts corresponding to iso-vector and iso-scalar states respectively for the np system are shown to be neatly intertwined in the entire elastic region. A similar correlation can also be established between the $^1S_0$ virtual state and the $^3S_1$ deuteron bound state.

### 3.3 Large $N_c$-OBE potential and Wigner symmetry

Our starting point is the field theoretical OBE model of the NN interaction [53] which includes all mesons with masses below the nucleon mass, i.e., $\pi$, $\sigma(600)$, $\eta$, $\rho(770)$ and $\omega(782)$.

For the purpose of discussing SU(4) Wigner symmetry within the OBE framework we will deal here with S-waves only, neglecting for the moment the S-D wave mixing stemming from the tensor force as...
required by Wigner symmetry. Non-central waves and the role of spin-orbit as well as tensor force will be considered as $SU(4)$-breaking perturbations. For the S-waves the NN potential reads

$$V = V_C + \tau W_C + \sigma V_S + \tau \sigma W_S ,$$

(3.16)

Thus, for the spin singlet $^1S_0$ and spin triplet $^3S_1$ states we get

$$V_\sigma = V_C + W_C - 3V_S - 3W_S ,$$

(3.17)

$$V_t = V_C - 3W_C + V_S - 3W_S .$$

(3.18)

We will discard terms in the potential which are phenomenologically small. Actually, according to Refs. [140, 141] in the leading $1/N_c$-expansion one has $V_C \sim W_S \sim N_c$ while $V_S \sim W_C \sim 1/N_c$. In terms of meson exchanges (see Sec. A.4 in Appendix A and Sec. B.4.2 in Appendix B) one has the contributions

$$V_\sigma(r) = V_t(r) = - \frac{g_{\sigma NN}^2 m_\sigma^2 \ e^{-m_\sigma r}}{16\pi M_N^2} \frac{e^{-m_\sigma r}}{r} - \frac{g_{\rho NN}^2 e^{-m_\rho r}}{4\pi} \frac{e^{-m_\rho r}}{r} + \frac{g_{\omega NN}^2 e^{-m_\omega r}}{8\pi M_N^2} \frac{f_{\rho NN}^2 m_\rho^2 e^{-m_\rho r}}{r} + \mathcal{O} \left( N_c^{-1} \right) ,$$

(3.19)

where $g_{\sigma NN}$ is a scalar type coupling, $g_{\rho NN}$ a pseudo-scalar derivative coupling, $g_{\omega NN}$ is a vector coupling and $f_{\rho NN}$ the tensor derivative coupling. Note that the scheme proposed in [36] of neglecting both energy and non-local corrections is realized explicitly. In fact, these relativistic corrections are sub-leading in the $1/N_c$-expansion (see Sec. B.4.2 in Appendix B). In principle the large $N_c$ limit contains infinitely many multi-meson exchanges which decay exponentially with the sum of the exchanged meson masses. However, NN scattering in the elastic region below pion production threshold probes c.m. momenta $p < p_{\text{max}} = 400 \text{ MeV}$. Given the fact that $1/m_\omega = 0.25\text{fm} < 1/p_{\text{max}} = 0.5\text{fm}$ we expect heavier meson scales to be irrelevant, an in particular $\omega$ and $\rho$ themselves, are expected to be at most marginally important. Note that, in any case, when $m_\omega = m_\rho$ the redundant combination $g_{\omega NN}^2 - f_{\rho NN}^2 m_\rho^2 / (2M_N^2)$ appears, indicating a further source of cancellation between $\rho$ and $\omega$ in this channel. Moreover, since the leading contributions to the potential are $\sim N_c$ and the sub-leading ones are $\sim 1/N_c$, the neglected terms are of relative $1/N_c^2$ order, so we might expect an a priori $\sim 10\%$, accuracy.

The coincidence between $^1S_0$ and $^3S_1$ potentials complies to the Wigner $SU(4)$ symmetry for the two-nucleon system. Modern high quality potentials [57] describing accurately NN scattering below pion production threshold show some traces of the symmetry for distances above $1.4 - 1.8\text{fm}$. We have seen that quenched lattice QCD evaluations of NN potentials for $m_\pi/m_\rho \sim 0.6$ [186] yield also similar $^1S_0$ and $^3S_1$ potentials for $r > 1.4\text{fm}$. At first sight one may conclude that Wigner symmetry holds when OPE dominates, and thus has a limited range of applicability. We will see however that this is not necessarily so, provided the relevant scales of symmetry breaking are properly isolated with the help of renormalization ideas.

---

\(^2\)This of course does not exclude explicit and leading $N_c$ uncorrelated multiple pion exchanges, i.e. background non-resonant contributions in $\pi\pi$ or $\pi\rho$ scattering. We expect them not to be dominant once $\sigma$, $\rho$ and $\omega$ are included.
3.4 Traditional approach vs. renormalization viewpoint

3.4.1 Traditional approach to OBE potentials

Within the standard approach to OBE potentials the scattering phase-shift $\delta_0(p)$ is computed by solving the (S-wave) Schrödinger equation in r-space

$$-u_p''(r) + M_N V(r) u_p(r) = p^2 u_p(r),$$

$$u_p(r) \rightarrow \frac{\sin (pr + \delta_0(p))}{\sin \delta_0(p)} ,$$

with a regular boundary condition at the origin $u_p(0) = 0$. Moreover, for a short range potential such as the one in Eq. (3.19) one also has the Effective Range Expansion (ERE)

$$pcot \delta_0(p) = -\frac{1}{\alpha_0} + \frac{1}{2} r_0 p^2 + \cdots ,$$

where the scattering length, $\alpha_0$, is defined by the asymptotic behavior of the zero energy wave function as

$$u_0(r) \rightarrow 1 - \frac{r}{\alpha_0} ,$$

and the effective range, $r_0$, is given by

$$r_0 = 2 \int_0^{\infty} dr \left[ \left( 1 - \frac{r}{\alpha_0} \right)^2 - u_0(r)^2 \right].$$

In the usual approach [53, 55] everything is obtained from the potential assumed to be valid for $0 \leq r < \infty$. We note incidentally that the Wigner symmetry relation, Eq. (3.19), holds at all distances $^3$. In addition, due to the unnaturally large NN $^1S_0$ scattering length ($\alpha_s \sim -23 \text{fm}$), any change in the potential $V \rightarrow V + \Delta V$ has a dramatic effect on $\alpha_0$, since one obtains

$$\Delta \alpha_0 = \alpha_0^2 M_N \int_0^{\infty} \Delta V(r) u_0(r)^2 dr ,$$

and thus the potential parameters must be fine-tuned, and in particular the short distance physics. As it was discussed in Refs. [187, 188] and will be done in Chapter 5 this short distance sensitivity is unnatural as long as the OBE potential does not truly represent a fundamental NN force at short distances. Indeed, the sensitivity manifests itself as tight constraints for the potential parameters when the $^1S_0$ phase shift is fitted resulting in incompatible values of the coupling constants as obtained from other sources as NN scattering. Of course, there is nothing wrong in the need of a fine-tuning as this is a unavoidable consequence of the large scattering length; the relevant point is whether this should be driven by a potential which will not be realistic at short distances.

In any case, in the traditional approach to NN potentials we are confronted with a paradox; on the one hand the symmetry seems to suggest that somewhere the phase shifts should coincide, while on the other hand a fine-tuning is required because of the large scattering lengths. In the standard approach,

$^3$In practice, strong form factors are included mimicking the finite nucleon size and reducing the short distance repulsion of the potential, but the regular boundary condition is always kept. Strong form factors will be re-analyzed in Chapter 5 using renormalization techniques.
if \( V_s(r) = V_t(r) \) then \( \delta_s(k) = \delta_t(k) \) and thus \( \alpha_s = \alpha_t \), as one naturally expects. A straightforward explanation, of course, is to admit that the symmetry is strongly violated. This would make difficult to understand how can SU(4) work at all for light nuclei if the simpler two nucleon system does not show manifestly the symmetry.

A good condition for an approximate symmetry is that it be stable under symmetry breaking, that is, a tiny perturbation \( V_s(r) - V_t(r) = \Delta V(r) \neq 0 \) should not yield a large change in the result. This suggests that we should provide a framework where the highly potential-sensitive scattering length becomes a variable independent of the potential. More generally, we want to avoid the logical conclusion that a symmetry of the potential is a symmetry of the S-matrix in the same spirit what happened in the case of anomalies in Quantum Field Theory where the parallel statement would be that a symmetry of the Lagrangean becomes a symmetry of the S-matrix.

The puzzle may be overcome by the concept of long distance symmetry, i.e., a symmetry which is only broken at short distances by a suitable boundary condition.

### 3.4.2 Renormalization viewpoint

To introduce what we call long distance symmetry, the must natural is to appeal to renormalization in coordinate space [127, 129]. This enables to disentangle short and long distances in a way that the symmetry is kept at all non-vanishing distances. The main idea is to fix the scattering length independently of the potential by means of a suitable short distance boundary condition. As a result the undesirable dependence of observables on the potential is reduced at short distances, precisely the region where a determination of the NN force in terms of hadronic degrees of freedom becomes less reliable.

In Ref. [133], it was shown for the case of S-waves that this renormalization procedure in coordinate space is fully equivalent to introduce one counter-term in the cut-off Lippmann-Schwinger equation in momentum space.

We first analyze scattering states. For the finite energy wave function, the superposition principle of boundary conditions implies,

\[
\begin{align*}
  u_k(r) &= u_{k,c}(r) + k \cot \delta_0 u_{k,s}(r), \\
  u_0(r) &= u_{0,c}(r) - \frac{1}{\alpha_0} u_{0,s}(r),
\end{align*}
\]

with \( u_{k,c}(r) \to \cos(kr) \) and \( u_{k,s}(r) \to \sin(kr)/k \) for \( r \to \infty \). At zero energy, \( k \to 0 \), and \( \delta_0(k) \to -\alpha_0 k \) yields

\[
  u_0(r) = u_{0,c}(r) - \frac{1}{\alpha_0} u_{0,s}(r),
\]

with \( u_{0,c}(r) \to 1 \) and \( u_{0,s}(r) \to r \) for \( r \to \infty \). Combining the zero and finite energy wave functions we get

\[
[u_k'(r)u_0(r) - u_0'(r)u_k(r)] \bigg|_{r_c}^\infty = k^2 \int_{r_c}^\infty u_k(r)u_0(r)dr,
\]

where \( r_c \) is a short distance cut-off radius which will be removed at the end. To calculate the contribution from the term at infinity we use the long distance behavior, Eq. (3.21). The integral and the boundary
term at infinity yield two canceling delta functions\(^4\) leaving only the boundary term at short distances. From orthogonality of different energy solutions and taking the renormalization limit \(r_c \to 0\) we get

\[
\lim_{r_c \to 0} [u_k'(r_c)u_0(r_c) - u_0'(r_c)u_k(r_c)] = 0.
\]

(3.29)

Note that the regular solution \(u_k(r_c) = u_0(r_c) = 0\) is a particular choice for \(r_c = 0\). Writing out the orthogonality condition via the superposition principle at finite and zero energies, Eq. (3.26) and Eq. (3.27) respectively, one gets

\[
0 = \int_0^\infty dr \left[ u_{0,c}(r) - \frac{1}{\alpha_0} u_{0,s}(r) \right] \times [u_{k,c}(r) + k\cot\delta_0(k)u_{k,s}(r)].
\]

(3.30)

Expanding the integrand and defining the universal functions,

\[
A(k) &= + \int_0^\infty dr u_{0,c}(r)u_{k,c}(r) = \lim_{r_c \to 0} \left[ u_{0,c}(r_c)u_{k,c}'(r_c) - u_{0,c}'(r_c)u_{k,c}(r_c) \right],
\]

\[
B(k) &= - \int_0^\infty dr u_{0,s}(r)u_{k,c}(r) = \lim_{r_c \to 0} \left[ u_{0,s}(r_c)u_{k,c}(r_c) - u_{0,s}'(r_c)u_{k,c}'(r_c) \right],
\]

\[
C(k) &= - \int_0^\infty dr u_{0,c}(r)u_{k,s}(r) = \lim_{r_c \to 0} \left[ u_{0,c}'(r_c)u_{k,s}(r_c) - u_{0,s}(r_c)u_{k,s}'(r_c) \right],
\]

\[
D(k) &= + \int_0^\infty dr u_{0,s}(r)u_{k,s}(r) = \lim_{r_c \to 0} \left[ u_{0,s}(r_c)u_{k,s}'(r_c) - u_{0,s}'(r_c)u_{k,s}(r_c) \right],
\]

(3.31)

we get the explicit formula

\[
k\cot\delta_0(k) = \frac{\alpha_0 A(k) + B(k)}{\alpha_0 C(k) + D(k)}.
\]

(3.32)

The functions \(A, B, C\) and \(D\) are even functions of \(k\) which depend only on the potential. Note that the dependence of the phase-shift on the scattering length is wholly explicit; \(\cot\delta_0\) is a bilinear rational mapping of \(\alpha_0\). Further, using Eq. (3.27), one gets the effective range

\[
r_0 = A + \frac{B}{\alpha_0} + \frac{C}{\alpha_0^2},
\]

(3.33)

where

\[
A = 2 \int_0^\infty dr (1 - u_{0,c}^2),
\]

(3.34)

\[
B = -4 \int_0^\infty dr (r - u_{0,c}u_{0,s}),
\]

(3.35)

\[
C = 2 \int_0^\infty dr (r^2 - u_{0,s}^2),
\]

(3.36)

\(^4\)This can be also understood if we apply a regulator to the integral,

\[
\int_0^\infty u_k(r)u_p(r)dr \to \lim_{\epsilon \to 0} \int_0^\infty u_k(r)u_p(r)e^{-\epsilon r}dr,
\]

so the term \([u_k'(r)u_0(r) - u_0'(r)u_k(r)]\) \(r=\infty\) is well defined and vanishes. The regulator does not affect the Lagrange identity as \(\epsilon \to 0\).
depend on the potential parameters only. Again, the interesting thing is that all explicit dependence on
the scattering length \( \alpha_0 \) is displayed by Eq. (3.33).

We turn now to discuss the case of a bound state corresponding to the case of negative energy \( E = -\gamma^2/M \)
where \( \gamma \) is the wave number. The wave function behaves asymptotically as
\[
u_\gamma(r) \to A_S e^{-\gamma r}, \tag{3.37}
\]
and is chosen to fulfill the normalization condition
\[
\int_0^\infty \nu_\gamma(r)^2 dr = 1. \tag{3.38}
\]
In principle, such a state would be unrelated to the scattering solutions. An explicit relation may be
determined from the orthogonality condition, which applied in particular to the zero energy state yields
\[
0 = \int_0^\infty dr \left[ u_{0,c}(r) - \frac{1}{\alpha_0} u_{0,s}(r) \right] u_\gamma(r). \tag{3.39}
\]
This generates a correlation between the scattering length, \( \alpha_0 \) and the bound state wave number, \( \gamma \),
\[
\alpha_0(\gamma) = \frac{\int_0^\infty dr u_\gamma(r) u_{0,s}(r)}{\int_0^\infty dr u_\gamma(r) u_{0,c}(r)}. \tag{3.40}
\]
We remind that the two independent zero energy solutions, \( u_{0,c}(r) \) and \( u_{0,s}(r) \) depend only on the
potential.

A trivial realization of the conditions Eqs. (3.30) and (3.40) is given by the case where there is no
potential, \( U(r) = 0 \). The general solution for a positive energy state \( E = k^2/M \) is given by
\[
u_k(r) = \cot \delta_0(k) \sin(kr) + \cos(kr), \tag{3.41}
\]
and using the low energy limit condition \( \delta_0(k) \to -\alpha_0 k \) we obtain
\[
u_0(r) = 1 - \frac{r}{\alpha_0}. \tag{3.42}
\]
Orthogonality between zero and finite energy states yields after evaluating the integrals Eqs. (3.31),
\[
k \cot \delta_0(k) = -\frac{1}{\alpha_0}, \tag{3.43}
\]
and as a consequence the effective range vanishes \( r_0 = 0 \), in accordance to the fact that the range of the
potential is zero. For a negative energy state \( E = -\gamma^2/M \) the normalized bound state is
\[
u_\gamma(r) = A_S e^{-\gamma r}, \quad A_S = 1/\sqrt{2\gamma}. \tag{3.44}
\]
Orthogonality between the zero energy and the bound state, again, yields the correlation
\[
\alpha_0 = 1/\gamma. \tag{3.45}
\]
A key point at this moment is whether we need to take the limit \( r_c \to 0 \), which corresponds to eliminating
the cut-off. We note that the potential, \( V(r) \), is used at all distances both in the standard approach,
which involves the regular solution only, and the renormalization approach, which requires the regular as well as the irregular solution. However, the sensitivity to the short distance behaviour of the potential is quite different; the standard approach displays much stronger dependence while the renormalization approach is fairly independent on the hardly accessible short distance region. This is in fact the key property that allows to eliminate the cut-off in the renormalization approach. Thus, removing the cut-off does not mean that the OBE potential is believed to hold all the way down to the origin.

Once the method has been established we now proceed to analyze the symmetry from a quantitative point of view.

### 3.5 Central phases and the deuteron

#### 3.5.1 Potential Parameters

First of all we need to fix the potential parameters keeping in mind that the leading $N_c$ nature of the potential embodies some systematic $1/N_c^2$ uncertainties. Of course, while we will use relations which are compatible with large $N_c$ scaling, the numerical values can only be fixed phenomenologically. The main point is that besides the $\sigma$-meson mass, we may choose quite natural values for the masses and couplings unlike the usual OBE potentials [53]. In fact, as already noted the standard approach suffers from tight constraints reflecting the unnatural short distance sensitivity. In the renormalization viewpoint the undesirable short distance sensitivity is largely removed, allowing for a determination of the potential parameters using independent sources.

For definiteness we take $g_{\sigma NN} = 13.1$ coming from phase shifts analyzes and quite close to the Goldberger-Treiman value for $\pi$, $g_{\sigma NN} = g_A M_N/f_\pi$ with $g_A = 1.29$. We also take the $SU(3)$ value $g_{\omega NN} = 3 g_{\rho NN} - g_{\sigma NN}$ which on the basis of the OZI rule, $g_{\sigma NN} = 0$, Sakurai’s universality $g_{\rho NN} = g_{\rho\pi\pi}/2$ and the KSFR relation $2 g_{\rho\pi\pi} f_\pi^2 = m_\rho^2$ yields $g_{\omega NN} = N_c m_\rho/(2\sqrt{2} f_\pi) = 8.8$. The $\rho$ tensor coupling is taken to be $f_{\rho NN} = \sqrt{2} M_N g_{\omega NN}/m_\rho = 15.5$ which cancels the vector meson contributions in the potential and yields $\kappa_\rho = f_{\rho NN}/g_{\rho NN} = 5.5$ a quite reasonable result [53]. Note that $1/N_c$ effects include not only mesons but also finite width effects of $\sigma$ and $\rho$ since for large $N_c$ one has stable mesons, $\Gamma_\sigma, \Gamma_\rho \sim 1/N_c$. For the masses we take $m_\sigma = 140 \text{MeV}$ and $m_\omega = 783 \text{MeV}$. This fixes all parameters except the $\sigma$ ones which we identify with the lightest $J^{PC} = 0^{++}$ meson $f_0(600)$. Actually from the Goldberger-Treiman relation for $\sigma$, $g_{\sigma NN} = M_N/f_\pi$ one has $g_{\sigma NN} = 10.1$. Its mass is, according to the recent analysis based on Roy equations $m_\sigma - i \Gamma_\sigma/2 = 441^{+16}_{-8} - i 272^{+9}_{-12} \text{MeV}$ [43]. A fit of the $\sigma$ parameters to the $np$ data of Ref. [57] in the $^1S_0$ channel yields $g_{\sigma NN} = 9(1)$ and $m_\sigma = 501(25) \text{MeV}$, where the error is statistical. The fitted mass value differs by about 10% from the location of the real part of the resonance, in harmony with the expected $1/N_c^2$ corrections. Although a more quantitative estimate of the large $N_c$ corrections to the potentials parameters would be very useful, for the present purposes of discussing Wigner symmetry on the light of large $N_c$ it is more than sufficient.

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5The reader can find a review of natural coupling constants and its sources in Sec. B.5 of Appendix B.

6Actually as shown in [187, 188] the net vector meson exchange contribution corresponding to the combined repulsive coupling $g_{\sigma NN}^2 - f_{\rho NN}^2 m_\rho^2/2M_N^2$ (referred there simply as $g_{\sigma NN}^2$) cannot be pinned down accurately from a fit to the $^1S_0$ phase shift being compatible with zero within errors. This is due to the short distance insensitivity embodied by the renormalization approach.

7Actually, our estimate of the $\sigma$-mass as a pole in the second Riemann sheet for $\pi\pi$ scattering for large $N_c$ [188] yields the value $m_\sigma \sim 507 \text{MeV}$.
3.5.2 Low energy parameters and phase shifts

Obviously, in the traditional approach if we have \( V_s(r) = V_t(r) \) and impose the regular boundary condition, \( u_s(0) = u_t(0) = 0 \), the only possible solution is \( \alpha_s = \alpha_t \), \( r_s = r_t \), and \( \delta_s(p) = \delta_t(p) \) which lead to the puzzle situation discussed in Sec. 3.2. However, in the renormalization approach we allow different short distance boundary conditions \( u_s'(0^+) / u_s(0^+) \neq u_t'(0^+) / u_t(0^+) \) \(^8\), and hence we may have \( \alpha_s \neq \alpha_t \). This corresponds to claim that the symmetry is broken at short distances but we postulate its validity at long distances. The previous equations imply straight away the following expressions for the effective ranges in the singlet and triplet channels,

\[
\begin{align*}
    r_s &= A + \frac{B}{\alpha_s} + \frac{C}{\alpha_s^2}, \\
    r_t &= A + \frac{B}{\alpha_t} + \frac{C}{\alpha_t^2}.
\end{align*}
\]

As already mentioned, the remarkable aspect of these two equations is the fact that the coefficients \( A, B, C \) are identical both in the triplet as well as in the singlet channels as long as \( V_s(r) = V_t(r) \), thus the only difference resides in the numerical values of the scattering lengths \( \alpha_s \) and \( \alpha_t \). Numerically we get (everything in fm)

\[
\begin{align*}
    r_0 &= 1.3081 \frac{4.5477}{\alpha_0} + \frac{5.1926}{\alpha_0^2} \, (\pi) \\
    &= 1.5089 \text{fm} \quad (\alpha_0 = \alpha_s) \quad (\exp.2.770\text{fm}) \\
    &= 0.6458 \text{fm} \quad (\alpha_0 = \alpha_t) \quad (\exp.1.753\text{fm}), \\
    r_0 &= 2.4567 \frac{5.5284}{\alpha_0} + \frac{5.7398}{\alpha_0^2} \, (\pi + \sigma) \\
    &= 2.6989 \text{fm} \quad (\alpha_0 = \alpha_s) \quad (\exp.2.770\text{fm}) \\
    &= 1.5221 \text{fm} \quad (\alpha_0 = \alpha_t) \quad (\exp.1.753\text{fm}),
\end{align*}
\]

where the corresponding numerical values when the experimental \( \alpha_s = -23.74\text{fm} \) and \( \alpha_t = 5.42\text{fm} \) are taken as well as the experimental values for the effective ranges have also been added. More generally, for any fixed potential the correlation of \( r_0 \) on \( 1/\alpha_0 \) is a parabola which we plot in Fig. 3.3 for the OPE and OPE+\( \sigma \).

This dependence is universal to all S-waves having the same long distance potential and there is nothing in this curve making unnaturally large scattering lengths particularly different from smaller ones. The present analysis, however, does not shed any light on the origin of the large size of the \( \alpha \)'s nor how \( \alpha_s \) and \( \alpha_t \) are interrelated. This is in fact a price we pay for the built-in short distance insensitivity. In any case, as we see from Fig. 3.3, the experimental values fall strikingly almost on top of the curve, pointing towards a correct interpretation of the underlying symmetry.

We turn next to the phase shifts. According to Eq. (3.32) they are given in terms of the universal functions \( A, B, C \) and \( D \) defined by Eqs. (3.31). These functions are plotted in Fig. 3.4 for completeness and as we see they are smooth. From them the corresponding singlet and triplet phase shifts are obtained

\(^{8}\)The limit from above, \( u(0^+) = \lim_{r \to 0^+} u(r_c) \) is really necessary to pick both the irregular and irregular solutions. If one starts exactly from the origin the only possible solution is the regular one.
Figure 3.3: Wigner correlation for the effective range $r_0$ (in fm) of a $np$ S-wave as a function of an arbitrary inverse scattering length $\alpha_0$ in the case of the OPE and OPE+$\sigma$ potentials. The parabolic shape is determined by a unique long distance potential. The points in the solid curve correspond to the two different values of the effective range $r_s$ in the singlet $^1S_0$ and $r_t$ in the triplet $^3S_1$ channels when the scattering length is taken to be $\alpha_s = -23.74$fm and $\alpha_t = 5.42$fm respectively. Experimental points are also shown for comparison.

Figure 3.4: Universal functions $A$, $B$, $C$ and $D$ defined by Eqs. (3.31) in appropriate length units as a function of the c.m. momentum $p$ (in MeV). These functions depend on the potential $V_s(r) = V_t(r)$ only but are independent of the scattering length. By

\[
\begin{align*}
  k \cot \delta_s &= \frac{\alpha_s A(k) + B(k)}{\alpha_s C(k) + D(k)}, \\
  k \cot \delta_t &= \frac{\alpha_t A(k) + B(k)}{\alpha_t C(k) + D(k)},
\end{align*}
\]

respectively. When the experimental scattering lengths $\alpha_s = -23.74$fm and $\alpha_t = 5.42$fm are taken we
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Figure 3.5: Phase shifts (in degrees) for the fitted \(^1S_0\) (left panel) and predicted \(^3S_1\) (right panel) channels as a function of the c.m. momentum (in MeV). In both cases the potential is the same, \(V_c(r) = V_t(r)\), while the only difference is in the scattering lengths. In the singlet channel \(\alpha_s = -23.74\text{fm}\) and in the triplet channel \(\alpha_t = 5.42\text{fm}\) corresponding to a different short distance boundary condition.

We also plot the cases with only \(1\sigma\)-exchange and \(1\pi\)-exchange for comparison. Data from [57].

can fit the potential (actually only the \(\sigma\)-meson parameters) to the singlet \(^1S_0\) channel and predict the triplet \(^3S_1\) channel. The result is shown in Fig. 3.5 and as we see the agreement is remarkably good taking into account that we have neglected the tensor force and the \textit{a priori} \(1/N^2_c\) systematic corrections to the potential. Note that the identity of the singlet and triplet potentials is not sufficient; the simple OPE fulfills this property but does not explain the neither phase-shifts. Actually, it shows that both failures are correlated \(^9\).

3.5.3 Renormalization group and scale invariance

It is worth to analyze our results from the point of view of the Renormalization Group (RG) on the light of Refs. [141, 189, 190] where a square well potential, PDS and sharp momentum cut-off were used respectively to model the short distance contact interactions arising when all exchanged particles are integrated out. Here we are interested in the dependence on the arbitrary renormalization scale separating the contact and the extended particle exchange interaction since they are not independent of each other; by keeping this scale dependence we may enter the interaction region where, as we will show now, the symmetry can be visualized. To do so we appeal to the coordinate space version of the renormalization group largely discussed in Refs. [125, 129] (see also Ref. [191] for a momentum space version), where the version of the Callan-Zymanzik equation for potential scattering reads

\[
R \xi_k^c(R) = \xi_k(R)(1 - \xi_k(R)) + (MV(R) - k^2)R^2, \quad (3.51)
\]

where \(\xi_k(R) = Ru_k^c(R)/u_k(R)\) is a suitable combination of the short distance boundary condition representing the physics below the scale \(R\) for momentum \(k\). The above equation provides the evolution of the boundary condition as a function of the distance \(R\) (the renormalization scale) in order to have a fixed scattering amplitude (see Ref. [129] for a thorough discussion). Clearly, at long distances \(r \gg 1/m_\pi\), or equally, in the infrared limit, \(R \to \infty\), the potential becomes negligible and the equation is scale invariant \((R \to \lambda R)\) at zero energy. The scale invariance is only broken by the renormalization condition which

\(^9\)The reason why OPE fails at much lower energies in the \(^1S_0\) channel than in the \(^3S_1\) channel is due to a stronger short distance sensitivity of the channel with larger scattering length.
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Figure 3.6: Zero energy S-wave radial functions for the singlet $^1S_0$ and triplet $^3S_1$ channels as a function of distance (in fm). The normalization is such that $u_{0,1}^{^1S_0} \rightarrow 1 - r/\alpha_s$ and $u_{0,3}^{^3S_1} \rightarrow 1 - r/\alpha_t$ with $\alpha_s = -23.74 \text{fm}$ and $\alpha_t = 5.42 \text{fm}$ the singlet and triplet scattering lengths respectively. The potentials generating these wave functions are the same $V_s(r) = V_t(r)$.

Virtual and bound states

Finally we want to analyze the kind of features that may result for the deuteron from this simplified picture without tensor force. The deuteron is determined by integrating in the Schrödinger equation with negative energy $E = -\gamma_d^2/M$ with $\gamma_d = 0.2316 \text{fm}^{-1}$ the wave number and imposing the long distance boundary condition, Eq. (3.37). We also compute the matter radius

$$r_m^2 = \frac{1}{4} \int_0^\infty r^2 u_d(r)^2.$$ (3.53)

and the $M_{M1}$ matrix element

$$A_S M_{M1} = \int_0^\infty dr u_d(r) u_{0,1}^{^1S_0}(r).$$ (3.54)
which corresponds to the dominant magnetic contribution to neutron capture process $np \rightarrow \gamma d$ in the range of thermal neutrons ($\sim$ KeV) in stars \textsuperscript{10}. For the experimental $\gamma_d = 0.2316$fm\textsuperscript{-1} we get $A_S = 0.8643$fm\textsuperscript{-1/2} (exp. $0.8846$fm\textsuperscript{-1/2}), $r_m = 1.9138$fm (exp. $1.9754$fm) and $M_{M1} = 4.0446$fm (exp. $3.979$fm). As mentioned above, orthogonality between the bound state and the zero energy state yields an explicit correlation between the triplet scattering length, $\alpha_t$ and the deuteron wave number, $\gamma$,

$$\alpha_t = \alpha_0(\gamma_d) = \frac{\int_0^\infty d\gamma u_\gamma(r)u_{0,t}(r)}{\int_0^\infty d\gamma u_\gamma(r)u_{0,e}(r)} \big|_{\gamma=\gamma_e}. \tag{3.55}$$

Since the two independent zero energy solutions, $u_{0,e}(r)$ and $u_{0,s}(r)$ depend only on the potential and hence are identical for the S-wave components of the singlet and triplet channels, this correlation is a consequence of the Wigner symmetry as well, as long as we take $V_s(r) = V_t(r)$. For the experimental $\gamma_d = 0.2316$fm\textsuperscript{-1} we get $\alpha_t = 5.32$fm. This value improves over the simple formula $\alpha_t = 1/\gamma = 4.31$fm obtained from the case without potential, or the single OPE case where $\alpha_t = 4.60$fm. Exploiting Wigner symmetry a similar relation between the virtual state energy, a purely exponentially growing wave function, $u_\gamma(r) \rightarrow e^{\gamma r}$, and the scattering length in the singlet channel, yields for $\alpha_s = -23.74$fm the value $\gamma_0 = 0.042$fm\textsuperscript{-1}. In other words, the function $\alpha_0(\gamma)$ fulfills $\alpha_0(\gamma_d) = \alpha_t$ and simultaneously $\alpha_0(\gamma_e) = \alpha_s$ as a consequence of Wigner symmetry. Numerically we get

$$\alpha_0(-0.042$fm\textsuperscript{-1}) = -23.74$fm, \tag{3.56}$$
$$\alpha_0(0.2265$fm\textsuperscript{-1}) = 5.42$fm. \tag{3.57}$$

In the region below 1fm the virtual state $u_\gamma(r)$ and the deuteron bound state $u_d(r)$ look very much alike the corresponding singlet and triplet zero energy wave functions respectively (see Fig. 3.6). Thus, $u_{0,1S_0}(r) \sim u_\gamma(r)$ and $u_{0,3S_1}(r) \sim u_d(r)$ are consequences the closeness of the poles to the real axis, either in the second or first Riemann sheets respectively. However, $u_{0,1S_0}(r) \sim u_{0,3S_1}(r)$ and $u_\gamma(r) \sim u_d(r)$ are further consequences of the identity of the potentials $V_s(r) = V_t(r)$.

As a function of the scattering length, the expression

$$M(\gamma,\alpha_0) = \int_0^\infty d\gamma u_\gamma(r)u_\alpha(r), \tag{3.58}$$

yields both the orthogonality relation as well as $M_{M1}$

$$M(\gamma_d,\alpha_t) = 0, \tag{3.59}$$
$$M(\gamma_d,\alpha_s) = M_{M1}. \tag{3.60}$$

The dependence on the inverse scattering length is a straight line which is shown in Fig. 3.7. As we see both conditions are very well fulfilled. Similarly to the previous case, the orthogonality between finite energy states and the deuteron corresponds to the magnetic contribution to the photodisintegration of the deuteron. The result, however does not differ much from the potential-less theory.

\textsuperscript{10}Here we are neglecting meson exchange currents in the calculation of $M_{M1}$ but its effects will be studied in Chapter 7.
Chapter 3. Wigner SU(4) symmetry as a long distance symmetry

3.6 Symmetry breaking

3.6.1 Symmetry breaking with two counter-terms

In the renormalization scheme we are dealing with, we parameterize the unknown short distance physics by fixing the scattering length because we allow, besides the regular solution, the irregular one and we need to fix an appropriate combination of both solutions. As a result we get an energy independent boundary condition. In fact it is possible to renormalize using energy dependent boundary conditions, a procedure essentially equivalent to imposing more renormalization conditions or counter-terms. The procedure in coordinate space turns out to be rather simple. In the case of two conditions we would fix the scattering length, \( \alpha_0 \), and the effective range \( r_0 \) independently of the potential. The coordinate space procedure [129, 133] consists of expanding the wave function in powers of the energy

\[
u_p(r) = u_0(r) + p^2 u_2(r) + \ldots, \tag{3.61}
\]

where \( u_0(r) \) and \( u_2(r) \) satisfy the following equations,

\[
\begin{align*}
- u_0''(r) + M V(r) u_0(r) &= 0, \\
u_0(r) &\to 1 - r/\alpha_0, \\
- u_2''(r) + M V(r) u_2(r) &= u_0(r), \\
u_2(r) &\to (r^3 - 3\alpha_0 r^2 + 3\alpha_0 r_0 r) / (6\alpha_0),
\end{align*}
\tag{3.62, 3.63, 3.64, 3.65}
\]

The asymptotic conditions correspond to fix \( \alpha_0 \) and \( r_0 \) as independent parameters (two counter-terms) and the matching condition at the boundary \( r = r_c \) becomes energy dependent [129]

\[
\frac{u_p'(r_c)}{u_p(r_c)} = \frac{u_0'(r_c)}{u_0(r_c)} + \rho^2 u_2'(r_c) + \ldots = \frac{u_0(r_c)}{u_0(r_c)} + \rho^2 u_2(r_c) + \ldots. \tag{3.66}
\]
The corresponding phase shift may be deduced by integrating in Eqs. (3.62) and (3.64) with the asymptotic conditions Eqs. (3.63) and (3.65), imposing Eq. (3.66) and integrating out the finite energy equation. Obviously, if $r_0$ is fixed from the start to their experimental values in the singlet and triplet channels, the Wigner correlation given by Eqs. (3.47) and generating the universal curve shown in Fig. 3.3 would not be predicted. The breaking of the symmetry with two counter-terms is a short distance one when the cut-off is eliminated, $r_c \to 0$, because the potential is kept fixed and $V_s(r) = V_t(r)$ for any non-vanishing distance, $r \geq r_c > 0$. Thus, if we write

$$r_0 = A + \frac{B}{\alpha_0} + \frac{C}{\alpha_0^2} + r_0^{\text{short}},$$

with $r_0^{\text{short}}$ the effect of the second counter-term, we would obtain

$$r_t - r_s \sim r_t^{\text{short}} - r_s^{\text{short}} + B \left[ \frac{1}{\alpha_t} - \frac{1}{\alpha_s} \right] + \ldots,$$

where small $1/\alpha^2$ terms have been neglected. This yields $r_t^{\text{short}} - r_s^{\text{short}} \sim 0.1\text{fm}$. Thus, while introducing no counter-term (trivial boundary condition) does not break the symmetry yielding identical phase shifts, $\delta_s(k) = \delta_t(k)$, introducing more than one counter-term (energy dependent boundary condition) breaks the symmetry at the $\sim 10\%$ level.

### 3.6.2 Symmetry breaking due to the tensor force

An interesting possibility is that of keeping the energy independence of the boundary condition and breaking the symmetry by introducing a long distance component of the potential, such as e.g. the tensor force, which would couples the $^3S_1$ and $^3D_1$ channels. Actually, this would correspond to take into account, as proposed in Ref. [140], the leading and complete large-$N_c$ NN potential. In fact, although Wigner symmetry implies a vanishing tensor force, leading large-$N_c$ does not necessarily implies the tensor force to be small. To analyze this potential source of conflict we consider the $^3S_1$ effective range parameter which incorporates the D-wave contribution coming from S-D tensor force mixing and is given by

$$r_t = 2 \int_0^\infty \left[ \left( 1 - \frac{r}{\alpha_t} \right)^2 - u_{0,\alpha}(r)^2 - w_{0,\alpha}(r)^2 \right] dr,$$

where the zero energy S-wave function $u_{0,\alpha}(r) \to u_{0,^3S_1}(r)$ (discussed above) and the D-wave function $w_{0,\alpha}(r) \to 0$ when the tensor force is switched off keeping $\alpha_t$ fixed. The corresponding tensor potential would include $\pi$ and $\rho$ exchange contributions characterized by the $g_{\pi NN}$ and $f_{\rho NN}$ couplings and diverges as $1/r^3$ at short distances\(^\text{11}\). Numerically, taking $f_{\rho NN} = 17$ and $g_{\omega NN} = 9.86$, we get,

$$r_t = 2.6199 - \frac{5.7843}{\alpha_t} + \frac{5.7608}{\alpha_t^2}.$$

which corresponds to a $\sim 10\%$ breaking due to the tensor force. As we see, the coefficients in Eq. (3.48) are not modified much despite the singularity of the tensor force and its dominance at short distances. Therefore, while from the large $N_c$ viewpoint a large tensor force is not forbidden, we find the effect in the S-wave to be numerically small, as implied by Wigner symmetry.

\(^\text{11}\)In Chapter 5 a detailed study of the OBE potential including the tensor force will be presented.
Chapter 3. Wigner SU(4) symmetry as a long distance symmetry

3.6.3 Symmetry breaking in non-central waves

Within the previous interpretation of the Wigner symmetry as a long distance one for the S-waves, we want to analyze what are the consequences for the phase shifts corresponding to partial waves at angular momentum larger than zero, $L > 0$. Unlike the S-waves we expect the dependence on the short distance behavior to be suppressed due to the centrifugal barrier, and the symmetry should become more evident.

In the two-nucleon system the Wigner symmetry implies the following relations for spin-isospin components of the antisymmetric sextet, $6_A$, and the symmetric decuplet, $10_S$, respectively (see Sec. 3.2) thus we should have

$$\delta_{LJ}^{01} = \delta_{LJ}^{10} = \delta_L, \quad \text{even} - L \quad \text{(3.71)}$$

$$\delta_{LJ}^{00} = \delta_{LJ}^{11} = \delta_L, \quad \text{odd} - L \quad \text{(3.72)}$$

For P-waves, for instance, we have the spin singlet state $1^1P_1$ and the spin triplets $3^3P_0, 3^1P_1$ and $3^3P_2$ which according to the symmetry should be degenerate as they belong to the $10_S$ super-multiplet. Inspection of the Nijmegen PWA [56] reveals that $1^1P_1$ is very similar to $3^3P_0$ at all energies, $|\delta_{1P_1} - \delta_{3P_0}| \sim 1^0$, but very different from the $3^3P_0$ and $3^3P_2$ phases. For D-waves, associated to a $6_A$ super-multiplet, we have a similarity between $1^1D_2$ and $3^1D_1$ phases $|\delta_{1D_2} - \delta_{3D_1}| \sim 1^0$ but, again, clear differences between the $3^1D_1$ and $3^3D_2$ ones. Clearly, the symmetry is broken in higher partial waves. In what follows we want to determine whether our interpretation of a long distance symmetry which worked so successfully for S-waves above (see Sect. 3.5) holds also for non-central phases.

As it is well-known the spin-orbit interaction lifts the independence on the total angular momentum, via the operator $L \cdot S$. Moreover, the tensor coupling operator, $S_{12}$, mixes states with different orbital angular momentum. The proper way of seek the symmetry is by looking at the center of the multiplet. We use first order perturbation theory, and the Wigner symmetric distorted waves as the unperturbed states. In Appendix F we show how is the procedure explicitly. To first order in spin-orbit and tensor force perturbation the following sum rule for the center of the $S = 1$ multiplet, denoted as $\delta_{L}^{10}$ and $\delta_{L}^{11}$, and the $S = 0$ states, denoted as $\delta_{L}^{01}$ and $\delta_{L}^{00}$, holds,

$$\delta_{L}^{10} \equiv \frac{\sum_{J=0}^{L+1} (2J+1)\delta_{LJ}^{10}}{(2L+1)3} = \delta_{L}^{01} \equiv \delta_{L}^{01}, \quad \text{(3.73)}$$

$$\delta_{L}^{11} \equiv \frac{\sum_{J=0}^{L+1} (2J+1)\delta_{LJ}^{11}}{(2L+1)3} = \delta_{L}^{00} \equiv \delta_{L}^{00}, \quad \text{(3.74)}$$

In terms of these mean phases, Wigner symmetry is formulated for non-central waves up to $L = 4$ as,

$$\delta_{1P_1} = \frac{1}{9}(3\delta_{P_1} + 3\delta_{3P_1} + 5\delta_{5P_1})$$,

$$\delta_{1D_2} = \frac{1}{15}(3\delta_{D_1} + 5\delta_{3D_1} + 7\delta_{5D_1})$$,

$$\delta_{1F_2} = \frac{1}{21}(5\delta_{F_2} + 7\delta_{3F_2} + 9\delta_{5F_2})$$,

$$\delta_{1G_4} = \frac{1}{27}(7\delta_{G_3} + 9\delta_{3G_3} + 11\delta_{5G_3})$$.

(3.75)

In Fig. 3.8 we show the l.h.s. and the r.h.s. of Eqs. (3.75). As we see the D-waves fulfill this relation rather accurately up to $p \sim 250$MeV and the G-waves up to $p \sim 400$MeV while the P- and F-waves fail
Wigner SU(4) symmetry as a long distance symmetry

Figure 3.8: Average values of the phase shifts [56] (in degrees) as a function of the c.m. momentum (in MeV). (Upper left panel) P-waves. (Upper right panel) D-waves. (Lower left panel) F-waves. (Lower right panel) G-waves. According to the Wigner symmetry $\delta_{1L} = \delta_{3L}$. Serber symmetry implies $\delta_{3L} = 0$ for odd-$L$. One sees that L-even waves satisfy Wigner symmetry while L-odd spin triplet waves satisfy Serber symmetry.

completely. Actually, at threshold, $\delta_{L} \rightarrow -\alpha_{LP}^{2L+1}$, and using the low energy parameters of the NijmII and Reid93 potentials [56] determined in Ref. [192] we get

$$\alpha_{1P} = \frac{1}{9}(\alpha_{0P} + 3\alpha_{2P} + 5\alpha_{3P}) ,$$

$$(-2.46\text{fm}^3) = (0.08\text{fm}^3) ,$$

$$\alpha_{1D} = \frac{1}{15}(3\alpha_{1D} + 5\alpha_{2D} + 7\alpha_{3D}) ,$$

$$(-1.38\text{fm}^5) = (-1.23\text{fm}^3) ,$$

(3.76) (3.77)

On the light of the previous discussions for the S-waves one reason for the discrepancy should be looked in a short distance breaking of the symmetry for the D-waves. Actually, the fact that D-waves violate the sum rule at $p \sim 250\text{MeV}$ while the G-waves show no violation up to $p \sim 400\text{MeV}$ agrees with our interpretation in the S-waves that the Wigner symmetry be a long distance one, since higher partial waves are less sensitive to short distance effects. The case of P-waves is different since the $^1P$-potentials and the $^3P$-potentials are very different. This pattern of symmetry breaking agrees with the findings of Ref. [140] based on the large $N_c$ expansion where the central potential preserves the symmetry in $L$-even partial waves while it breaks the symmetry in the $L$-odd partial waves, since at leading order and
neglecting the tensor force one has,

\[ V(r) = V_C(r) + \sigma W_S(r) + \mathcal{O}(1/N_c), \]  

(3.78)

so that for the lower L-channels we have

\[ V_1S = V_3S, \quad V_1P = V_C(r) - 3W_S(r) + \mathcal{O}(1/N_c), \]  

(3.79)

\[ V_3P = V_C(r) + W_S(r) + \mathcal{O}(1/N_c), \]  

(3.80)

\[ V_1D = V_3D = V_C(r) - 3W_S(r) + \mathcal{O}(1/N_c), \]  

(3.81)

so as we see \( V_3P \neq V_1P \), and thus it is obvious that \( \delta_{3P} \neq \delta_{1P} \).

It is important to note that the initial claim of Ref. [141] on the validity of the Wigner symmetry based on the large \( N_c \) expansion was restricted to purely center potentials, which do not faithfully distinguish the two irreducible representations, \( 10_s \) and \( 6_A \), of the \( SU(4) \) group for the NN system. Later on, the issue was qualified by a more complete study carried out in Ref. [140] which in fact could not justify the Wigner symmetry in odd-L partial waves, even when the tensor force was neglected. Although this appeared as a puzzling result, our calculations clearly show that the pattern of \( SU(4) \)-symmetry breaking supports a weak violation in even-L partial waves and a strong violation in the odd-L partial waves, exactly as the large \( N_c \) expansion suggests.

Finally, an important feature of Fig. 3.8 is that for odd-waves the mean triplet phase is close to null. One might attribute this feature to an accidental symmetry where the odd-waves potentials are negligible. From a large \( N_c \) perspective this means \( V_C + W_S \ll V_C + 9W_S \), a fact which is well verified in fact. For instance at short distances the Yukawa OBE potentials have Coulomb like behavior \( V \to C/(4\pi r) \) with the dimensionless combinations

\[ C_{V_C + W_S} = -g_{\sigma NN}^2 + g_{\omega NN}^2 + \frac{f_{\rho NN}^2 m_\rho^2}{6 M_N^2}, \]  

(3.83)

\[ C_{V_C + 9W_S} = -g_{\sigma NN}^2 + g_{\omega NN}^2 + \frac{3 f_{\rho NN}^2 m_\rho^2}{2 M_N^2}, \]  

(3.84)

where the small OPE contribution has been dropped. Numerically we get \( C_{V_C + W_S} \sim 10 \) and \( C_{V_C + 9W_S} \sim 300 \) for reasonable natural couplings. This approximate vanishing of triplet odd-wave potentials although is not a consequence of large \( N_c \) it is nevertheless reminiscent of the old and well-known Serber force as we will see at length in the next chapter.

### 3.7 Conclusions

Low energy NN interactions are dominated by two S-waves in different channels where spin-isospin \((S,T)\) are interchanged, \((1,0) \leftrightarrow (0,1)\). Wigner \( SU(4) \) symmetry implies that the potentials in the \( ^1S_0 \) and \( ^3S_1 \) channels coincide and the tensor force vanishes, while the corresponding phase shifts from Partial Wave Analyses are actually very different at all energies and show no evident trace of the identity of the potential. Nevertheless, the nuclear force at short distances is evidently unknown and the validity of the symmetry to all distances is at least questionable. This lack of knowledge of the short distance physics
should not be crucial at low energies, where the phase shifts are indeed quite dissimilar. For that reason we have proposed to regard Wigner $SU(4)$ as a long distance symmetry which might be strongly broken at short distances and weakly broken at large distances. We have seen that using renormalization ideas where the short distance insensitivity is manifest, the standard Wigner correlation between potentials indeed predicts one phase shift from the other in a non-trivial and successful way. Using the One Boson Exchange potential constrained to have a well large-$N_c$ scaling we have proven that if one channel is described successfully the other channel is unavoidably well reproduced within uncertainties which might be compatible with the disregard of the tensor force and the $1/N_c^2$ corrections to the potential. The effects of symmetry breaking at long and short distances have been analyzed and the extension to higher partial waves has also been discussed, where a relation for phase shifts has been deduced. We have shown that the pattern of symmetry breaking is indeed in agreement with what the large-$N_c$ expansion suggests.
Chapter 4

Serber symmetry and Yukawa-like OBE Potentials

4.1 Serber symmetry

It is well known that the nuclear force has saturation properties which is manifest by two facts, the density of nucleons is roughly equal for all nuclei (saturation of density), and the binding energy per nucleon is roughly equal for all nuclei (saturation of $E/A$). In fact, the existence of these two saturation phenomena has decisive implications for the nature of nuclear forces and led Heisenberg and Majorana to introduce the hypothesis of exchange forces. The basic ingredient of these exchange forces is the assumption that the force, or part of it, between two nucleons is sometimes attractive and some others repulsive, depending on the state of the two nucleons respect to each other\(^1\). One of these exchange forces is the well-known Serber force, postulated by R. Serber based on the observation that at low energies the proton-proton and neutron-proton differential cross sections are symmetric functions in the c.m. scattering angle around 90°. This force is defined as,

$$V_{Serber} = \frac{1}{2} (1 + P_M) V(r),$$

(4.1)

where $V(r)$ may be a spin-dependent potential which is assumed to be attractive for all $r$. The Majorana exchange operator $P_M$ acting on the two-nucleon wave function exchanges positions,

$$P_M \psi(r_1, \xi_1, \eta_1; r_2, \xi_2, \eta_2) = \psi(r_2, \xi_1, \eta_1; r_1, \xi_2, \eta_2),$$

(4.2)

being $\xi$ and $\eta$ the spin and isospin of the nucleons respectively. This operator can also be written in terms of spin and isospin operators as,

$$P_M = -\frac{1}{4} (1 + \sigma_1 \cdot \sigma_2)(1 + \tau_1 \cdot \tau_2).$$

(4.3)

Since $P_M$ is $+1$ for even-L waves and $-1$ for odd-L waves, the Serber force acts only on states of even orbital angular momentum. This fact has consequences in the scattering amplitude. Separating $f(\theta)$

\(^{1}\)A review of the exchange forces in nuclear physics can be found for instance in Ref. [193].
into a symmetric and an anti-symmetric part around $\theta = \pi/2$, one has,

$$ f(\theta) = \frac{1}{2} [f(\theta) + f(\pi - \theta)] = f_{\text{even}} $$

$$ + \frac{1}{2} [f(\theta) - f(\pi - \theta)] = f_{\text{odd}}, \quad (4.4) $$

and given the fact that scattering in states of even-$L$ contributes to $f_{\text{even}}$ and scattering in states of odd-$L$ contributes to $f_{\text{odd}}$, it follows that the scattering amplitude $f(\theta)$ produced by a Serber force is an even function of $\theta$, i.e., $f(\theta) = f(\pi - \theta)$ and as a consequence the scattering cross section is an even function of $\theta$ around $\theta = \pi/2$. We call this property *Serber symmetry* for definiteness.

Specific attempts were directed towards the verification of such a property [194]. This symmetry was shown to hold for the NN system, up to relatively high energies [195]. However, such a force was also found to be incompatible with the requirement of nuclear matter saturation [196] as well as with the underlying meson forces mediated by one and two pion exchange [197]. These puzzling inconsistencies were cleared up when it was understood that only singular Serber forces could provide saturation [198].

Old phase shift analyses [200] confirm the rough Serber exchange character of nuclear forces. Many nuclear structure [201], nuclear matter [199], nuclear reactions [202–204], use Serber forces both for their simplicity as well as their phenomenological success in the low and medium energy region. Actually the symmetry has been implemented in modern versions (SLy4) of the Skyrme effective interactions [205] and recently a novel fitting strategy [206] for the coupling constants of the nuclear energy density functional, which focus on single-particle energies rather than ground-state bulk properties, yields naturally an almost perfect fulfillment of Serber symmetry.

Possibly one of the most vivid evidences of the Serber symmetry is the $np$ differential cross section which is shown in Fig. 4.1 where we plot several c.m. momenta using the Partial Wave Analysis and the high quality potentials [56, 57] carried out by the Nijmegen group (http://nn-online.org). Although discrepancies between forward and backward directions show that this symmetry breaks down at short distances, the intermediate region however does exhibit a clear Serber symmetry. While these are well established features of the NN interaction, even if it does not hold in the entire range, it is amazing that this symmetry has no obvious explanation from the more fundamental and QCD motivated side.

As an advance, appealing to the phase shifts sum rules derived in Chapter 3 Serber symmetry implies,

$$ \delta_{3P} \equiv \frac{1}{9} (3\delta_{3P_0} + 3\delta_{3P_1} + 5\delta_{3P_2}) = 0, \quad (4.5) $$

$$ \delta_{3F} \equiv \frac{1}{21} (5\delta_{3P_3} + 7\delta_{3P_2} + 9\delta_{3P_1}) = 0, \quad (4.6) $$

which is well fulfilled by the phase shifts [56, 57] as shown in Fig. 3.8 where $\delta_{3P} \ll \delta_{1P}$ and $\delta_{3F} \ll \delta_{1F}$.

This evidence for both even-$L$ Wigner and odd-$L$ Serber symmetries need for an explanation more closely

\[ 2 \text{This can be understood if we note that } f(\theta, p) \text{ has an expansion in spherical harmonics,} \]

\[ f(\theta, p) = \sum_{l=0}^{\infty} (2L + 1)f_l(p)F_L(\cos \theta) \]

and even (odd) spherical harmonics are even (odd) functions of $\theta$.

\[ 3 \text{It is not difficult to see this statement in nuclear matter [199] where for a Serber force like Eq. (4.1) one obtains for the energy density per nucleon,} \]

\[ E/A = \frac{1}{A} \langle \bar{\Omega} \hat{H} \Omega \rangle = \frac{3}{2} \frac{k_F^2}{2M_N} + \frac{k_F^4}{4\pi^2} \left\{ \int d^3r V(r) + \int d^3r V(r) \left[ \frac{3\mu(k_Fr)}{k_F} \right]^2 \right\} \]

which diverges in the limit $k_F \to \infty$ if $\int d^3r V(r) < \infty$. 

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Serber symmetry and Yukawa-like OBE Potentials

Based on our current knowledge of strong interactions and QCD. Again the analysis will be done in the spirit of the $1/N_c$ expansion. I would like to point out that while the pattern of $SU(4)$-symmetry breaking complies to the large $N_c$ expectations as we have seen in Chapter 3, Serber symmetry, while not excluded for odd-L waves, is not a necessary consequence of large $N_c$. The search for an explanation of the Serber force requires more detailed information than in the case of Wigner symmetry.

4.2 Serber symmetry as a long distance one

In order to analyze the symmetry it is important to delimitate the scale where it actually operates. As we see from Fig. 4.1, Serber symmetry does not work all over the range and equally well for all energies. Thus, we expect to see the symmetry in the medium and long distance region.

Suppose that the NN potential is separable as the sum of central components and non-central ones which will be assumed to be small,

$$V_{NN} = V_0 + V_1,$$

(4.7)

where $[\mathbf{L}, V_0] = 0$ whereas $[\mathbf{J}, V_1] = 0$ and $[\mathbf{L}, V_1] \neq 0$. Specifically, for the central part we take

$$V_0 = V_C + \sigma_1 \cdot \sigma_2 V_S + (W_C + \sigma_1 \cdot \sigma_2 W_S) \tau_1 \cdot \tau_2,$$

(4.8)

while the non-central part is

$$V_1 = (V_T + \tau_1 \cdot \tau_2 W_T) S_{12} + (V_{LS} + \tau_1 \cdot \tau_2 W_{LS}) \mathbf{L} \cdot \mathbf{S}.$$

(4.9)

where $\sigma_i$ and $\tau_i$ are the Pauli matrices representing the spin and isospin respectively of the nucleon $i$. The tensor operator is $S_{12} = 3\sigma_1 \cdot \hat{x} \sigma_2 \cdot \hat{x} - \sigma_1 \cdot \sigma_2$ while $\mathbf{L} \cdot \mathbf{S}$ corresponds to the spin-orbit term. The total potential commutes with the total angular momentum $J = L + S$. However, we will start assuming that the potential is central, and that the breaking in orbital angular momentum is small.
If we only consider the central potential it is clear that partial waves do not depend on the total angular momentum and we would have e.g. $\delta_{3P_0} = \delta_{3P_1} = \delta_{3P_2}$ and so on, in complete contradiction to the data. To account for the $J$-dependence we use first order perturbation theory in the spin-orbit and tensor potentials with the orbital symmetric distorted waves as the unperturbed states just as it was done in Chapter 3. As it was shown there, the phase shift for the center of the multiplet can be written as,

$$\bar{\delta}_{L}^{ST} = \frac{\sum_{J=J=L-1}^{L+1}(2J+1)\delta_{J}^{ST}}{(2L+1)3} = \delta_{L}^{ST}, \quad (4.10)$$

Therefore, to first order we may define a common mean phase obtained as the one obtained from a mean potential

$$\bar{V}_{L}(r) = \frac{\sum_{J=J=L-1}^{L+1}(2J+1)V_{LJ}(r)}{3(2L+1)}. \quad (4.11)$$

It is in terms of these potentials where we expect to formulate the verification of a given symmetry. This is nothing but the standard procedure of verifying a symmetry between multiplets by defining first the center of the multiplet. Now, Serber symmetry requires

$$V_{1L}(r) = V_{3L}(r) = 0 \quad \text{odd } - L, \quad (4.12)$$

while Wigner symmetry requires

$$V_{3L}(r) = V_{1L}(r) \quad \text{all } - L. \quad (4.13)$$

Clearly these two requirements are incompatible except when all potentials vanish. In Fig. 4.2 we plot the Argonne V-18 potentials \cite{58} for the center of the orbital multiplets.

Thus the potentials suggest instead, that for $r > 1.5$fm,

$$V_{3L}(r) \ll V_{1L}(r) \quad \text{odd } - L \quad (4.14)$$

$$V_{3L}(r) \sim V_{1L}(r) \quad \text{even } - L \quad (4.15)$$

i.e. Wigner symmetry is fulfilled for even-$L$ states while Serber symmetry holds for odd-$L$ triplet states at distances above 1.5fm in agreement with the expectations of a long distance symmetry. We obtain the same pattern of fulfillment and breaking than the one for phase shifts (see Fig. 3.8). The remarkable aspect is that the symmetry pattern while incompatible with Wigner symmetry for odd-$L$ states is fully compatible with large $N_c$ expectations. It does not explain, however, why Serber symmetry is a good one.

### 4.3 Looking for clues of the symmetry in NN

Most modern high-precision models of the NN interaction include OPE as the dominant longest range contribution. However, they vary substantially in their treatment of the intermediate-range attraction and the short-range repulsion where many effects compete and even are written in quite different forms (energy dependent, momentum dependent, angular momentum dependent, etc.). These differences
should be somehow related because all of them describe the same NN data. Actually the different parameterizations are not fully independent of each others \cite{207,208}. The relevant issue within the present context and which we analyze below regards the range and form of current NN interactions from the viewpoint of long distance symmetries. Any potential fitting the elastic scattering data must possess the symmetries displayed by the phase-shifts sum-rules as displayed in Fig. 3.8. However, it is not obvious that potentials display the symmetry explicitly.

### 4.3.1 One Pion Exchange

The OPE potential reads

\[
V^\pi(r) = \{\sigma_1 \cdot \sigma_2 \ W_\pi^S(r) + S_{12}W_\pi^T(r)\} \ \tau_1 \cdot \tau_2.
\]  
(4.16)
While OPE complies to the Wigner symmetry it does not embody exactly the Serber symmetry. Actually we get for even-L waves

\[ V_{\pi}^S(r) = V_{\pi}^D(r) = V_{\pi}^G(r) = -3W_{\pi}^S(r), \quad (4.17) \]

\[ V_{\pi}^3(r) = V_{\pi}^3(r) = V_{\pi}^3(r) = -3W_{\pi}^3(r), \quad (4.18) \]

while for odd-L waves we have

\[ V_{\pi}^P(r) = V_{\pi}^F(r) = V_{\pi}^H(r) = 9W_{\pi}^S(r), \quad (4.19) \]

\[ V_{\pi}^3(r) = V_{\pi}^3(r) = V_{\pi}^3(r) = W_{\pi}^S(r). \quad (4.20) \]

Nevertheless one can say that the factor 9 for the singlet to triplet ratio is a close approximation to the Serber limit in a region where the potential is anyhow small. These OPE relations are verified in practice for distances above \(3 - 4\)fm. As we see from Fig. 4.2 the vanishing of \(3P\) potential happens down to the region around 1.5fm. For lower distances, potential models start deviating from each other (see e.g. [57]) but this vanishing of \(3P\) potential is a common feature which occurs beyond the validity of OPE.

### 4.3.2 Boundary conditions (alias \(V_{\text{high}}\))

We now analyze the symmetry for the highly successful PWA [56] of the Nijmegen group. There, a OPE potential is used down to \(r_c = 1.4\)fm and the interaction below that distance is represented by a boundary condition determined by an energy dependent square well potential,

\[ 2\mu V_{\pi,\beta}(k^2) = \sum_{n=0}^{N} a_{n,\beta}k^{2n}, \quad (4.21) \]

where \(\beta\) stands for the corresponding channel, so that the total potential reads

\[ V_{\beta}(r) = \left[ V_{\pi,\beta}^r(r) + V_{\pi,\beta}^{\text{int}}(r) \right] \theta(r - r_c) + V_{\pi,\beta}(k^2) \theta(r_c - r), \quad (4.22) \]

where \(V_{\pi,\beta}^{\text{int}}(r)\) is a phenomenological intermediate range potential accounting for heavier-boson-exchanges and which acts in the region \(1.4\)fm \(\leq r \leq 2.0\)fm. Then, for the center of the L-multiplets (with \(V\) in MeV and \(k\) in fm ) we have

\[ V_{S,1P}(k^2) = 139.448 - 23.417k^2 + 2.479k^4, \quad (4.23) \]

\[ V_{S,3P}(k^2) = 14.666 + 0.92k^2 + 0.029k^4, \quad (4.24) \]

\[ V_{S,1F}(k^2) = 248.73, \quad (4.25) \]

\[ V_{S,3F}(k^2) = -33.08 + 5.90k^2, \quad (4.26) \]

where, again, we see that Serber symmetry takes place since \(V_{S,3F}(k^2) \ll V_{S,1P}(k^2)\) and \(V_{S,3F}(k^2) \ll V_{S,1F}(k^2)\). Actually, the factor is strikingly similar to the \(1/9\) of the OPE interaction, which in the analysis holds down to \(r_c = 1.4\)fm. Thus, in the Nijmegen PWA decomposition of the interaction we find the remarkable relation

\[ V_{3L}(r) \ll V_{1L}(r) \quad \text{odd} - L, \quad \text{all } r, \quad (4.27) \]
Chapter 4. Serber symmetry and Yukawa-like OBE Potentials

showing that there is Serber symmetry in the short range piece of the potential. On the other hand, the even partial waves yield

\[ V_{S;S}(k^2) = -17.813 - 1.016k^2 + 2.564k^4, \]  
\[ V_{S;3S}(k^2) = -40.955 + 4.714k^2 + 1.779k^4, \]  
\[ V_{S;1D}(k^2) = 61.42 - 15.678k^2, \]  
\[ V_{S;3D}(k^2) = 28.869 - 3.579k^2, \]  
\[ V_{S;1G}(k^2) = 466.566, \]  
\[ V_{S;3G}(k^2) = 0, \]

where we clearly see the expected violation pattern of Wigner symmetry at short distances,

\[ V_{3L}(r) \sim V_{1L}(r) \text{ even } -L r \geq r_c. \]  

This simple analysis suggests that Serber symmetry, when it works, holds to shorter distances than the Wigner symmetry which is in fully agreement to the previous in terms of phase shifts. Indeed, higher partial waves with angular momentum \( l \) are necessarily small at small momenta due to the well known \( \delta_l(p) \sim -\alpha_l p^{2l+1} \) threshold behavior. In fact, this is the case for \( \delta_{1P} \) and \( \delta_{1F} \). However, Serber symmetry implies that \( \delta_{3P} \) and \( \delta_{3F} \) are rather small not only in the threshold region but also in the entire elastic region as can be clearly seen from Fig. 3.8.

4.3.3 Model-independent low momentum potential \( V_{lowk} \)

In Ref. [209] a model-independent low momentum NN interaction, the so-called \( V_{lowk} \), was derived from a Renormalization Group (RG) analysis. The \( V_{lowk} \) approach takes a Wilsonian point of view of integrating out high energy components and working within an effective Hilbert space. The out-coming potentials are smooth and more amenable to mean field and perturbative treatments at the expense of introducing scale dependent three- and higher-body forces. This potential corresponds to integrating out high momentum modes below a given cut-off \( k \leq \Lambda \) in the Lippmann-Schwinger equation by demanding RG invariance. In the case of the two-body problem the \( V_{lowk} \) approach [210] replaces the (half off-shell) coupled channel Lippmann-Schwinger equation for the bare potential

\[ T(k', k; k^2) = V(k', k) + \frac{2}{\pi} \int_0^\infty dq \frac{q^2}{k^2 - q^2} V(k', q) T(q, k; k^2), \]  

by the equivalent \( V_{lowk} \) potential defined by the equation restricted to \( (k, k') \leq \Lambda \),

\[ T(k', k; k^2) = V_{lowk}(k', k) + \frac{2}{\pi} \int_0^\Lambda dq \frac{q^2}{k^2 - q^2} V_{lowk}(k', q) T(q, k; k^2). \]  

For instance, in perturbation theory the \( V_{lowk} \) potential becomes

\[ V_{lowk}(k', k) = V(k', k) + \frac{2}{\pi} \int_\Lambda^\infty dq \frac{q^2}{k^2 - q^2} V(k', q) V(q, k) + \ldots, \]
which clearly shows a departure from the original potential. This of course becomes small for \( \Lambda \to \infty \). In the opposite limit \( \Lambda \to 0 \) one may Taylor expand the effective S-wave interaction

\[
V_{\text{lowk}}(k, k') = C_0 + C_2(k^2 + k'^2) + \ldots
\]  

(4.38)

Plugging this ansatz into Eq. (4.36) and fixing the \( T \) matrix at low energies to the scattering length \( a_0 \), effective range \( r_0 \), one gets (for \( C_2 = 0 \)) a result depending on the scale \( \Lambda \),

\[
C_0(\Lambda) = \frac{a_0}{1 - 2a_0\Lambda/\pi}.
\]  

(4.39)

Therefore, this effective low energy interaction depends on the scale \( \Lambda \). It was found that high quality potential models, i.e. fitting the NN data to high accuracy and also incorporating OPE, collapse into a unique self-adjoint nonlocal potential for \( \Lambda \sim 400\text{MeV} \). This is, in fact, a not unreasonable result since all the potentials provide a rather satisfactory description of elastic NN scattering data up to \( p \sim 400\text{MeV} \). Moreover, the potential which comes out from eliminating high energy modes can be accurately represented as the sum of the truncated original potential (suggested by the large \( \Lambda \) limit, Eq. (4.37)) and a polynomial in the momentum (suggested by the small \( \Lambda \) limit, Eq. (4.38)) \[210\],

\[
V_{\text{lowk}}(k', k) = V_N(k', k) + V_{\text{CT}}^\Lambda(k', k), \quad (k, k') \leq \Lambda,
\]  

(4.40)

where \( V_N(k', k) \) is the original potential in momentum space for a given partial wave channel and \( V_{\text{CT}}^\Lambda(k', k) \) is the effect of the high energy states,

\[
V_{\text{CT}}^\Lambda(k', k) = \frac{k}{k'} \left[ C_0^\Lambda(\Lambda) + C_2^\Lambda(\Lambda)(k^2 + k'^2) + \ldots \right],
\]  

(4.41)

where the coefficients \( C_0^\Lambda(\Lambda) \) play the role of counterterms. It should be noted that here \( V_N(k', k) \) is cut-off \emph{independent} whereas \( V_{\text{CT}}^\Lambda(k', k) \) does depend on \( \Lambda \). The cut-off dependent \( V_{\text{lowk}} \) potential is obtained by solving the RG equation \( \frac{d}{d \Lambda} V_{\text{lowk}} = \beta(V_{\text{lowk}}; \Lambda) \). When the potential given by Eq. (4.40) is plugged into the truncated Lippmann-Schwinger equation, i.e. intermediate states \( q \leq \Lambda \), the phase shifts corresponding to the full original potential \( V_N(k', k) \) are reproduced. In Fig. 4.3 the corresponding diagonal \( V_{\text{lowk}}(p, p) \) mean potentials are plotted for the Argonne-V18 force \[58\]. As we see both Wigner and Serber symmetries are, again, vividly seen.

Nevertheless, the separation assumed by Eq. (4.40) does not manifestly display the symmetry due to possible polynomial contributions in \( V_N N \). A more convenient representation would be to separate off all polynomial dependence explicitly from the original potential

\[
V_{\text{lowk}}(k', k) = \tilde{V}_N(k', k) + \tilde{V}_{\text{CT}}^\Lambda(k', k), \quad (k, k') \leq \Lambda,
\]  

(4.42)

so that if \( \tilde{V}_{\text{CT}}^\Lambda(k', k) \) contains up to \( \mathcal{O}(p^n) \) then \( \tilde{V}_N(k', k) \) starts off at \( \mathcal{O}(p^{n+1}) \), i.e. the next higher order. This way the departures from a pure polynomial may be viewed as explicit effects due to the original potential. In terms of these polynomials, Wigner and Serber symmetries are formulated from the coefficients

\[
\tilde{C}_0 = C_0 + C_0^{\text{high}}(\Lambda)
\]  

(4.43)
constructed from the sum of the potential and the integrated out contribution below a cut-off Λ, namely

\[
C_{0,1,\ell} = C_{0,3,\ell}, \quad \text{even } - L,
\]

\[
C_{0,2,\ell} = 0, \quad \text{odd } - L.
\]

in the spirit of the \( V_{\text{high } R} \) approach.

### 4.3.4 Nuclear Potentials

From the previous discussions one can note that in the momentum space \( V_{\text{low } k} \) approach [210] the long distance behaviour of the theory is not determined by the low momentum components of the original potential only, one has to add virtual high energy states which also contribute to the interaction at low energies in the form of counterterms, as outlined by Eqs. (4.40) and (4.42). Alternatively the more conventional coordinate space boundary condition (alias \( V_{\text{high } R} \)) method shows that the low energy behaviour of the theory is not determined only by the long distance behaviour of the potential, one has to include the contribution from integrated out short distances in the form of boundary conditions. However, the low momentum features of the interaction in the \( V_{\text{low } k}(p,p) \) potential can be reflected into long distance characteristics of the potential \( V(r) \). In fact, in Ref. [210] it is suggested that the \( V_{\text{low } k} \) is a viable way of determining the effective interactions which could be further used in shell model calculations for finite nuclei. Actually, this interpretation when combined with our observation of Fig. 4.3 that Serber symmetry shows up quite universally has interesting consequences. Schematically, this can
be implemented as a Skyrme type effective (pseudo)potential \[ V(\vec{r}) = t_0(1 + x_0 P_\sigma)\delta^{(3)}(\vec{r}) + t_1(1 + x_1 P_\sigma)\left\{-\nabla^2, \delta^{(3)}(\vec{r})\right\} - t_2(1 + x_2 P_\sigma)\nabla\delta^{(3)}(\vec{r})\nabla + \ldots \] (4.46)

where \( P_\sigma = (1 + \sigma_1 \cdot \sigma_2)/2 \) is the spin exchange operator with \( P_\sigma = 1 \) for spin single \( S = 0 \) and \( P_\sigma = 1 \) for spin triplet \( S = 1 \) states. The dots stand for spin-orbit, tensor interaction, etc. It should be noted the close resemblance of the momentum space version of this potential

\[ V(p', p) = t_0(1 + x_0 P_\sigma) + t_1(1 + x_1 P_\sigma)(p'^2 + p^2) + t_2(1 + x_2 P_\sigma)p' \cdot p + \ldots \] (4.47)

with Eq. (4.42) after projection onto partial waves, where only S- and P-waves have been retained. Traditionally, binding energies have been used to determine the parameters \( t_i \) and \( x_i \) within a mean field approach and many possible fits arise depending on the chosen observables (see e.g. Ref. [205]) possibly displaying some spurious short distance sensitivity beyond the range of applicability of Eq. (4.46).

Actually, inclusion of tensor force and a new fitting strategy to single particle energies [206] yields \( x_2 = -0.99 \) which is an almost perfect Serber force for spin-triplets \( (P_\sigma = 1) \) and reproduces the so-called SLy4 form of the Skyrme functional [205]. It seems quite natural then that Serber symmetry becomes manifest directly from a coarse graining of the NN interaction.

### 4.3.5 Chiral two pion exchange potential

The chiral Two Pion Exchange (TPE) potentials computed in Ref. [111] are understood as direct consequences of the spontaneous chiral symmetry breaking in QCD. The TPE contribution takes over the OPE one at about \( r = 2 \text{ fm} \). The TPE potential at long distances is,

\[ V_{2\pi}^{\text{ChPT}}(r) = (1 + 2 \tau_1 \cdot \tau_2) \frac{e^{-2m_\pi r}}{r} \frac{3g_A^2 m_\pi^5}{1024 f_\pi^4 M_N^2 \pi^2} + \ldots , \] (4.48)

with \( m_\pi \) and \( M_N \) the pion and nucleon masses, \( g_A \) the axial coupling constant and \( f_\pi \) the pion weak decay constant. As we see Serber symmetry is completely broken already at long distances because \( \tau_1 \cdot \tau_2 \) can be only \(-3\) for isosinglet channels and \(+1\) for isotriplets. Generally, these chiral potentials are supplemented by counterterms or equivalently boundary conditions when discussing NN scattering and generating phase shifts (see e.g. Ref. [212]). Given that these NN phase shifts do fulfill the symmetry (see Fig. 3.8) it is reasonable to expect that the breaking of the symmetry at long distances must be compensated by the counterterms which encode the unknown short distance physics [212]. In fact, the \( V_{\text{low } k} \) potential corresponding to the Next-to-next-to-next-to-leading order (N^3LO) chiral potential which contains its own cut-off parameter of \( \Lambda_\chi = 500 \text{ MeV} \) [121] does also fulfill the same pattern of symmetry breaking and fulfillment. This potential contains OPE and describes successfully the data so falls into the universality class of high-quality potentials [213] when the common \( V_{\text{low } k} \) cut-off scale \( \Lambda = 400 \text{ MeV} \) is used. If the chiral potential is slightly detuned by taking \( \Lambda_\chi = 600 \text{ MeV} \) one sees a low momentum violation of the Wigner symmetry in Fig. 4.4 in contradiction with what one expects from an asymptotically OPE dominance.
4.3.6 Resonance saturation

Inspired by the success of the resonance saturation hypothesis of the exchange forces in $\pi\pi$ scattering (see e.g. [214]), the OBE picture [53] low momentum contributions from the exchange of heavier mesons have been identified as generating the counter-terms for chiral potentials [190]. Taking e.g. a heavy scalar meson with mass $m \gg \Lambda$ and coupling $g$ one can use Eq. (4.37) and the potential reads

$$\frac{g^2}{(p'-p)^2 + m^2} = \frac{g^2(p' - p)^2}{m^2} + \cdots = C_0 + C_2 \left( p^2 + p'^2 \right) + C_1 p \cdot p' + \cdots \quad (4.49)$$

Note that here $C_0 = g^2/m^2$ does not depend on $\Lambda$, unlike Eq. (4.39). The OBE resonance matching to chiral potentials besides identifying terms scaling differently in $N_c$, $C_0^{\text{OBE}} \sim N_c$ vs $C_0^{\text{Chiral}} \sim g^4_A/f^2 \sim N_c^2$ do not comply to Serber symmetry for P-waves as it happens for the $V_{\text{lowk}}$ determination [215]. This does not mean, however, that the effective interaction cannot be represented in the polynomial form of Eq. (4.49), but rather that the coefficients cannot be computed directly and generically as the Fourier components of the potential, since the corrections are not necessarily small (unlike the $\pi\pi$ case).

4.4 Serber force from a large-$N_c$ perspective

So far, we have provided evidences that long distance symmetries such as Wigner’s and Serber’s do really take place in the two nucleon system. The question it should be asked is whether those symmetries are purely accidental ones or obey some pattern closely related with QCD. As it was already shown in Chapter 3 the $1/N_c$ expansion provides a rationale for Wigner symmetry. The fact that Serber symmetry holds when Wigner symmetry fails suggests analyzing the large-$N_c$ consequences more thoroughly.

One of the main features which makes large-$N_c$ expansion useful for the NN problem is that it does not specialize hold for long or short distances. Actually, although the large-$N_c$ scaling behavior and spin-flavor structure of the NN potential is directly established in terms of quarks and gluons (see Sec. A.4 in Appendix A), quark-hadron duality at distances larger than the confinement scales requires an identification with meson exchanges and then a link to the OBE picture is provided. Another important
advantage of taking the large-$N_c$ limit is that nucleons become infinitely heavy, so if their momentum is taken to be $p \sim N_c^0$, the non-relativistic potential is a well-defined quantity.

Based on the contracted $SU(4)$ symmetry of the large-$N_c$ the spin-flavour structure of the NN interaction was analyzed by Kaplan, Savage and Manohar [140, 141] who found that the leading $N_c$ nucleon-nucleon potential indeed scales as $N_c$ and has the structure (see Sec. A.4 in Appendix A)

$$V_{NN} = V_C + \tau_1 \cdot \tau_2 [W_S \sigma_1 \cdot \sigma_2 + W_T S_{12}].$$

It is important to note that the tensor force appears at the leading order in the large $N_c$ expansion. We should stress here that this potential only corresponds to the $NN \rightarrow NN$ elastic channel. Actually, one should also, at least, consider the $NN \rightarrow N\Delta$ and $NN \rightarrow \Delta\Delta$ processes as dynamical coupled channels because in the large-$N_c$ the nucleon and $\Delta$ become degenerate $M_\Delta - M_N \sim \mathcal{O}(N_c)$. Nevertheless, we have a potential which is well-defined (although incomplete) in the large-$N_c$ limit.

From the large $N_c$ potential, Eq. (4.50), we have for the center of multiplet potentials the sum rules

$$V_{1L} = V_{s_L} = V_C(r) - 3W_S(r), \quad (-1)^L = +1,$$
$$V_{1L} = V_C(r) + 9W_S(r), \quad (-1)^L = -1,$$
$$V_{5L} = V_C(r) + W_S(r), \quad (-1)^L = -1,$$

where as we see $V_{1L} = V_{s_L}$ for even-L and $V_{1L} \neq V_{s_L}$ for odd-L. Thus, large $N_c$ implies Wigner symmetry in even-L channels and allows a violation of Wigner symmetry in odd-L partial waves while it allows a violation of Serber symmetry in spin singlet channels. The question is whether or not large $N_c$ implies Serber symmetry in spin triplet channels as we observe both for the potentials in Fig. 4.2 as well as for the phase shifts in Fig. 3.8. From the odd-waves we see from Fig. 3.8 that the mean triplet phase is close to null, thus one might attribute this feature to an accidental symmetry where the odd-waves potentials are likewise negligible. In the large $N_c$ limit this means $V_C + 9W_S \gg V_C + W_S$.

To proceed further we write the potentials in terms of leading single meson exchanges [142] (see Table A.1). One has the Yukawa like potentials,

$$V_C(r) = -\frac{g_{\pi NN}^2}{4\pi} \frac{e^{-m_\pi r}}{r} + \frac{g_{\sigma NN}^2}{4\pi} \frac{e^{-m_\sigma r}}{r},$$
$$W_S(r) = \frac{1}{12} \frac{g_{\pi NN}^2 m_\pi^2}{4\pi \Lambda_N^2} \frac{e^{-m_\pi r}}{r} + \frac{1}{6} \frac{f_{\rho NN}^2}{4\pi \Lambda_N^2} \frac{m_\omega^2 e^{-m_\omega r}}{r},$$

where $\Lambda_N = 3M_N/N_c$ is a scale $\mathcal{O}(N_c^0)$ which is numerically equal to the nucleon mass. All meson couplings scale as $g_{\pi NN}, g_{\pi NN}, g_{\omega NN}, f_{\rho NN} \sim \sqrt{N_c}$ whereas all meson masses scale as $m_\pi, m_\sigma, m_\rho, m_\omega \sim N_c^0$. In principle there would be infinitely many contributions but we stop at the vector mesons. A relevant question which will be postponed to the next section regards what values of Yukawa masses should one take. This is particularly relevant for the $m_\sigma$ case. Note that the next-to-leading order correction of the potential Eq. (4.50) is $\mathcal{O}(N_c^{-1})$. This leaves room for $\mathcal{O}(N_c^0)$ corrections to the NN potential without generating new dependences.

As we have mentioned above, to obtain Serber symmetry we must get a large cancellation. First, let us look at short distances. At short distances the Yukawa OBE potentials have Coulomb like behavior.
V \to C/(4\pi r)$ with the dimensionless combinations

$$\begin{align*}
C_{Vc+w_s} &= -g_{\sigma NN}^2 + g_{\omega NN}^2 + \frac{f_{\rho NN}^2 m_{\rho}^2}{6M_N^2} \\
C_{Vc+g w_s} &= -g_{\sigma NN}^2 + g_{\omega NN}^2 + \frac{3f_{\rho NN}^2 m_{\rho}^2}{2M_N^2}
\end{align*}$$  

(4.54)

(4.55)

where the small OPE contribution has been dropped. To resemble Serber symmetry we should have $C_{Vc+w_s} \ll C_{Vc+g w_s}$. If we impose the OPE $1/9$-rule for the full potential we would have $g_{\sigma NN}^2 = g_{\omega NN}^2$. Using $SU(3)$, $3g_{\rho NN} = g_{\omega NN}$, Sakurai’s universality $g_{\rho NN} = g_{\rho \pi \pi}/2$, the current-algebra KSFR relation, $\sqrt{2}g_{\rho \pi \pi} f_\pi = m_\rho$, and the scalar Goldberger-Treiman relation, $g_{\sigma NN} f_\pi = M_N$, one would get $M_N = M_{c}m_{\rho}/(2\sqrt{2})$ a not unreasonable result. But this only would explain the cancellation at short distances. The cancellation is more effective at intermediate distances if $m_\sigma$ and $m_\rho$ would be numerically closer. Actually, there are various schemes where an identity between scalar and vector meson masses are explicitly verified [216–218] although in reality the scalar and vector masses are sufficiently different $m_\sigma = 444\text{MeV}$ vs $m_\rho = 770\text{MeV}$. In the next section we will analyze this apparent contradiction.

### 4.5 From $\pi\pi$ resonances to NN Yukawa potentials

#### 4.5.1 Correlated two pion exchange

As it has been mentioned in Sec. 4.3.5 that TPE dominates OPE at intermediate distances. It is well known that the iterated TPE may become an enhancement in the $t$-channel, which is consider as either a $\sigma$ or a $\rho$ resonance for isoscalar and isovector states respectively. While the interactions leading to this resonances are controlled by chiral symmetry [219–221], the resulting contributions to the NN potential in terms of boson exchanges bear a very indirect relation to it. Actually, the relation of the ubiquitous scalar meson and NN forces in terms of correlated two pion exchange although has been pointed out many times in the literature [36, 53, 111, 222–226] still remain unclear. Chiral lagrangeans have been also consider in nuclear physics [59–61] but the implications for the OBE potential are meager despite the fact that useful relations among couplings can be deduced $^4$. As we will see, there are complementary information in large-$N_c$ requirements.

Note that the leading term generating the scalar meson scale like $g_{\rho}^4 / f_\rho^6 \sim N_c$ but occurs first at $N_c^3$LO in the chiral counting. Actually, the central potential in coordinate space reads [111, 224–226],

$$V_N(r) = -\frac{32\pi}{3m_{\pi}^2} \int \frac{d^3q}{(2\pi)^3} e^{iqx} \left[\sigma_{\pi N}(-q^2)\right]^2 t_{00}(-q^2),$$  

(4.56)

where $\sigma_{\pi N}(s)$ is the $\pi N$ sigma term and $t_{00}(s) = (e^{2i\delta_{\pi N}(s)} - 1)/2i\rho_{\pi\pi}(s)$ the $\pi\pi$ scattering amplitude in the $I = J = 0$ channel as a function of the $\pi\pi$ c.m. energy $\sqrt{s}$ with $\rho_{\pi\pi}(s) = \sqrt{1-4m_{\pi}^2/s}$. Under the inclusion of $\Delta$ resonance contributions Eq. (4.56) is modified by an additive redefinition of $\sigma_{\pi N}$ to include those $\Delta$-states [224]. In the large $N_c$ limit, $t_{\pi\pi}(s) \sim 1/N_c$ while $\sigma_{\pi N}(s) \sim N_c$ yielding $V_N \sim N_c$ as expected [140]. Actually, at the sigma pole

$$\frac{32\pi}{3m_{\pi}^2} \left[\sigma_{\pi N}(s)\right]^2 t_{\pi\pi}^I(s) \to \frac{g_{\sigma NN}^2}{s - (m_\sigma - i\Gamma_\sigma/2)^2} \to \frac{g_{\sigma NN}^2}{s - m_\sigma^2},$$  

(4.57)

$^4$See for example the review of coupling constants summarized in Sec. B.5 of Appendix B.
where in the second step we have taken the large-$N_c$ limit. The “fictitious” narrow $\sigma$ exchange has been attributed to $N\Delta + \Delta\Delta$ intermediate states \[111\], to $2\pi$ iterated pions \[226\] or both \[224\]. This identification is based on fitting the resulting r-space potentials to a Yukawa function in a reasonable distance range.

### 4.5.2 Exchange of pole resonances

In this section we separate the resonance contribution to the NN potential from the background, neglecting for simplicity the vertex correction in Eq. (4.56). The most obvious definition of the $\sigma$ or $\rho$ propagator is via the $\pi\pi$ scattering amplitude in the scalar-isoscalar and vector-isovector channels, $(J,I) = (0,0)$ and $(J,I) = (1,1)$ respectively. Using the definition

$$t_{IJ}(s) = \frac{1}{2i\rho_{\pi\pi}(s)} \left( e^{2i\delta_{IJ}(s)} - 1 \right),$$

(4.58)

with $\rho_{\pi\pi}(s) = \sqrt{1 - 4m_{\pi}^2/s}$ the phase space in our notation. Taking into account the fact that on the second Riemann sheet (taking $\sigma$ as an example) the amplitude has a pole

$$t_{00}(s) \to R_\sigma \sqrt{s - s_\sigma},$$

(4.59)

with $\sqrt{s_\sigma} = M_\sigma - i\Gamma_\sigma/2$ the pole position and $R_\sigma$ the corresponding residue. Here we define, as usual, the analytical continuation as

$$t_{00}(s) \to t_{00}(s \pm i0^+).$$

(4.60)

By continuity $t_{00}(s) \equiv t_{00}(s \pm i0^+) = t_{00}(s \mp i0^+)$ and thus unitarity requires $\rho_{\pi\pi}(s + i0^+) = -\rho_{\pi\pi}(s - i0^+)$. One has for the (suitably normalized) scalar propagator

$$D_S(s) \equiv \frac{t_{00}(s)}{|R_\sigma|},$$

(4.61)

in the whole complex plane. In particular

$$D_S^H(s) = \frac{t_{00}^H(s)}{|R_\sigma|} \to \frac{e^{i\varphi_\sigma}}{s - (M_\sigma - i\Gamma_\sigma/2)^2}.$$

(4.62)

where the phase $\varphi_\sigma$ is defined as $e^{i\varphi_\sigma} = R_\sigma/|R_\sigma|$ and is related to the background, i.e. the non-pole contribution. In Appendix G we discuss a toy model for $\pi\pi$ scattering \[227\] which proves quite useful to fix ideas. The function $D_S(s)$ is analytic in the complex $s$-plane with a $2\pi$ right cut along the $(4m_\pi^2, \infty)$ line stemming from unitarity in $\pi\pi$ scattering and a left cut running from $(-\infty,0)$ due to particle exchange in the $u$ and $t$ channels. Assuming the scattering amplitude to be proportional to this propagator the corresponding $\pi\pi$ phase shift is then given by

$$e^{2i\delta_{\infty}(s)} = \frac{t_{00}(s + i0^+)}{t_{00}(s - i0^+)} = \frac{D_S(s + i0^+)}{D_S(s - i0^+)}.$$

(4.63)

The propagator satisfies the un-subtracted dispersion relation \[158\],

$$D_S(q^2) = \int_{4m_\pi^2}^{\infty} d\mu^2 \frac{\rho_S(\mu^2)}{\mu^2 - q^2},$$

(4.64)
where the spectral function is related to the discontinuity across the unitarity branch cut \(^5\)

\[
\rho_S(s) = \frac{1}{2i\pi} \text{Disc} D_S(s + i0^+) = \frac{1}{\pi |R_\sigma|} \rho_{\pi\pi}(s)|t_{00}(s)|^2 ,
\]

which satisfies the normalization condition

\[
\int_{4m_\pi^2}^{\infty} d\mu^2 \rho_S(\mu^2) = Z_\sigma .
\]

where \(Z_\sigma\) is the integrated strength. Thus, the Fourier transformation of the propagator is

\[
D_S(r) = \int \frac{d^3 q}{(2\pi)^3} e^{i\mathbf{q}\cdot \mathbf{r}} D_S(-\mathbf{q}^2) = -\frac{1}{4\pi^2} \int_{4m_\pi^2}^{\infty} d\mu^2 \rho_S(\mu^2)e^{-\mu r} .
\]

According to Eq. (4.67), \(D_S(r) \sim -Z_\sigma/(4\pi r)\) for small distances. We define the analytic function \(\rho_S(z)e^{-\sqrt{\pi}r}\) for \(r > 0\) in the cut plane without \((-\infty,0)\) and \((4m_\pi^2,\infty)\) where

\[
\rho_S(z) = \frac{1}{\pi |R_\sigma|} \rho_{\pi\pi}(z) t_{00}(z) t_{00}^\dagger (z) ,
\]

and fulfilling the boundary value condition \(\rho_S(s) = \rho_S(s + i0^+)\). This function has a pole at the complex point \(z = s_\sigma = (M_\sigma - i\Gamma_\sigma/2)^2\) and a square root branch cut at \(z = 4m_\pi^2\) triggered by the phase space factor only since \(t_{00}^\dagger(z)t_{00}(z)\) is continuous, so that \(\rho_S(s + i0^+) = -\rho_S(s - i0^+)\). Thus, we can write the spectral integral, Eq. (4.68) as running below the unitarity cut and by suitably deforming the contour in the fourth quadrant in the second Riemann Sheet, as done in Appendix G for a toy model, we can separate explicitly the contribution from the pole and the \(2\pi\) background yielding

\[
D_S(r) = D_\sigma(r) + D_{2\pi}(r) .
\]

While, in principle, both contributions are complex, the total result must be real and their imaginary parts cancel identically (see Appendix G for a specific example). Since Eq. (4.60) implies \(2it_{00}(s_\sigma)\rho_{\pi\pi}(s_\sigma) = 1\) the \(\sigma\)-pole contribution is effectively given by

\[
\text{Re} D_\sigma(r) = -\frac{Z_\sigma e^{-M_\sigma r}}{4\pi r} \left[ \cos \left( \frac{r\Gamma_\sigma}{2} \right) - \tan \varphi_\sigma \sin \left( \frac{r\Gamma_\sigma}{2} \right) \right] ,
\]

which is an oscillating function damped by an exponential. In the narrow resonance limit, \(\Gamma_\sigma \to 0\), one has \(\varphi_\sigma = \mathcal{O}(\Gamma_\sigma)\) yielding

\[
\text{Re} D_\sigma(r) \sim -\frac{Z_\sigma e^{-M_\sigma r}}{4\pi r} \left[ 1 + \mathcal{O}(\Gamma_\sigma^2) \right] ,
\]

which is a Yukawa potential. The \(2\pi\) background reads

\[
\text{Re} D_{2\pi}(r) = -\frac{1}{4\pi r} \left[ \int_0^\infty dy \rho_S(4m_\pi^2 - iy)e^{-r\sqrt{4m_\pi^2 - iy}} \right. \\
- \left. \int_0^\infty dy \rho_S(4m_\pi^2 + iy)e^{-r\sqrt{4m_\pi^2 + iy}} \right] .
\]

\(^5\)Defined as \(\text{Disc} t(s + i0^+) = t(s + i0^+) - t(s - i0^+) = 2i \text{Im} t(s)\) for a real function below \(\pi\pi\) threshold, \(0 < s < 4m_\pi^2\).
At large distances the integral is dominated by the small $y$ region, and we get the distinct TPE behaviour $\sim e^{-2m_y r}$. The pre-factor is obtained by expanding at small $y$ and using that unitarity imposes the spectral density to be proportional to the phase space factor, Eq. (4.66). Close to threshold, $s \to 4m^2_{\pi}$, involves the $\pi\pi$ scattering length $a_{00}$ defined as $\delta_{00}(s) \sim a_{00}\sqrt{s-4m^2_{\pi}}$ yielding

$$
\rho_s(s) = \frac{2ma^2_{00}}{\pi|R_s|}\sqrt{s-4m^2_{\pi}} + \ldots
$$

We therefore get

$$
D_{2\pi}(r) = -\frac{K_y(2m_\pi r)}{r^2} \frac{4m^3_{\pi}a^2_{00}}{\pi^2|R_\pi|} + \ldots
\sim -\frac{e^{-2m_\pi r}}{r^2} \frac{2a^2_{00}m^2_{\pi}}{\pi|R_\sigma|}.
$$

In Appendix G the pole-background decomposition in Eq. (4.70) is checked explicitly in a toy model. The resonance produces a Yukawa tail with an oscillatory modulation which alternates between attraction and repulsion, although the region where the oscillation is relevant depends largely on $\varphi_\sigma$.

### 4.5.3 $\pi\pi$ resonances at large $N_c$

The large $N_c$ analysis is also useful for a better understanding of the role played by the ubiquitous scalar meson. The scalar meson is actually an essential ingredient accounting phenomenologically for the mid range nuclear attraction and which, with a mass of $\sim$ 500MeV, was originally proposed in the fifties [33] to provide saturation and binding in nuclei. Along the years, there has always been some arbitrariness on the “effective” or “fictitious” scalar meson mass and coupling constant to the nucleon, partly stimulated by lack of other sources of information. During the last decade, the situation has been steadily changing, having finally culminated with the inclusion of the $0^{++}$ resonance (commonly denoted by $\sigma$) in the PDG [41] as the $f_0(600)$ seen in $\pi\pi$ scattering. The PDG gives a wide spread of values ranging from 400 – 1200MeV for the mass and 600 – 1200MeV for the width [42] but these uncertainties have recently been sharpened by a benchmark determination based on Roy equations and chiral symmetry [43] yielding the value $m_\sigma - i\Gamma_\sigma/2 = 441^{+16}_{-8} - i272^{+9}_{-12}$MeV. Clearly, these numbers represent the value for $N_c = 3$, but large $N_c$ counting requires that for mesons $m_\sigma \sim N^0_c$ and $\Gamma_\sigma \sim 1/N_c$. In this regard the large $N_c$ analysis may provide a clue of what value should be taken for the $\sigma$ mass.

Actually large $N_c$ studies in $\pi\pi$ scattering based on scaling and unitarization with the Inverse Amplitude Method (IAM) of ChPT amplitudes provide results which in the scalar meson case depend on details of the scheme used. While the one loop coupled channel approach [28] yields any possible $m_\sigma$ and a large width (in apparent contradiction with standard large $N_c$ counting), the two loop approach [29], yields a large mass shift (a factor of 2) for the scalar meson when going from $N_c = 3$ to $N_c = \infty$ having $m_\sigma \to 900$MeV, but a small shift in the case of the $\rho$ meson. However the large uncertainties that appear in the two loop IAM method [221] is something to keep in mind. Furthermore, although results

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6For instance, in the very successful Charge Dependent (CD) Bonn potential [55] any partial wave $2S+1L_J$-channel is fitted with a different scalar meson mass and coupling.

7From one side, the formerly fictitious $\sigma$-meson has become a real and well determined lowest resonance of the QCD spectrum since then, but the problem of its nature has been a headache for theoretical physicist in the last few years. The problem is far from being over. For example, structures of the type tetra-quark or glueball, etc, have been proposed in Ref. [45]. Lattice QCD calculations for the scalar [47, 48] have found that the mass of the lightest $0^{++}$ meson is suppressed relative to the mass of the $0^{++}$ glueball in quenched QCD at an equivalent lattice spacing [46].
of Ref. [230] confirm and explain the different behavior, in agreement with Refs. [228, 229], of the ρ and σ for $N_c$ not far from 3, when extending the analysis to the large $N_c$ limit, no robust conclusions of the σ pole behavior can be inferred from the one loop IAM only. This is due to a large cancellation existing in the combination of LEC’s that govern the scalar channel.

If we make use of the large $N_c$ expansion according to the standard assumption ($M^{(k)} \sim N_c^{-k}$)

$$M_\sigma = M_\sigma^{(0)} + M_\sigma^{(1)} + \mathcal{O}(N_c^{-2}),$$

$$\Gamma_\sigma = \Gamma_\sigma^{(1)} + \mathcal{O}(N_c^{-2}),$$

the pole contribution becomes

$$D_\sigma(r) = \frac{e^{-m_\sigma r}}{4\pi r} + \mathcal{O}(N_c^{-2}),$$

where $m_\sigma = M_\sigma^{(0)} + M_\sigma^{(1)}$, representing the resonance mass to NLO in the $1/N_c$ expansion, should be used. Note that the width does not contribute to this order. Thus, for all purposes we may use a Yukawa potential to represent the exchange of a wide resonance. However, what is the numerical value of this $m_\sigma$ one should use for the NN problem?

According to Ref. [231] the effect of a meson width in the Yukawa-like potential is

$$V(r) = -\frac{g^2}{4\pi} \left(1 - \frac{\Gamma_\sigma}{m_\sigma \pi}\right) e^{-(m_\sigma + \Gamma_\sigma/\pi)r},$$

which corresponds to a NLO large $N_c$ renormalization of the mass and coupling providing a $\mathcal{O}(N_c^0)$ correction to the central potential. This analysis is based on separating the integrand into different intervals which become dominant at large distances. Our analysis separates first the pole contribution form the background and then studies each contribution separately.

As it is shown in Ref. [232] (see Sec. G.3 in Appendix G for a summary) the large $N_c$-NLO pole contribution could be replaced by the equivalent Breit-Wigner resonance mass to the same accuracy, since we have,

$$\delta_{00}(m_\sigma^2) = \frac{\pi}{2} + \mathcal{O}(1/N_c^3)$$

(4.80)

Thus, at LO and NLO in the large $N_c$ limit the exchange of a resonance between nucleons can be represented at long distances as a Yukawa potential with the Breit-Wigner mass to $\mathcal{O}(N_c^{-2})$. The vertex correction $\sigma_{\pi N}$, e.g. in Eq. (4.56), just adds a coupling constant yielding

$$V_\sigma(r) = -\frac{g^2_{\pi NN}}{4\pi} \frac{e^{-m_\sigma r}}{r} + \mathcal{O}(1/N_c).$$

(4.81)

Of course, the same type of arguments apply to the ρ-meson exchange, with the only modification

$$\delta_{11}(m_\rho^2) = \frac{\pi}{2} + \mathcal{O}(1/N_c^3)$$

(4.82)

where now $m_\rho = M_\rho^{(0)} + M_\rho^{(1)}$. In Fig. 4.5 we show the data for $\pi\pi$ phase shifts, where we see that the true Breit-Wigner masses or not very different. Of course, these arguments do not imply that the Yukawa masses should exactly coincide, but at least suggest that one should expect a large shift for the
σ mass from the pole position and a very small one for the ρ meson mass when the next-to-leading $1/N_c$ correction to the pole masses are considered. The identity of scalar and vector masses has been deduced from several scenarios based on algebraic chiral symmetry [216, 233]. Actually, it has been shown that $m_\rho = m_\sigma$ without appealing to the strict large $N_c$ limit but assuming the narrow resonance approximation. Moreover, in Ref. [230] the Inverse Amplitude Method (IAM) is used to unitarize the one-loop $SU(2)$ ChPT amplitude in $\pi\pi$ scattering. They found, by requiring consistence between resonance saturation and unitarization, a scenario where the masses of the $\sigma$ and $\rho$ become degenerate in the large-$N_c$ limit. In fact, the two-loop calculation of Ref. [229] supports this result.

\section*{4.6 Conclusions}

We have analyzed what we have called Serber symmetry from the viewpoint of long distance symmetries, a concept which has proven useful in the study of Wigner $SU(4)$ spin-flavour symmetry in the last chapter. This symmetry comes out from the old, and well-known in nuclear physics, Serber force, which predicts a vanishing interactions in odd-$L$ partial waves. Serber symmetry is clearly seen in the np differential cross section and implies a set of sum rules for the partial wave phase shifts which are well verified to a few percent level in the entire elastic region.

We have formulated these sum rules at the level of high quality potentials, i.e. potentials which describe elastic NN scattering with $\chi^2/DOF \sim 1$ which are also well verified at distances above 1fm. This suggests that a coarse graining of the NN interaction might also display the symmetry. The equivalent momentum space Wilsonian viewpoint is implemented explicitly by the $V_{lowk}$ approach by integrating all modes below a certain cut-off $\Lambda \sim 400$ MeV. By analyzing existing $V_{lowk}$ calculations for high quality potentials we have shown that Serber symmetry is indeed fulfilled to a high degree. A surprising finding has been that chiral potentials, while implementing important QCD features, do not fulfill the symmetry to the same degree as current high quality potentials. This effect must necessarily be compensated by similar symmetry violations in the counterterms encoding the non-chiral and unknown short distance interaction and needed to describe NN phase shifts where the symmetry does indeed happen.

Inspired by our previous findings in describing Wigner $SU(4)$ symmetry from a large $N_c$ perspective, an attempt to provide a more fundamental understanding of the so far accidental Serber symmetry has
been carried out. As we saw in Chapter 3 large $N_c$ predicts the NN channels where Wigner symmetry works and fails phenomenologically. Actually we found that when Wigner symmetry fails, as allowed by large $N_c$ considerations, Serber symmetry holds instead. In this chapter we have verified this at the level of potentials at large distances or using $V_{\text{lowk}}$ potentials. It is obvious then wondering whether Serber symmetry could be justified from a large $N_c$ viewpoint.

In practical terms we have shown that within a One Boson Exchange framework, the fulfillment of the symmetry at the potential level is closely related to having similar values of $\sigma$ and $\rho$ meson masses as they appear in Yukawa potentials. These mesons are associated to resonances in $\pi\pi$ scattering and can be uniquely defined as poles in the second Riemann sheet of the scattering amplitude at the invariant mass $\sqrt{s} = M_R - i\Gamma_R/2$. However, assuming the standard mass $m_R \sim N_c^0$ and width $\Gamma_R \sim 1/N_c$ scaling, we have found that, provided we keep terms in the potential to NLO, meson widths do not contribute to the NN potential, as they are $O(N_c^{-1})$, i.e. a relative $1/N_c^2$ correction to the LO contribution. This justifies using a Yukawa potential where the mass corresponds to an approximation to the pole mass $m_R = M_R^{(0)} + M_R^{(1)}$ which cannot be distinguished from the Breit-Wigner mass up to $O(N_c^{-2})$. This suggests that the masses $m_\sigma$ and $m_\rho$ which appear in the OBE potential could be interpreted as an approximation to the pole mass rather than its exact value. The question on what numerical value should be used for the Yukawa mass is a difficult one, and at present we know of no other direct way than NN scattering fits for which $m_\sigma = 501(25)$MeV might be acceptable when the uncorrelated $2\pi$ contribution is disregarded.

We point out that the $V_{\text{lowk}}$ interpretation unveils important symmetries of the effective NN interaction relevant to Nuclear Structure. Given a symmetry group with a generic element $G$, a standard symmetry means that $[V,G] = 0$ implies $[V_{\text{lowk}},G] = 0$. The reverse, however, is not true. We might define a long distance symmetry as a symmetry of the effective interaction, i.e. $[V_{\text{lowk}},G] = 0$ but $[V,G] \neq 0$. From a renormalization viewpoint that corresponds to a symmetry of the potential broken only by counter-terms. The non-trivial fact that the Wigner and Serber symmetries occur unequivocally for $V_{\text{lowk}}$ and not so much for the bare $V$ reinforces the $V_{\text{lowk}}$-approach as an efficient and symmetry-based coarse grained interaction. Once the symmetries emerge, simplifications are expected, and this affects the very definition of effective interactions. Recently [241] it has been found that these old symmetries are saturated at rather low scales.
Chapter 5

Renormalization of One Boson Exchange potentials

5.1 Introduction

The OBE potential has been a cornerstone for Nuclear Physics during many years. It represents the natural generalization of the OPE potential proposed by Yukawa \[4\] with inclusion of the scalar and vector mesons as well. In general, the field theoretical OBE model of the NN interaction \[53, 54\] includes all mesons with masses below the nucleon mass, i.e., \(\pi, \eta, \rho(770)\) and \(\omega(782)\). These models have been traditionally used to describe elastic NN scattering. In such a regime relative NN de Broglie wavelengths larger than half a fermi are probed; a factor of two bigger scale than the Compton wavelengths of the vector and heavier mesons. However, while from this simple minded argument we might expect those mesons to play a marginal role, OBE potentials have traditionally been sensitive to short distances requiring a unnatural fine-tuning of the vector meson coupling. As a consequence there has been some inconsistency between the couplings required from meson physics, \(SU(3)\) or chiral symmetry on the one hand and those from NN scattering fits on the other hand (see also \[59–61\]). Part of the disagreement could only be overcome after even shorter scales were explicitly considered \[62, 63\].

Furthermore, OBE models have also included strong form factors to incorporate finite nucleon size using parameterizations which are loosely related to the field theoretical meson-baryon Lagrangian from which the meson exchange picture is derived. In fact, the use of strong form factor is twofold. As we will see the meson-nucleon Lagrangian itself while providing the NN OBE potential from the Born approximation, \textit{does not predict} the NN S-matrix and the deuteron unambiguously beyond perturbation theory from the OBE potential. This can be seen as due to singularities appearing at short distances which must be overcome with the introduction of new information which is not implicit in the Lagrangean itself. The traditional philosophy has been the introduction of vertex form factors \textit{ad hoc}.

Because of their fundamental character and the crucial role played in NN calculations there have been countless attempts to evaluate strong form factors by several means, mainly \(\Gamma_{\pi NN}(q^2)\). These include meson theory \[242–247\], Regge models \[248\], chiral soliton models \[249–252\], QCD sum rules \[253\] the Goldberger-Treimann discrepancy \[254\] or lattice QCD \[255, 256\] and quark models \[257\] (see also the discussion in Chapter 2). Most calculations yield rather small values \(\Lambda_{\pi NN} \sim 800\,\text{MeV}\) generating the
soft form factor puzzle for the OBE potential for several years since the cut-off could not be lowered below \( \Lambda_{\pi NN} = 1.3\text{GeV} \) without destroying the quality of the fits and the description of the deuteron \([53]\). The contradiction was solved by including either \( \rho \pi \) exchange \([258]\), a strongly coupled excited \( \pi'(1300) \) state \([259]\), two pion exchange \([260]\) or three pion exchange \([261]\). Some of these ways out of the paradox assume the meson exchange picture seriously to extremely short distances. However, as noted in Ref. \([252]\) the contradiction is misleading since a large cut-off is needed just to avoid a sizable distortion of the OBE potential in the region \( r > 0.5\text{fm} \) which can also be achieved by choosing a suitable shape of the form factor. This point was explicitly illustrated by using the Skyrme soliton model form factors \([252]\). In fact, this conclusion is coherent with the early hard core regularizations \([28, 34, 35]\), recent lattice calculations \([256]\) (where an extremely hard \( \Lambda_{\pi NN} \sim 1.7\text{GeV} \) and a rather flat behaviour are found) and, as we will show, with the renormalization approach we advocate.

In this chapter we propose another manner of dealing with the problem which consists on a decoupling of short distance components of the interaction at the energies involved in NN elastic scattering by using a suitable set of renormalization conditions. This, of course, does not mean that strong form factors have not a physical meaning but rather that a suitable set of renormalization conditions minimizes their impact in the NN problem because they can implement a desirable short-distance insensitivity.

In order to facilitate and simplify the analysis we will use large \( N_c \) relations for meson-nucleon couplings \([137]\) which are well satisfied phenomenologically and pick the leading tensorial structures for the OBE potential. The OBE component is dominated by the \( \pi, \sigma, \rho, \omega \) and \( a_1 \) mesons \([140, 142]\). Further motivation for using this large \( N_c \) approximation have been stressed in previous chapters in regard to Wigner and Serber symmetries, in particular the fact that relativistic, spin-orbit and meson widths corrections are suppressed by a relative \( 1/N_c^2 \) factor suggesting a bold 10% accuracy. However, we hasten to emphasize that despite the use of this appealing and simplifying approximation in the OBE potential we do not claim to undertake a complete large \( N_c \) calculation since multiple meson exchanges and \( \Delta \) intermediate states should also be implemented \([142]\). In spite of this, some of our results fit naturally well within naive expectations of the large \( N_c \) approach. The coordinate space renormalization scheme is particularly suited within the large \( N_c \) framework, where non-localities in the potential are manifestly suppressed, and an internally consistent multi-meson exchange scheme is possible if energy independent potentials are used \([262, 263]\).

### 5.2 OBE potentials and the need for renormalization

#### 5.2.1 OBE potentials at leading \( N_c \)

The well known process of deriving the OBE potential from a nucleon-meson Lagrangian is done in details in Appendix B. Our potential starts from a chiral Lagrangian as done in Refs. \([59–61]\) where the
relevant nucleon-meson interactions are,

\[ L_{\sigma NN} = -g_{\sigma NN} \bar{N} \sigma N, \]  
(5.1)

\[ L_{\pi NN} = -\frac{g_{\pi NN}}{2 \Lambda_N} \bar{N} \gamma_5 \gamma_\mu \tau \cdot \partial^\mu \pi N, \]  
(5.2)

\[ L_{\omega NN} = -g_{\omega NN} \bar{N} \gamma_\mu \omega^\mu N - \frac{f_{\omega NN}}{2 \Lambda_N} \bar{N} \sigma_{\mu \nu} \partial^\mu \omega^\nu N, \]  
(5.3)

\[ L_{\rho NN} = -g_{\rho NN} \bar{N} \gamma_\mu \rho^\mu N - \frac{f_{\rho NN}}{2 \Lambda_N} \bar{N} \sigma_{\mu \nu} \tau \cdot \partial^\mu \rho^\nu N, \]  
(5.4)

Here, \( \Lambda_N \) is a mass scale which we take as \( \Lambda_N = 3M_N/N_c \) with \( N_c \) the number of colours in QCD and an overview of estimates of couplings from several sources can be found in Sec. B.5 of Appendix B. Now, in the large \( N_c \) limit the Lagrangian simplifies tremendously since one has the following scaling relations [137] 1

\[
M_N \sim N_c, \quad \Lambda_N \sim N_c^0, \\
g_{\pi NN} \sim g_{\sigma NN} \sim g_{\omega NN} \sim f_{\rho NN} \sim \sqrt{N_c}, \\
f_{\omega NN} \sim g_{\rho NN} \sim 1/\sqrt{N_c}, \\
m_\pi \sim m_\sigma \sim m_\rho \sim m_\omega \sim N_c^0, \\
\Gamma_\sigma \sim \Gamma_\rho \sim 1/N_c. 
\]  
(5.5)

In the large \( N_c \) limit it is convenient to use a heavy baryon formalism. Thus, the LO meson-nucleon Lagrangian in the \( 1/N_c \) expansion becomes (see Sec. A.4 Appendix A),

\[ \mathcal{L} = -g_{\pi NN} s B^\dagger B + g_{\omega NN} s_0 B^\dagger B \\
+ g_{\pi NN} B^\dagger \sigma \tau_a B \partial^\mu \phi^a + \frac{f_{\omega NN}}{2 \Lambda_N} s^{ijk} B^\dagger \sigma \tau_a B \partial^\mu \rho^{ka}. \]  
(5.6)

From the heavy-baryon Lagrangian, Eq. (5.6), the calculation of the NN potential in momentum space is straightforward and the corresponding coordinate space potential is given by Fourier transform (see Appendix B). According to their increasing mass the leading \( N_c \) contributions to the OBE potentials read

\[ V_\pi(r) = \frac{1}{12} g_{\pi NN}^2 m_\pi^2 \left[ Y(m_\pi r) \sigma_1 \cdot \sigma_2 + T(m_\pi r) S_{12}(\vec{r}) \right] \tau_1 \cdot \tau_2, \]  
(5.7)

\[ V_\sigma(r) = -\frac{g_{\sigma NN}^2}{4\pi} m_\sigma Y(m_\sigma r), \]  
(5.8)

\[ V_\rho(r) = \frac{1}{12} f_{\rho NN}^2 m_\rho^3 \left[ 2 Y(m_\rho r) \sigma_1 \cdot \sigma_2 - T(m_\rho r) S_{12}(\vec{r}) \right] \tau_1 \cdot \tau_2, \]  
(5.9)

\[ V_\omega(r) = \frac{g_{\omega NN}^2}{4\pi} m_\omega Y(m_\omega r), \]  
(5.10)

\[ 1 \] There should be no confusion in forthcoming sections when we take \( N_c = 3 \) and \( \Lambda_N = M_N \).
where we have defined,

$$
Y(x) = \frac{e^{-x}}{x}, \quad (5.11)
$$

$$
T(x) = \left(1 + \frac{3}{x} + \frac{3}{x^2}\right)Y(x), \quad (5.12)
$$

$$
S_{12}(\hat{r}) = 3\sigma_1 \cdot \hat{r} \sigma_2 \cdot \hat{r} - \sigma_1 \cdot \sigma_2. \quad (5.13)
$$

This way, the structure of the leading large $N_c$-OBE potential is compatible with the spin-flavor structure found in Refs. [140, 141],

$$
V_{NN}(r) = V_C(r) + \tau_1 \cdot \tau_2 [W_S(r) \sigma_1 \cdot \sigma_2 + W_T(r)S_{12}], \quad (5.14)
$$

with the only non-vanishing components,

$$
V_C(r) = -\frac{g_{\pi NN}^2}{4\pi} e^{-m_\rho r} + \frac{g_{\pi NN}^2}{4\pi} \frac{e^{-m_\rho r}}{r}, \quad (5.15)
$$

$$
W_S(r) = \frac{1}{12} \frac{g_{\pi NN}^2 m_\pi^2}{4\pi \Lambda_N^2} \frac{e^{-m_\rho r}}{r} + \frac{1}{6} \frac{f_{\rho NN}^2 m_\rho^2}{4\pi \Lambda_N^2} \frac{e^{-m_\rho r}}{r}, \quad (5.16)
$$

$$
W_T(r) = \frac{1}{12} \frac{g_{\pi NN}^2 m_\pi^2}{4\pi \Lambda_N^2} \frac{e^{-m_\rho r}}{r} \left[1 + \frac{3}{m_\rho r} + \frac{3}{(m_\rho r)^2}\right] - \frac{1}{12} \frac{f_{\rho NN}^2 m_\rho^2}{4\pi \Lambda_N^2} \frac{e^{-m_\rho r}}{r} \left[1 + \frac{3}{m_\rho r} + \frac{3}{(m_\rho r)^2}\right]. \quad (5.17)
$$

At short distances we have

$$
V_C(r) \to \frac{g_{\pi NN}^2}{4\pi} e^{-1/r}, \quad (5.18)
$$

$$
W_S(r) \to \frac{1}{12} \frac{g_{\pi NN}^2 m_\pi^2}{4\pi \Lambda_N^2} \frac{1}{r}, \quad (5.19)
$$

$$
W_T(r) \to \frac{1}{4} \frac{g_{\pi NN}^2}{4\pi \Lambda_N^2} \frac{1}{r^3}. \quad (5.20)
$$

As we see, the potential is singular at short distances except for the very special value $f_{\rho NN} = g_{\pi NN} \frac{e}{2}$. While the central $V_C$ and spin $W_S$ contributions present a mild Coulomb singularity, the tensor force component $W_T$ develops a more serious type of singularity, a situation that appeared already for the simpler OPE potential [126].

### 5.2.2 The OBE potential and ambiguities in the S-matrix

Now we will show that in the non-exceptional situation $g_{\pi NN} \neq f_{\rho NN}$ the S-matrix associated to the OBE potential is necessarily ambiguous, precisely because of the short distance $1/r^3$ singularity. In other words, the standard regularity conditions for the wave function do not uniquely determine the solution of the Schrödinger equation.

At short distances, i.e. much smaller than meson masses, $r \ll 1/m$, the NN problem due to the OBE potential corresponds to solving the following Schrödinger equation,

$$
-\nabla^2 \Psi_k(r) + U_S(r) \Psi_k(r) = p^2 \Psi_k(r), \quad r \ll 1/m, \quad (5.21)
$$

2Actually quite often the $\rho$-exchange is used as a regulator of the $\pi$-exchange. The exceptional case, $g_{\pi NN} = f_{\rho NN}$, is not far from phenomenological values and for that reason will be treated latter.
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where the reduced short distance potential is given by

\[ U_S(r) = MV_S(r) = \pm \frac{R}{r^2} (3\hat{\sigma}_1 \cdot \hat{\sigma}_2 \cdot \hat{r} - \sigma_1 \cdot \sigma_2) , \quad (5.22) \]

with \( R \) a length scale that in our particular case is,

\[ \pm R = \frac{M}{16\pi \Lambda_N^2} (g_{\pi NN}^2 - f_{\rho NN}^2) , \quad (5.23) \]

with the positive or negative sign depending on whether \( g_{\pi NN} > f_{\rho NN} \) or \( g_{\pi NN} < f_{\rho NN} \) respectively.

In the basis with good total angular momentum \( J \) the states are labeled by the spectroscopic notation \( 2S+1L_J \). For spin singlet states \( S = 0 \) and \( J = L \), the parity is natural \( U_P = (-1)^J \) and one has \( S_{12} = 0 \). For un-coupled spin triplet states \( S = 1 \) one has \( J = L \), natural parity \( U_P = (-1)^L \) and \( S_{12} = 2 \). For coupled spin triplet states \( S = 1 \) one has \( L = J \pm 1/2 \), unnatural parity \( U_P = (-1)^{J+1} \) and

\[ S_{12} = \left( \frac{-2(J-1)}{2J+1} \frac{6\sqrt{J(J+1)}}{2J+1} \right) \quad (5.24) \]

For the un-coupled spin-triplet channel we have

\[ -v''_J(r) + \left[ \pm \frac{2R}{r^2} + \frac{J(J+1)}{r^2} \right] v_J(r) = p^2 v_J(r) . \quad (5.25) \]

At very short distances we may neglect the centrifugal barrier and the energy yielding

\[ -v''_J(r) \pm \frac{2R}{r^2} v_J(r) = 0, \quad r \ll 1/m, R, 1/p . \quad (5.26) \]

The general solution can be written in terms of Bessel functions. Using their asymptotic expansions we may write at short distances \(^3\)

\[ v_{+J}(r) \rightarrow \left( \frac{r}{R} \right)^{3/4} \left[ C_{1R} e^{+i\sqrt{\pi} r} + C_{2R} e^{-i\sqrt{\pi} r} \right] , \quad (5.27) \]

\[ v_{-J}(r) \rightarrow \left( \frac{r}{R} \right)^{3/4} \left[ C_{1A} e^{-i\sqrt{\pi} r} + C_{2A} e^{+i\sqrt{\pi} r} \right] . \]

Clearly, in the repulsive case the regularity condition fixes the coefficient of the diverging exponential to zero, \( C_{1R} = 0 \), whereas in the attractive case both linearly independent solutions are regular and the solution is not unique.

\(^{3}\)The solutions of \(-y''(x) - y(x)/x^3 = 0\) are

\[ \sqrt{x} J_1(2/\sqrt{x}) = -\frac{x^{\frac{1}{4}}}{\sqrt{\pi}} \cos(\pi/4 + 2/\sqrt{x}) + \ldots \]

\[ \sqrt{x} Y_1(2/\sqrt{x}) = -\frac{x^{\frac{1}{4}}}{\sqrt{\pi}} \cos(\pi/4 - 2/\sqrt{x}) + \ldots \] \[ \sqrt{x} K_1(2/\sqrt{x}) = \frac{1}{\sqrt{\pi}} x^{-\frac{1}{4}} e^{-2/\sqrt{x}} + \ldots \]

\[ \sqrt{x} I_1(2/\sqrt{x}) = \frac{1}{2\sqrt{\pi}} x^{\frac{1}{4}} e^{2/\sqrt{x}} + \ldots \]
neglecting centrifugal barrier and energy, the system of two coupled differential equations becomes

\[
\begin{pmatrix}
-u''_J(r) \\
-w''_J(r)
\end{pmatrix} \pm \frac{R}{r^3} \begin{pmatrix}
-\frac{2(J-1)}{2J+1} & \frac{6\sqrt{J(J+1)}}{2J+1} \\
\frac{6\sqrt{J(J+1)}}{2J+1} & -\frac{2(J+2)}{2J+1}
\end{pmatrix} \begin{pmatrix}
u_J(r) \\
w_J(r)
\end{pmatrix} = 0.
\] (5.28)

This system can be diagonalized by going to the rotated basis

\[
\begin{pmatrix}
v_1, J(r) \\
v_2, J(r)
\end{pmatrix} = \begin{pmatrix}
\sqrt{\frac{J}{2J+1}} & -\sqrt{\frac{J+1}{2J+1}} \\
\sqrt{\frac{J+1}{2J+1}} & \sqrt{\frac{J}{2J+1}}
\end{pmatrix} \begin{pmatrix}
u_J(r) \\
w_J(r)
\end{pmatrix},
\] (5.29)

where the new functions satisfy

\[
-v''_{1, J}(r) \pm \frac{4R}{r^3} v_{1, J}(r) = 0,
\] (5.30)

\[
-v''_{2, J}(r) \pm \frac{8R}{r^3} v_{2, J}(r) = 0.
\] (5.31)

Note that here the signs are alternate, i.e. when one of the short-distance eigenpotentials is attractive the other one is repulsive and vice-versa, and hence the type of solutions in Eq. (5.27) can be applied. This means that in general there will be solutions which are not necessarily fixed by the regularity condition at the origin, and thus the OBE potential does not predict the $S$–matrix uniquely. As a consequence some additional information should be given. The traditional way is to introduce form factors which besides implementing the finite nucleon size have the additional benefit of killing the singularity so that the regularity condition fixes the solution uniquely. Another way, which is more model independent, is to fix directly the integration constants from data with or without form factors. As we will show this new way of proceeding does not make much difference showing a marginal influence of form factors and hence reducing their impact.

5.3 The standard approach with vertex form factors

5.3.1 Features of Vertex Functions

A physically motivated way out to avoid the singularities is to implement vertex functions in the OBE potentials corresponding to the replacement (here $q^2 = q^2_0 - q^2$ is the 4-momentum transfer)

\[
V_{mNN}(q) \rightarrow V_{mNN}(q) \left[ \Gamma_{mNN}(q^2) \right]^2.
\] (5.32)

Note that this assumes i) off-shell independence and ii) that the form factor is accurately known. Standard choices are to take form factors of the monopole \([53]\) and exponential \([145]\) parameterizations

\[
\Gamma_{mNN}^{\text{mon}}(q^2) = \frac{\Lambda^2 - m^2}{\Lambda^2 - q^2},
\] (5.33)

\[
\Gamma_{mNN}^{\text{exp}}(q^2) = \exp \left[ \frac{q^2 - m^2}{\Lambda^2} \right],
\] (5.34)

\footnote{Note that the exponential form was already studied in Chapter 2.}
fulfilling the normalization condition \( \Gamma_{mNN}(m^2) = 1 \). These forms are so constructed as to have the same slope at small values of \( q^2 \) in the large cut-off expansion
\[
\Gamma_{mNN}(q^2) = 1 + \frac{q^2 - m^2}{\Lambda^2} + \mathcal{O}(\Lambda^{-4}).
\]
so that the meaning for the cut-off is similar at low energies. In coordinate space we have to Fourier transform with the corresponding vertex function. For the leading \( N_c \) OBE potential, the following Fourier transforms appear,
\[
\bar{O}_1(r,m,\Lambda) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2 + m^2} \left[ \Gamma_{mNN}(q^2) \right]^2 e^{+iqr},
\]
\[
\bar{O}_2(r,m,\Lambda) = \int \frac{d^3q}{(2\pi)^3} \frac{\sigma_1 \cdot q(\sigma_2 \cdot q)}{q^2 + m^2} \left[ \Gamma_{mNN}(q^2) \right]^2 e^{+iqr},
\]
\[
\bar{O}_3(r,m,\Lambda) = \int \frac{d^3q}{(2\pi)^3} \frac{q^2}{q^2 + m^2} \left[ \Gamma_{mNN}(q^2) \right]^2 e^{+iqr}.
\]

- **Monopole form factor**

In general we can write the above Fourier transforms as,
\[
\bar{O}(r,m,\Lambda) = \int \frac{d^3q}{(2\pi)^3} O(q,m) \left( \frac{\Lambda^2 - m^2}{\Lambda^2 - q^2} \right)^2 e^{+iqr}
\]
\[
= -\frac{(\Lambda^2 - m^2)^2}{2\Lambda} \frac{d}{d\Lambda} \left\{ \frac{1}{\Lambda^2 - m^2} \left[ O(r,m) - O(r,\Lambda) \right] \right\},
\]
where we have defined,
\[
O(r,m) = \int \frac{d^3q}{(2\pi)^3} O(q,m) e^{+iqr}.
\]
The result is,
\[
\bar{O}^{\text{mon}}_1(r,m,\Lambda) = e^{+mr} - \frac{e^{-\Lambda r}}{4\pi r} \left[ 1 + \frac{\Lambda^2 - m^2}{2\Lambda} \right],
\]
\[
\bar{O}^{\text{mon}}_2(r,m,\Lambda) = \frac{m^2}{12\pi} \frac{e^{-mr}}{r} \left[ \sigma_1 \cdot \sigma_2 + \left( 1 + \frac{3}{mr} + \frac{3}{(mr)^2} \right) S_{12}(\hat{r}) \right]
\]
\[
+ \frac{e^{-\Lambda r}}{24\pi r^3} \left[ S_{12}(\hat{r}) \left( 6 - (mr)^2 (1 + r\Lambda) + r\Lambda(6 + r\Lambda(3 + r\Lambda)) \right) \right.
\]
\[
\left. + \sigma_1 \cdot \sigma_2 \left( r\Lambda^3 + (rm)^3 (2 - r\Lambda) \right) \right],
\]
\[
\bar{O}^{\text{mon}}_3(r,m,\Lambda) = \frac{m^2}{4\pi} \frac{e^{-mr}}{r} + \frac{e^{-\Lambda r}}{4\pi r} \left( \frac{r\Lambda^3}{2} + m^2 - \frac{r\Lambda}{2} m^2 \right).
\]
The regularized potentials become finite at short distances, i.e., \( r\Lambda \ll 1 \). For instance, in the case of the regularized Yukawa we obtain,
\[
\bar{O}^{\text{mon}}_1(r,m,\Lambda) = \frac{(m - \Lambda)^2}{8\pi \Lambda} + \mathcal{O}(r^2).
\]
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- **Exponential form factor**

The exponential form factor was used in Chapter 2 to regularize the $1/r^6$ divergence at short distances. There, the Fourier transform Eq. (5.37) was written in terms of the radial function,

$$F(r) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2 + m^2} e^{-2q^2/L^2} e^{iqr}$$

$$= \frac{e^{-mr}}{8\pi r} \left\{ 1 + \text{Erf} \left( \frac{r\Lambda^2 - 4m}{2\sqrt{2}\Lambda} \right) - e^{2mr} \text{Erfc} \left( \frac{r\Lambda^2 + 4m}{2\sqrt{2}\Lambda} \right) \right\} e^{2m^2/L^2},$$

(5.45)

and its derivatives, where Erf and Erfc are the error and complementary error function respectively. Using this result it is straightforward to arrive at,

$$\tilde{O}_1^{\exp}(r, m, \Lambda) = \frac{e^{-mr}}{8\pi r} \left\{ 1 + \text{Erf} \left( \frac{r\Lambda^2 - 4m}{2\sqrt{2}\Lambda} \right) - e^{2mr} \text{Erfc} \left( \frac{r\Lambda^2 + 4m}{2\sqrt{2}\Lambda} \right) \right\},$$

(5.46)

$$\tilde{O}_2^{\exp}(r, m, \Lambda) = \frac{e^{-2m^2/L^2}}{3} \left[ \left( \frac{F'(r)}{r} - F''(r) \right) S_{12}(r) - \left( \frac{2}{r} F'(r) + F''(r) \right) \sigma_1 \cdot \sigma_2 \right],$$

(5.47)

$$\tilde{O}_3^{\exp}(r, m, \Lambda) = e^{-2m^2/L^2} \left( R(r) - m^2 F(r) \right),$$

(5.48)

with $R(r)$ defined as,

$$R(r) = \int \frac{d^3q}{(2\pi)^3} e^{-2q^2/L^2} e^{iqr} = \frac{\Lambda^3}{16\sqrt{2\pi}^3} e^{-\frac{1}{4}r^2\Lambda^2}.$$  

(5.49)

Again at short distances these regularized potentials become finite. For the exponentially regularized Yukawa potential we have the finite result at $\Lambda r \ll 1$,

$$\tilde{O}_1^{\exp}(r, m, \Lambda) = \frac{e^{-2m^2/L^2}}{\sqrt{2\pi}^4} \frac{\Lambda}{2\pi} - m^2 \text{Erfc} \left( \frac{\sqrt{2}m}{\Lambda} \right) + O(r^2),$$

(5.50)

which diverges linearly for $\Lambda \to \infty$. In the limit $\Lambda r \gg 1$ behaves as

$$\tilde{O}_1^{\exp}(r, m, \Lambda) = \frac{e^{-mr}}{4\pi r} - \frac{e^{-\frac{1}{4}r^2\Lambda^2}}{\sqrt{2\pi} \Lambda r^2} e^{-2m^2/L^2} + \ldots$$

(5.51)

and the distortion of the original Yukawa potential is much more suppressed in the exponential than in the case of monopole form factor.

An amazing feature is that the form factors have a radically different effect on different components of the potential. While $V_C$ and $W_S$ with a mild $\sim 1/r$ short distance behaviour become finite, the tensor force behaving as $W_T \sim 1/r^3$ vanishes at the origin after the regularization, $W_T^{\text{mon}}(0) = W_T^{\exp}(0) = 0$. This can be seen from the expression

$$\lim_{r \to 0} \int \frac{d^3q}{(2\pi)^3} e^{iqr} \frac{\sigma_1 \cdot q \sigma_2 \cdot q}{q^2 + m^2} \left[ \Gamma_{mNN}(q^2) \right]^2$$

$$= \frac{1}{3} \sigma_1 \cdot \sigma_2 \left[ \int \frac{d^3q}{(2\pi)^3} \left[ \Gamma_{mNN}(q^2) \right]^2 - m^2 \tilde{O}_1(0, m, \Lambda) \right],$$

(5.52)

They are defined as

$$\text{Erf}(z) = 1 - \text{Erfc}(z) = \frac{2}{\sqrt{\pi}} \int_0^z dt e^{-t^2} = 1 - \frac{e^{-z^2}}{\sqrt{\pi} z} [1 + O(z^{-1})].$$
which corresponds to a vanishing angular average at short distances for the tensor component. This feature suggests that the impact of the tensor force at short distances should be small and looks clearly against the result of the short distance analysis outlined in Section 5.2 where there is a strong mixing at short distances. This will be analyzed in Sec. 5.6 within the renormalization approach.

We show in Fig. 5.1 the potentials $V_C(r)$, $W_S(r)$ and $W_T(r)$ in MeV as a function of the distance (in fm). We also include the effect of both exponential, Eq. (5.34), and monopole Eq. (5.33) form factors for $\Lambda_{\pi NN} = 1.3\text{GeV}$ and $\Lambda_{\pi NN} = 2\text{GeV}$. All other cut-offs are kept to $\Lambda_{\sigma NN} = \Lambda_{\rho NN} = \Lambda_{\omega NN} = 2\text{GeV}$. As we see, the distortion of the tensor component due to the strong form factor takes place already at $r \sim 1\text{fm}$ for softest cut-off $\Lambda_{\pi NN} = 1.3\text{GeV}$. The key issue here is to decide to what extent and for what distances this distortion faithfully represents the true physical effect due to the finite nucleon size. Obviously arises the question of weather one can visualize finite nucleon size effects when the probing wavelength is not shorter than $0.5\text{fm} \leq r \leq 1\text{fm}$.  

5.3.2 Short distance sensitivity vs spurious bound states  

The advantage of using vertex functions is that they make the OBE non-singular at short distances. As a consequence, the choice of the regular solution determines the solution of the scattering problem uniquely. However, as we will see if one use form factors as customarily employed in NN calculations based on the OBE potential important and unavoidable consequences arise. We will see that for natural choices of meson-nucleon parameters (see Sec. B.5 in Appendix B), the NN potential displays short distance insensitivity and at the same time spurious deeply bound states. However, if we insist on not having spurious bound states the resulting description is highly short distance sensitive. In fact as we have mentioned, NN scattering in the elastic region below pion production threshold involves c.m. momenta $p < p_{\text{max}} = 400\text{ MeV}$ and given the fact that $1/m_\omega \sim 1/m_\rho \sim 0.25\text{fm} \ll 1/p_{\text{max}} = 0.5\text{fm}$ we should expect heavier mesons to be irrelevant, and $\omega$ and $\rho$ to be marginally important, even in s-waves, which are most sensitive to short distances. This property has not been fulfilled in the traditional approach to OBE potentials.
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Figure 5.2: The \(^1S_0\) potential \(V_{1S_0}(r)\) in MeV as a function of the distance (in fm) for the different scenarios with large and small \(\omega\)–couplings. We include the effect of both exponential, Eq. (5.34), and monopole Eq. (5.33) form factors for \(\Lambda_{\pi NN} = 1.3\text{GeV}\). All other cut-offs are kept to \(\Lambda_{\sigma NN} = \Lambda_{\rho NN} = \Lambda_{\omega NN} = 2\text{GeV}\).

In order to illustrate this, we consider the \(^1S_0\) channel, where the potential (without form factor) is

\[
V_{1S_0}(r) = V_C(r) - 3W_\Sigma(r)
= -\frac{g_\pi^2NN m_\pi^2}{16\pi M_N^2} \frac{e^{-m_\pi r}}{r} - \frac{g_\sigma^2NN}{4\pi} \frac{e^{-m_\sigma r}}{r} + \frac{g_\omega^2NN}{4\pi} \frac{e^{-m_\omega r}}{r} - \frac{f_\rho^2NN m_\rho^2}{8\pi M_N^2} \frac{e^{-m_\rho r}}{r}.\tag{5.53}
\]

We take \(m_\pi = 138\text{MeV}, M_N = 939\text{MeV}, m_\rho = 770\text{MeV}, m_\omega = 783\text{MeV}\) and \(g_\pi^2NN = 13.1\), which seem firmly established, and treat \(m_\sigma, g_\sigma^2NN\) and \(g_\omega^2NN\) and \(f_\rho^2NN\) as fitting parameters just to see the role of vector mesons. Note the redundant combination of coupling constants \(g_\omega^2NN - f_\rho^2NN m_\rho^2/(2M_N^2)\) which appears in the \(^1S_0\) potential when we take \(m_\rho = m_\omega\), a tolerable approximation within the present context. To avoid unnecessary strong correlations we define the effective coupling

\[
g_{\omega NN}^* = \sqrt{g_\omega^2NN - \frac{f_\rho^2NN m_\rho^2}{2M_N^2}}.\tag{5.54}
\]

Natural values for the coupling constants from \(SU(3)\) symmetry or from the radiative decay \(\omega \to e^+ e^-\) are (see Sec. B.5 in Appendix B) \(g_{\omega NN} = 9 - 10.5\) and \(f_{\rho NN} = 14 - 18\) implying \(g_{\omega NN}^* = 0 - 7\). In Fig. 5.2 the potential without and with monopole and exponential vertex functions is depicted for several values of \(g_{\omega NN}^*\). As we see, the differences start below 1fm where the standard short distance repulsive core is achieved by large and unnatural values of \(g_{\omega NN}^*\), and not so much depending on the form factors.

On the other hand, if we use the regularized \(^1S_0\) potential at \(r = 0\) and take the natural values for the coupling constants \(g_{\omega NN} = 9 - 10.5\) and \(f_{\rho NN} = 14 - 18\) the potential at the origin becomes

\[
V_{1S_0}(0) = -(1000 - 3000)\text{MeV}.\tag{5.55}
\]

which is huge and attractive. The number of states is approximately given by the WKB estimate

\[
N_B \sim \frac{1}{\pi} \int_0^\infty \sqrt{-MV_1S_0(r)} dr.\tag{5.56}
\]
which yields numbers around unity. In fact the potential accommodates a deeply bound state, at about

$$E_B = -(500 - 2000) \text{MeV},$$

within the natural parameter range. This state does not exist in nature and should clearly be ruled out from the description on a fundamental level. On the other hand, we do not expect such state to influence the low energy properties below the inelastic pion production threshold $E_{CM} = 175$ MeV in any significant manner.

In the standard approach the scattering phase-shift $\delta_0(p)$ is computed by solving the (s-wave) Schrödinger equation in r-space

$$-u''_p(r) + M_N V(r) u_p(r) = p^2 u_p(r),$$

$$u_p(r) \rightarrow \frac{\sin (pr + \delta_0(p))}{\sin \delta_0(p)},$$

with a regular boundary condition at the origin

$$u_p(0) = 0.$$ 

This boundary condition obviously implies a knowledge of the potential in the whole interaction region, and it is equivalent to solve the Lippmann-Schwinger equation in p-space. In the usual approach $[53, 55]$ everything is obtained from the potential assumed to be valid for $0 \leq r < \infty$. In practice, and as mentioned above, strong form factors are included implementing the finite nucleon size and reducing the short distance repulsion of the potential, but the regular boundary condition is always kept. However, as was pointed out in Eq. (3.25), due to the unnaturally large NN $^1S_0$ scattering length ($\alpha_0 \sim -23\text{fm}$), any change in the potential $V \rightarrow V + \Delta V$ has a dramatic effect on $\alpha_0$, i.e., a quadratic effect in the large $\alpha_0$. This implies that potential parameters must be fine tuned, and in particular the short distance physics.
Note that despite the repulsive short distances repulsion (with or without form factors) in the solution with and a spurious bound state. The net short distance repulsion comes about only in the solution with without vertex functions $V\rightarrow 0$.

One can clearly understand the large values of the $m$ of $g_{\omega NN}$ and $g_{\omega NN}^*$. We use $\Lambda_{NN} = 1.3\text{GeV}$ and $\Lambda_{NN}^* = \Lambda_{NN} = 138.03\text{MeV}$, and fit $m_\rho = 770\text{MeV}$ and fit $m_\rho$, $g_{\rho NN}$ and $g_{\rho NN}^*$. We use $\Lambda_{NN} = 1.3\text{GeV}$ and $\Lambda_{NN}^* = \Lambda_{NN} = 138.03\text{MeV}$. $E_B$ represents the energy of the (spurious) bound state when it does exist.

Table 5.1: Fits to the $^1S_0$ phase shift of the Nijmegen group [57] using the OBE potential without or with strong exponential and monopole form factor. We take $m = 138.03\text{MeV}$, and $g_{\rho NN} = 13.1083$ [264] and $m_\rho = m_\rho = 770\text{MeV}$ and fit $m_\rho$, $g_{\rho NN}$ and $g_{\rho NN}^*$. We use $\Lambda_{NN} = 1.3\text{GeV}$ and $\Lambda_{NN}^* = \Lambda_{NN} = 138.03\text{MeV}$. $E_B$ represents the energy of the (spurious) bound state when it does exist.

To illustrate this we make a fit the $np$ data of Ref. [57]. The results using the OBE potential without or with strong exponential and monopole form factor $^6$ are presented in Table 5.1. In all cases we have at least two possible but mutually incompatible scenarios. An extreme situation corresponds to the case with no form factors $^7$. The small errors should be noted, in harmony with the fine-tuning displayed by Eq. (3.25) the corresponding couplings and scalar mass are determined to high accuracy but turn out to be incompatible. We may regard then these fits, despite their success in describing the data, as unnatural.

The ambiguity in these solutions has to do with the number of bound states allowed by the potential. Actually, this can be seen from Fig. 5.3 where the zero energy wave function is represented. According to the oscillation theorem, the number of interior nodes determines the number of bound states. Thus, the larger values of $g_{\omega NN}^*$ correspond to a situation with no-bound states since $u_0(r)$ does not vanish, whereas for the smaller $g_{\omega NN}^*$ values one has a bound state as $u_0(r)$ has a zero, which energy can be looked up in Table 5.1. Of course, such a bound state does not exist in nature and it is thus spurious.

On the other hand the spurious bound states always take place at more than twice the maximum energy probed in NN scattering, $E_{CM} = 175\text{MeV}$, and we should not expect any big effect from such an state. Note that despite the repulsive $\omega$-vector and attractive $\rho$-tensor couplings, the total potential does not present short distances repulsion (with or without form factors) in the solution with natural couplings and a spurious bound state. The net short distance repulsion comes about only in the solution with unnaturally large coupling (see Fig. 5.2).

From Table 5.1 one can clearly understand the large values of the $g_{\omega NN}$ coupling constant needed in OBE models as compared to those from $SU(3)$ symmetry $g_{\omega NN} \sim 9$ or from the radiative decay $\omega \rightarrow e^+e^-$ yielding $g_{\omega NN} = 10.2(4)$. Using the definition of $g_{\omega NN}^*$, Eq. (5.54), we get for $f_{\rho NN} = 14-18$ large values of $g_{\omega NN} = 20-22$ for the case with no bound state, whereas more natural values $g_{\omega NN} = 8.5-10.5$ are obtained for the case with one (spurious) bound state.

### 5.4 Renormalization with BC and completeness

According to the discussion of Sec. 5.2.2 the short distance $1/r^3$ singularity of the OBE potential makes the solution ambiguous, and thus there is a flagrant need of additional information not encoded in the potential itself. Once one recognizes this need can proceed by choosing suitable linear combinations of independent solutions. To do so we use boundary condition renormalization which has been shown to

$^6$In this particular channel the regularity condition, Eq. (5.60) determines the solution completely since the potential without vertex functions $V_{\rho NN}(r) \sim 1/r$ is not singular at short distances in the sense that $\lim_{r\rightarrow 0}2\rho V(r)r^2 = 0$ [131, 132].

$^7$Strong non-linear and well determined correlations have been found making a standard error analysis inapplicable. In this situation we prefer to quote errors by varying independently the fitting variables $g_{\omega NN}$, $m_\rho$, and $g_{\omega NN}^*$ until $\Delta \chi^2 = 3.53$ as it corresponds to 3 degrees of freedom.
be a useful tool implementing short distance insensitivity. It consists of choosing a physical constant as an independent parameter of the potential which encodes our lack of knowledge at short distances. This parameter just fix the linear combination of solutions at long distances. Orthogonality between different energy solutions is implemented by choosing a common boundary condition at a given renormalization scale \( r_c \). In the case of renormalization with one counterterm that means imposing (see Sec. E.4) \((\tilde{x} = x/r)\),

\[
\lim_{r_c \to 0} L_p(\tilde{x}r_c) = \lim_{r_c \to 0} L_0(\tilde{x}r_c),
\]

(5.61)

where \( L_p(\tilde{x}r_c) \) is defined as \( \partial_r \Psi_p(\tilde{x}r_c) = L_p(\tilde{x}r_c)\Psi_p(\tilde{x}r_c) \), with \( \Psi_p(\tilde{x}r_c) \) the solution of the Schrödinger equation in the outer region \( r > r_c \). For scattering \( \Psi_k(\tilde{x}r_c) \) satisfy

\[
-\frac{1}{M} \nabla^2 \Psi_k(x) + V(x) \Psi_k(x) = E_k \Psi_k(x),
\]

(5.62)

and goes at long distances as,

\[
\Psi_k(x) \to \left[ e^{ikx} + f(k', k) \frac{e^{ikr}}{r} \right] \chi_{s,ms}^{s',ms'},
\]

(5.63)

with \( f(k', k) \) the scattering matrix amplitude. For bound states (like the deuteron) \( \Psi_d(\tilde{x}r_c) \) satisfy,

\[
-\frac{1}{M} \nabla^2 \Psi_d(x) + V(x) \Psi_d(x) = -\frac{\gamma^2}{M} \Psi_d(x),
\]

(5.64)

with the behaviour at long distances,

\[
\Psi_d(x) \to \frac{A_S}{\sqrt{4\pi r}} e^{-\gamma r} \left[ 1 + \frac{\eta}{\sqrt{8}} S_{12} \right] \chi_{np}^{sm},
\]

(5.65)

For example Eq. (5.61) allows us to extract the phase shifts from the zero energy scattering state or the deuteron as a referent state. This process will be clear in subsequent sections.

Assuming the renormalization limit \( r_c \to 0 \) the completeness relation reads

\[
\int \frac{d^3k}{(2\pi)^3} \Psi_k(x) \Psi_k^\dagger(x') + \sum_{E_n < 0} \Psi_n(x) \Psi_n^\dagger(x') = \delta(x - x') \mathbf{1}.
\]

(5.66)

Besides the deuteron, the sum over negative energy states contains most frequently spurious bound states, and for the singular potential such as the one we are dealing here there are infinitely many. They show up as oscillations in the wave function at short distances, and are a consequence of extrapolating the long distance potential to short distances. On the other hand, from the above decomposition one may write a dispersion relation for the scattering amplitude \(^8\) of the form

\[
f(k', k) = f_B(k', k) - \frac{M}{4\pi} \sum_{E_n < 0} \frac{\langle k|V|\Psi_n(\xi)|\Psi_n|V|k \rangle}{E - E_n} - \frac{M}{4\pi} \int \frac{d^3q}{(2\pi)^2} \frac{\langle k'|V|\Psi_q(\xi)|\Psi_q|V|k \rangle}{E - q^2/M},
\]

(5.67)

\(^8\)This is done by using the Lippmann-Schwinger equation in the form \( T = V + GV \) with \( G = (E - H)^{-1} \), and normalization \( \langle k|x \rangle = e^{ikx} \) and \( \langle \Psi_k|x \rangle = \Psi_k(x) \) whence \( f(k', k) = -M/(4\pi)|k'|T(E)|k \).
where \( f_B(\hat{k}', \hat{k}) \) is the Born amplitude. The physical and spurious bound states occur as poles in the scattering matrix at negative energies \( E = E_n \) and the discontinuity cut along the real and positive axis is given by the second term only. Clearly, the influence of these spurious bound states poles is suppressed if their energy \( E_n \ll E_d \). Given the fact that these states do occur in practice it is mandatory to check their precise location to make sure that they do not influence significantly the calculations, or else one should study the dependence of the observables on the short distance cut-off \( r_c \) starting from a situation where it is small but still large enough as to prevent the occurrence of spurious bound states. It should be noted, however, that in no case can the spurious states occur in the first Riemann sheet of the complex energy plane. This restriction complies to causality, and implies in particular the fulfillment of Wigner inequalities as was discussed for the \(^1\)S\(_0\) channel in Ref. [127].

In the next sections we apply this renormalization program to the OBE potential of Sec. 5.2.1. As was done in Chapter 3 we will use the singlet channel to fix the \( \sigma \)-meson parameters. After this is done we can compute the deuteron properties and the \(^3\)S\(_1\) – \(^3\)D\(_1\) phase shifts.

### 5.5 The singlet channel

#### 5.5.1 Equations and boundary conditions

The \(^1\)S\(_0\) wave function in the np c.m. system can be written as

\[
\Psi(\mathbf{x}) = \frac{1}{\sqrt{4\pi r}} u(r) \chi_{np}^{s=0, m_s=0},
\]

(5.68)

with the total spin \( s = 0 \) and \( m_s = 0 \). The function \( u(r) \) is the reduced S-wave function, satisfying

\[
-u_p''(r) + U_{1S_0}(r) u_p(r) = p^2 u_p(r),
\]

(5.69)

where one has

\[
U_{1S_0} = M \left( V_C - 3W_S \right).
\]

(5.70)

At short distances the OBE potential behaves as a Coulomb type interaction,

\[
U_{1S_0}(r) \to \pm \frac{1}{R r},
\]

(5.71)

where

\[
\pm \frac{1}{R} = \frac{M}{4\pi} \left[ 2 g_{\pi NN}^2 - f_{\pi NN}^2 m_\pi^2 - g_{\sigma NN}^2 - f_{\sigma NN}^2 \right].
\]

(5.72)

Here, \( f_{\pi NN} = g_{\pi NN} m_\pi / (2M_N) \). The potential is then regular and the repulsive or attractive character of the interaction depends on a balance among coupling constants. The short distance solution can be

---

\(^9\)Causality violations, i.e. poles in the first Riemann sheet of the complex energy plane are easy to encounter (see e.g. [127]), particularly with energy dependent boundary conditions. A prominent example is an s-wave without potential and having \( u_p'(0)/u_p(0) = \cot \delta(0) = -1/\alpha_0 + r_0 p^2/2 + v_2 p^4 \), which for the \(^1\)S\(_0\)-channel values of parameters \( \alpha_0 = -23.74 \text{fm}, \ r_0 = 2.75 \text{fm} \) and \( v_2 = -0.48 \text{fm}^2 \) yields besides the well-known virtual state in the second Riemann sheet \( E_v = -0.066 \text{MeV} \) a spurious bound state at \( E_B = -18.37 \text{MeV} \) and an unphysical pole at \( E = 128.88 \pm 146.45 \text{MeV} \). However, finite cut-offs and energy dependent independent boundary conditions are guaranteed not to exhibit these problems, while some spurious bound states may be removed.
written as a linear combination of the regular and irregular solutions at the origin of the form,

$$u_p(r) \rightarrow c_1(p) + c_2(p)r/R, \quad (5.73)$$

where in principle the arbitrary constants $c_1(p)$ and $c_2(p)$ depend on energy. To fix the undetermined constants we impose orthogonality for $r > r_c$ between two different energy states,

$$u'_p(r_c)u_k(r_c) = (k^2 - p^2) \int_{r_c}^{\infty} u_k(r)u_p(r)dr = 0, \quad (5.74)$$

Taking the limit $r_c \rightarrow 0$ implies the following energy independent combination

$$\frac{c_1(p)}{c_2(p)} = \frac{c_1(k)}{c_2(k)} = \frac{c_1(0)}{c_2(0)}, \quad (5.75)$$

leaving one fixed ratio which can be determined from e.g. the zero energy state or any other reference state.

### 5.5.2 Phase shifts

For a finite energy scattering state we solve for the OBE potential with the normalization

$$u_p(r) \rightarrow \frac{\sin(pr + \delta_0(p))}{\sin \delta_0(p)}, \quad (5.76)$$

with $\delta_0(p)$ the phase shift. For a potential falling off exponentially $\sim e^{-m \pi r}$ at large distances, one has the effective range expansion at low energies, $|p| < m \pi/2$,

$$p \cot \delta_0(p) = -\frac{1}{\alpha_0} + \frac{1}{2} r_0 p^2 + v_2 p^4 + \ldots \quad (5.77)$$

with $\alpha_0$ the scattering length and $r_0$ the effective range. The phase shift is determined from Eq. (5.76). Thus, for the zero energy state we solve

$$-u_0''(r) + U_{1,0}(r)u_0(r) = 0, \quad (5.78)$$

with the asymptotic normalization at large distances, obtained from Eq. (5.76),

$$u_0(r) \rightarrow 1 - \frac{r}{\alpha_0}, \quad (5.79)$$

In this equation $\alpha_0$ is an input, so one integrates in Eq. (5.78) from infinity to the origin. Then, the effective range defined as

$$r_0 = 2 \int_0^\infty dr \left[ \left(1 - \frac{r}{\alpha_0}\right)^2 - u_0(r)^2 \right] \quad (5.80)$$

can be computed.
To determine the phase shift $\delta_0(p)$ one proceeds as follows. From Eq. (5.79) and integrating in Eq. (5.78) one determines $c_1(0)$ and $c_2(0)$ and uses Eq. (5.75) to determine the ratio $c_1(p)/c_2(p)$ \(^{10}\) then integrates out Eq. (5.69) matching Eq. (5.76). This way the phase shift $\delta_0(p)$ is determined from the potential and the scattering length as independent parameters. As it was shown in Ref. [133] this procedure is completely equivalent to renormalize the Lippmann-Schwinger equation with one counterterm.

5.5.3 Fixing of scalar parameters

As we have said we will fix our parameters in such a way that the $^1S_0$ phase shift is reproduced. This has the advantage that the scalar meson parameters are determined for the rest of observables. Thus, fixing the scattering length $\alpha_0 = -23.74$ fm and the OPE potential parameters $g_{\pi NN} = 13.1$ and $m_\pi = 138.04$ MeV we fit $g_{\sigma NN}$ and $m_\sigma$ to the $^1S_0$ phase shift of the Nijmegen group [57]. In the absence of vector meson contributions, i.e. taking $g_{\omega NN} = f_{\rho NN} = 0$ the fit yields

$$g_{\sigma NN} = 9(1) \quad m_\sigma = 501(25)\text{MeV}$$  \hspace{1cm} (5.81)

with a $\chi^2/DOF = 0.13$. As we see from Fig. 5.4 (left panel) there is a large, in fact linear, correlation, between the scalar coupling and mass, while the fit is quite good. The resulting $^1S_0$ phase shifts for this fit where shown in Fig. 3.5. For comparison we also showed the result with OPE which, despite reproducing the threshold behaviour does a poor job elsewhere. The effective range values from the low energy theorem for this election of couplings where displayed in Eq. (3.48). Actually, the linear $g_{\sigma NN} - m_\sigma$ correlation can be established solely by requiring that the effective range, say the Nijmegen value $r_0 = 2.67$ fm or the experimental one $r_0 = 2.77(5)$ fm, be reproduced [187]. This is shown in Fig 5.4 (right panel) where the linear correlation is more than evident. On the other hand, Eq. (5.81), yields the combination $C_\sigma = g_{\sigma NN}^2/m_\sigma^2 = 323(50)$ GeV\(^{-2}\) which is fixed by the effective range and not by the scattering length. This is in contrast with the resonance saturation viewpoint adopted in Ref. [190] where this combination fixes the scattering length.

Now, it is interesting to analyze the dependence of the fitted scalar parameters on the short distance cut-off radius, $r_c$. \textit{A priori} we should see the $\sigma$ exchange for $r_c \leq 1/m_\sigma = 0.4$ fm. In Fig. 5.5 we show the masses and couplings providing an acceptable fit $\chi^2/DOF < 1$ for which a reliable error analysis may be undertaken. As we see this happens for $r_c < 0.6$ fm and two stable plateau regions yield two potentially conflicting central $m_\sigma$ values. An error analysis both at a finite cut-off value $r_c = 0.4$ fm and the renormalized cut-off limit $r_c = 0$ fm gives two overlapping and hence compatible bands. This shows that in this case the data do not discriminate below $r_c = 0.5$ fm. Much above that scale, the $\sigma$ meson becomes nearly irrelevant, as the coupling becomes rather small.

Alternatively, we may treat the cut-off itself as a fitting parameter. To avoid the large $m_\sigma - g_{\sigma NN}$ correlations displayed in Fig. 5.4 we fix the coupling constant to its central value $g_{\sigma NN} = 9.1$ and get then $r_c = 0.10^{+0.13}_{-0.07}$ fm and $m_\sigma = 500(3)$ MeV. This shows that removing the cut-off is not only a nice theoretical requirement, but also a preferred phenomenological choice.

\(^{10}\)This is to say that we are imposing the condition

$$\frac{u_p'(r_c)}{u_p(r_c)} = \frac{u_0'(r_c)}{u_0(r_c)}.$$
Finally, to analyze now the role of vector mesons we note, as already discussed in Section 5.3.2, the redundant combination of coupling constants \( g_{\omega NN}^2 - f_{\rho NN}^2 m_\rho^2 / (2 M_N^2) \) which appears in the \(^1S_0\) potential when we take \( m_\rho = m_\omega \). We thus define the effective coupling \( g_{\omega NN}^* \) as in Eq. (5.54). This combination is responsible for the repulsive contribution to the potential in the \(^1S_0\) channel. From typical values of the couplings \( g_{\omega NN} = 9 - 10.5 \) and \( f_{\rho NN} = 15 - 17 \) we expect \( g_{\omega NN}^* \) to be effectively small. We show in Fig. 5.6 the corresponding \( \chi^2 / DOF \) as well as the readjusted scalar mass \( m_\sigma \) and coupling \( g_{\sigma NN} \) as a function of the effective combination of coupling constants, \( g_{\omega NN}^* \). As we see, the fit is rather insensitive but actually slightly worse than without vector mesons when their contribution is repulsive. Thus, we will fix this effective coupling to zero which corresponds to take

\[
g_{\omega NN}^2 = \frac{f_{\rho NN}^2 m_\rho^2}{2 M_N^2} \tag{5.82}
\]

This choice has the practical advantage of fixing \( g_{\sigma NN} \) and \( m_\sigma \) to the values provided in Eq. (5.81) also when the leading \( N_\chi \) vector meson contributions are included. Moreover, it is also phenomenologically satisfactory as we have discussed above. In Sec. 5.6 we will also see that deuteron properties or triplet \(^3S_1 - ^3D_1\) phase shifts do not fix the deviations from the relation given by Eq. (5.82).
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5.5.4 Discussion: Short range repulsion vs spurious bound states

Our calculation shows that an accurate fit without explicit contribution of the vector mesons is possible. In particular, our potential does not exhibit any repulsive region. This is in apparent contradiction with the traditional viewpoint that the $\omega$-meson is responsible for the short range repulsion of the nuclear force.

To understand this issue we plot in Fig. 5.7 the zero energy wave function obtained by integrating in with the physical scattering length $\alpha_0$. As we see, there appear two zeros indicating, according to the oscillation theorem, the existence of two negative energy spurious bound states. To compute such a states we solve Eq. (5.69) with negative energy $E_B = -\gamma_B^2/M$, for an exponentially decaying wave function, $u_B(r) \to A_B e^{-\gamma_B r}$ (normalized to one), and impose orthogonality to the zero energy state, namely

$$u_0 u_B' - u_B u_0' \bigg|_{r=r_e} = 0,$$  \hspace{0.5cm} (5.83)

from which $\gamma_B$ can be determined. A direct calculation yields $E_{B1} = -777$MeV and $A_{B1} = 15.64$fm$^{-1/2}$, and $E_{B2} = -11077$MeV and $A_{B2} = 27.43$fm$^{-1/2}$. If we regard the scattering amplitude as a function of
In order to discuss this point further we may try several ways of removing the unwanted poles and to quantify the effect on the results. Unitarity implies the usual relation between the partial wave amplitude and the phase shift

\[ [f_0(p)]^{-1} = p \cot \delta_0(p) - ip. \]  

(5.84)

Actually, the contribution of a negative energy state to the s-wave scattering amplitude is a pole contribution

\[ f_0(p)|_B = -\frac{A_B^2}{M E + |E_B|} = -\frac{A_B^2}{p^2 + \gamma_B^2}. \]  

(5.85)

A simple way of subtracting such a bound state without spoiling unitarity and preserving the value of the amplitude at threshold \( f_0(0) = F_0(0) = -\alpha_0 \) is to modify the real part of the inverse amplitude as follows,

\[ \frac{1}{F_0(p)} = \frac{1}{f_0(p)} - \frac{p^2}{A_B^2}, \]  

(5.86)

which has no pole at \( E = -|E_B| \), since \( F_0(i\gamma_B) = A_B^2/\gamma_B^2 \). Using the relation between amplitude and phase shift \( F_0(p) = 1/(p \cot \Delta_0(p) - ip) \) we get the modified phase shift,

\[ p \cot \Delta_0(p) = p \cot \delta_0(p) + \frac{p^2}{A_B^2}, \]  

(5.87)

which corresponds to a change in the effective range

\[ \Delta r_0|_B = \frac{2}{A_B^2}. \]  

(5.88)

For the values of the two spurious bound states we get \( \Delta r_0|_B = 0.008, 0.002 \text{fm} \), a tiny amount. The change in the phase shift never exceeds \( 0.1^0 \). Of course, this is not the only procedure to remove spurious bound states, but the result indicates that the effect should be small.

Another practical way to verify this issue is to study the influence of changing the cut-off \( r_c \) from the lowest value not generating any spurious bound state and the origin, corresponding to look for \( u_0(a) = 0 \). This point is clearly identified as the outer zero of the wave function, which takes place at about \( a = 0.5 \text{fm} \). Thus, if we choose \( r_c = a \), there will not be any bound state. For this particular point, the orthogonality of states, Eq. (5.74), implies that \( u_p(a) = 0 \), resembling the standard hard core picture, if we assume \( u_p(r) = 0 \) for \( r \leq a \). Thus, at this \( r_c \) our method would correspond to infinite repulsion below that scale. In other words, the boundary condition does incorporate some effective repulsion which need not be necessarily visualized as a potential. The advantage of using a boundary condition is that we need not require modeling nor deep understanding on the inaccessible and unknown short distance physics.

The contribution to the effective range from the origin to the “hard core” radius \( a \) is \( r_c^{in} \sim 0.04 \text{fm} \), while the change in the phase shift at the maximum energy due to the inner region \( 0 \leq r \leq a \) is \( \Delta \delta_0 = 0^0 \) to
be compared with the error estimate $\Delta \delta_0 = 0.7^0$ from the PWA analysis of the Nijmegen group [56] or the $\Delta \delta_0 = 2^0$ from the corresponding high quality potentials [57]. If we identify this hard core radius to the breakdown scale of the potential, these differences might be interpreted as a systematic error of the renormalization approach for our OBE potential and, as we see, they turn out to be rather reasonable.

5.6 The triplet channel

5.6.1 Equations and boundary conditions

The $^3S_1 - ^3D_1$ wave function in the np c.m. system can be written as

$$\Psi(\vec{x}) = \frac{1}{\sqrt{4\pi r}} \left[ u(r) \sigma_p \cdot \sigma_n + \frac{w(r)}{\sqrt{8}} (3 \sigma_p \cdot \hat{x} \sigma_n \cdot \hat{x} - \sigma_p \cdot \sigma_n) \right] \chi_{np}^{s_m},$$

(5.89)

with the total spin $s = 1$ and $m_s = 0, \pm 1$ and $\sigma_p$ and $\sigma_n$ the Pauli matrices for the proton and the neutron respectively. The functions $u(r)$ and $w(r)$ are the reduced S- and D-wave components of the relative wave function respectively. They satisfy the coupled set of equations in the $^3S_1 - ^3D_1$ channel

$$-u''(r) + U_{3S_1}(r)u(r) + U_{E_1}(r)w(r) = MEu(r),$$

$$-w''(r) + U_{E_1}(r)u(r) + \left[U_{3D_1}(r) + \frac{6}{r^2}\right] w(r) = MEw(r),$$

(5.90)

with $U_{3S_1}(r)$, $U_{E_1}(r)$ and $U_{3D_1}(r)$ the corresponding matrix elements of the coupled channel potential

$$U_{3S_1} = M(V_C - 3W_S),$$

(5.91)

$$U_{E_1} = -6\sqrt{2}MW_T,$$

(5.92)

$$U_{3D_1} = M(V_C - 3W_S + 6W_T).$$

(5.93)

At short distances one has the leading singularity

$$U_{3S_1} = O(r^{-1}),$$

(5.94)

$$U_{E_1} = -\frac{4\sqrt{2}R}{r^3} + O(r^{-1}),$$

(5.95)

$$U_{3D_1} = -\frac{12R}{r^3} + O(r^{-1}).$$

(5.96)

where

$$\pm R = \frac{g_{\pi NN}^2 - f_{\rho NN}^2}{32\pi M_N}.\quad (5.97)$$

This is very similar to the pure OPE case treated in Ref. [126] but with the important technical difference that for $f_{\rho NN} < g_{\pi NN}$ and $f_{\rho NN} > g_{\pi NN}$ there is a turn-over of repulsive-attractive eigenchannels since the effective short distance scale $R$ changes sign. Thus, we must distinguish two different cases $^{11}$.

$^{11}$The exceptional case, $g_{\pi NN} = f_{\rho NN}$ corresponds to a regular potential and will be treated latter
short distances we have for $g_{\pi NN} > f_{\rho NN}$ the plus sign in Eq. (5.97) yielding

\begin{align}
  u_A(r) &= \sqrt{\frac{2}{3}} u(r) + \frac{1}{\sqrt{3}} w(r), \\
  u_R(r) &= -\frac{1}{\sqrt{3}} u(r) + \sqrt{\frac{2}{3}} w(r),
\end{align}

(5.98)

whereas for $g_{\pi NN} < f_{\rho NN}$ the minus sign in Eq. (5.97) is taken and the solutions are interchanged

\begin{align}
  u_R(r) &= \sqrt{\frac{2}{3}} u(r) + \frac{1}{\sqrt{3}} w(r), \\
  u_A(r) &= -\frac{1}{\sqrt{3}} u(r) + \sqrt{\frac{2}{3}} w(r),
\end{align}

(5.100)

(5.101)

yielding an attractive singular potential $U_A \to -4R/r^3$ for $u_A$ and $U_R \to 8R/r^3$ for $u_R$, which solutions are

\begin{align}
  u_R(r) &\to \left(\frac{r}{R}\right)^{3/4} \left[C_{1R} e^{+4i\sqrt{\gamma}} + C_{2R} e^{-4i\sqrt{\gamma}}\right], \\
  u_A(r) &\to \left(\frac{r}{R}\right)^{3/4} \left[C_{1A} e^{-4i\sqrt{\gamma}} + C_{2A} e^{4i\sqrt{\gamma}}\right].
\end{align}

(5.102)

(5.103)

The constants $C_{1R}, C_{2R}, C_{1A}$ and $C_{2A}$ depend on both $\gamma$ and $\eta$ and the OBE potential parameters. As it was discussed in Ref. [126] the normalizability of the wave function at the origin requires

\[ C_{1R}(\gamma, \eta) = 0, \]

(5.104)

which is a relation between $\eta$ and $\gamma$. This means that the deuteron binding energy can be used as an independent parameter and that by integrating in from infinity and imposing the regularity condition Eq. (5.104) we can extract $\eta = \eta(\gamma)$. But some additional condition arises by requiring flux conservation. The flux at a point $r$ is given by

\begin{align}
  iJ(r) &= u^*(r)u(r) - u^*(r)u'(r) \\
  &+ w^*(r)w(r) - w^*(r)w'(r),
\end{align}

(5.105)

so that current probability conservation at the origin implies

\[ |C_{1A}|^2 - |C_{2A}|^2 = 2\sqrt{2i} \left(C_{1R}^* C_{2R} - C_{2R}^* C_{1R}\right). \]

(5.106)

Thus, if we set $C_{1R} = 0$ there is no condition on $C_{2R}$ and one has $C_{1A} = C_A e^{i\varphi}$ and $C_{2A} = C_A e^{-i\varphi}$ with $C_A$ and $\varphi$ real. So, we have three constants, $C_{2R}(\gamma)$, $C_A(\gamma)$ and $\varphi(\gamma)$, characterizing the normalizable solutions at short distances for a given value of the deuteron wave number $\gamma$,

\begin{align}
  u_R(r) &\to C_R(\gamma) \left(\frac{r}{R}\right)^{3/4} e^{-4i\sqrt{\gamma}}, \\
  u_A(r) &\to C_A(\gamma) \left(\frac{r}{R}\right)^{3/4} \sin \left[4\sqrt{\frac{R}{r}} + \varphi(\gamma)\right].
\end{align}

(5.107)

(5.108)

Orthogonality between different energy solutions impose the short distance phase $\varphi$ to be energy independent [126]. Then we could match the numerical solutions to the short distance expanded ones, but in practice this is a cumbersome procedure [126]. It is far more convenient to use an equivalent short distance
cut-off method with a boundary condition, i.e., to use a given boundary condition which be equivalent to Eq. (5.104). Thus, at the cut-off boundary, \( r = r_c \) we can impose a suitable regularity condition depending on the sign of \( g^2_{\pi NN} - f^2_{\rho NN} \). A set of possible auxiliary boundary conditions was discussed in Ref. [126], showing that the rate of convergence was depending on the particular choice. Actually, there are infinitely many auxiliary boundary conditions which converge towards the same renormalized value, as we discuss below.

### 5.6.2 The deuteron

In this case we have a negative energy state

\[
E = -\frac{\gamma^2}{M},
\]

(5.109)

and we look for regular solutions of the coupled equations (5.90) normalized to unity,

\[
\int_0^\infty dr \left[ u(r)^2 + w(r)^2 \right] = 1,
\]

(5.110)

which asymptotically behave as

\[
\begin{align*}
    u_\gamma(r) &\to A_S e^{-\gamma r}, \\
    w_\gamma(r) &\to A_S \eta e^{-\gamma r} \left( 1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right),
\end{align*}
\]

(5.111)

where \( A_S \) is the asymptotic wave function normalization and \( \eta \) is the asymptotic D/S ratio. To solve this problem it is useful to invoke the superposition principle, as suggested in [126].

The short distance regularity conditions must be imposed an a cut-off radius \( r_c \) in order to determine the value of \( \eta(r_c) \). Then, for a given solution we compute several properties as a function of the cut-off radius, \( r_c \). From the normalization condition, Eq. (5.110), in \( r_c \leq r \leq \infty \) we get \( A_S(r_c) \). We also compute the matter radius,

\[
r_m^2 = \frac{(\langle r^2 \rangle)}{4} = \frac{1}{4} \int_{r_c}^\infty r^2 (u(r)^2 + w(r)^2) dr,
\]

(5.112)

the quadrupole moment (without meson exchange currents)

\[
Q_d = \frac{1}{20} \int_{r_c}^\infty r^2 w(r)(2\sqrt{2}u(r) - w(r)) dr,
\]

(5.113)

the \( D \)-state probability

\[
P_D = \int_{r_c}^\infty w(r)^2 dr,
\]

(5.114)

which in the impulse aproximation and without meson exchange currents can be related to the deuteron magnetic moment. Finally, we also compute the inverse moment

\[
\langle r^{-1} \rangle = \int_{r_c}^\infty r^{-1} (u(r)^2 + w(r)^2) dr,
\]

(5.115)

which appears, e.g., in the multiple expansion of the \( \pi \)-deuteron scattering length.
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Figure 5.8: Short distance cut-off dependence of the asymptotic $D/S$-ratio for the case with $\pi + \sigma + \rho + \omega$. We show the dependence of the asymptotic $D/S$ normalization $\eta$ for several choices of the arbitrary and auxiliary short distance condition $\sin \alpha u(r_c) + \cos \alpha w(r_c) = 0$ for several values of $\alpha$.

As mentioned, there are infinitely many possible auxiliary conditions. This is an important point which we wish to illustrate. For instance, we could take

$$\sin \alpha u(r_c) + \cos \alpha w(r_c) = 0,$$  \hspace{1cm} (5.116)

where we may choose the parameter $\alpha$ arbitrarily. This is illustrated in Fig. 5.8. Note that despite possible wild behaviour all choices converge to the same value, although at a quite different rate. This is indeed another reason for removing the cut-off although it may be appealing and less demanding to choose one particular scheme where stability is found at the largest possible distances.

Here we will take the smoothest auxiliary condition (labeled as BC6 in Ref. [126])

$$u'(r_c) - \sqrt{2} w'(r_c) = 0, \quad g_{\pi NN}^2 - f_{\rho NN}^2 > 0,$$  \hspace{1cm} (5.117)

$$\sqrt{2} u'(r_c) + w'(r_c) = 0, \quad g_{\pi NN}^2 - f_{\rho NN}^2 < 0.$$  \hspace{1cm} (5.118)

Clearly, for the values that we will be using the convergence is determined by the size of the short distance scale characterizing the most singular component of the potential. As we see from Eq. (5.97) it depends strongly on the combination $g_{\pi NN}^2 - f_{\rho NN}^2$. This is an important point since the short distance cut-offs, $r_c$, for which convergence is achieved may change by orders of magnitude. An additional numerical problem arises due to undesired amplification of the short distance growing exponential, setting some limitations to the numerics due to roundoff errors. In all our calculations we have paid particular attention to these delicate issues.

The cut-off dependence of these observables is shown in Fig. 5.9, for the case of $\pi$ only (Ref. [126]), $\pi + \sigma$ and $\pi + \sigma + \rho + \omega$ and as we see good convergence can be achieved as $r_c \to 0$. As already mentioned, the rate of convergence depends on the scale of the singularity.

\footnote{This arbitrariness is not exclusive to this boundary condition, it is also present when the standard from factor regularization is introduced. The exponential, Eq. (5.34), and monopole Eq. (5.33) form factors are just two possible choices which do not cover the most general form which might allow a theoretical estimate on the systematic error.}

\footnote{An extreme example is given by the exceptional case $f_{\rho NN} = g_{\pi NN}$ since the $1/r^3$ singularity turns into a slowly and logarithmically converging Coulomb singularity. This case is treated specifically in Sec 5.8.}
The resulting coordinate space deuteron wave functions, $u$ and $w$, are depicted in Fig. 5.10 for the case of $\pi$ only (Ref. [126]), $\pi + \sigma$ and $\pi + \sigma + \rho + \omega$ compared to the wave functions of the high quality Nijmegen potential [57]. As we see, after inclusion of the scalar and vector mesons, the agreement is quite remarkable in the region above $1.4 - 1.8$fm, their declared range of validity. Similarly to the singlet case, we observe oscillations in the region below 1 fm. The first node is allowed since we are dealing with a bound state, the second node occurs already below 0.5fm indicating, similarly to the $^1S_0$ channel, the appearance of infinitely many spurious bound states, as we see from the short distance oscillatory behaviour of the wave function, Eq. (5.108). To compute such states we proceed similarly to the singlet channel. We solve Eq. (5.90) with negative energy $E_B = -\gamma_B^2 / M$, the asymptotic behaviour in Eq. (5.111) and impose the regularity conditions Eqs (5.117) and (5.118) to obtain $\eta(r_z)$. Then,
the dependence of (renormalized) deuteron properties as a function of the renormalization scale, determined to be $\eta_{\pi} = 0.2633$ and $\eta_{\sigma\omega\rho} = 0.2597$ (see table 5.2).

orthogonality to the deuteron state, namely

$$u_{\gamma}u'_{\gamma} - w_{\gamma}w'_{\gamma} \bigg|_{r=r_c} = 0,$$

(5.119)
determines $\gamma_B$. For instance, for the scalar parameters in Eq. (5.81) and $f_{\rho NN} = 15.5$ we identify the first spurious bound state $(u_{B1}, w_{B1})$ having one node less than the deuteron wave functions $(u_d, w_d)$ taking place at $\gamma_{B1} = 3.438\text{fm}^{-1}$. The corresponding energy is $E_{B1} = -\gamma_{B1}^2/M = -490\text{MeV}$, S-wave normalization $A_{B1} = 13.58\text{fm}^{-1/2}$ matter radius $r_{B1} = 0.49\text{fm}$ and asymptotic D/S ratio $\eta_{B1} = 0.1656$. This state is clearly beyond the range of applicability of the present framework. Subtracting this pole to the $^3S_1$ amplitude would result, according to Eq. (5.88), in $\Delta r_0 = 0.01\text{fm}$. The next spurious state has $E_{B2} = -18\text{GeV}$!. Note that if the scale where the second unphysical node takes place was to be interpreted as a (“hard core”) breakdown distance scale of our approach for the deuteron, it is certainly beyond the accessible region at the maximal energy in elastic NN scattering. This issue is relevant for the calculation of phase shifts where such oscillations also occur. The variation of the observables from this breakdown scale to the origin, could be interpreted as a source of systematic error coming from the fact that there is only one bound state and not infinitely many. As we see from Fig. 5.9 the effect is indeed small.

Numerical results for renormalized quantities can be looked up in Table 5.2. As we see, the inclusion of $\sigma$ provides some overall improvement while $\rho$ and $\omega$ yield a fairly accurate description of the deuteron for the choice $f_{\rho NN} = 15.5$ and $g_{\omega NN} = 9$ (this latter value complies to the $SU(3)$ relation $g_{\omega NN} = 3g_{\rho NN}$ when $g_{\rho NN} \sim 2.9$).

We show in Fig. 5.11 the dependence of (renormalized) deuteron properties as a function of $f_{\rho NN}$ for several values of the effective coupling constant $g_{\omega NN}^*$ featuring the strong correlation in the $^1S_0$ channel pointed out in Sec. 5.5. The scalar coupling $g_{\rho NN}$ and scalar mass $m_{\sigma}$ are always readjusted to fit the $^1S_0$ phase shift since the corresponding potential depends on $g_{\omega NN}^*$. As we see, for the asymptotic D/S ratio, there is a wide range of possible values within the experimental uncertainties but we obtain the bounds $f_{\rho NN} \leq 15$ and $g_{\omega NN} \leq 15$. It is amazing that the value of the tensor-$\rho$ coupling is so well determined to be $f_{\rho NN} \sim 16-17$ and corresponds to the strong $\kappa_\rho$ situation described by Machleidt and Brown [266]. Note that results depend in a moderate fashion on $f_{\rho NN}$ for not too large values, as one would expect from the short range of the $\rho$–meson.
5.6.3 Zero energy

At zero energy, the asymptotic solutions to the coupled equations (5.90) are given by

\[
\begin{align*}
  u_{0,\alpha}(r) & \rightarrow 1 - \frac{r}{\alpha_0}, \\
  w_{0,\alpha}(r) & \rightarrow \frac{3\alpha_0^2}{\alpha_0 r^2}, \\
  u_{0,\beta}(r) & \rightarrow \frac{r}{\alpha_0}, \\
  w_{0,\beta}(r) & \rightarrow \left(\frac{\alpha_2}{\alpha_0^2} - \frac{\alpha_0^2}{\alpha_0}\right) \frac{3}{r^3} - \frac{r^3}{15\alpha_0^2},
\end{align*}
\]

(5.120)
where $\alpha_0$, $\alpha_2$ and $\alpha_{02}$ are low energy parameters obtained from the phase shifts (see Sec. 5.6.4). Using these zero energy solutions the $^3S_1$ effective range

$$\left(\frac{r_0}{\alpha_0}\right)^2 = \int_0^{\infty} \left(1 - \frac{r}{\alpha_0}\right)^2 - u_{0,\alpha}(r)^2 - w_{0,\alpha}(r)^2 \right) \, dr,$$

(5.121)

can be determined. Moreover, the orthogonality constraints between the deuteron and the zero energy $\alpha$ and $\beta$ states read in this case

$$u_{\gamma}u'_{0,\alpha} - u'_{\gamma}u_{0,\alpha} + w_{\gamma}w'_{0,\alpha} - w'_{\gamma}w_{0,\alpha} \bigg|_{r=r_c} = 0,$$

(5.122)

$$u_{\gamma}u'_{0,\beta} - u'_{\gamma}u_{0,\beta} + w_{\gamma}w'_{0,\beta} - w'_{\gamma}w_{0,\beta} \bigg|_{r=r_c} = 0.$$

(5.123)

A further condition which should be satisfied is the $\alpha - \beta$ orthogonality

$$u_{0,\alpha}u'_{0,\beta} - u'_{0,\alpha}u_{0,\beta} + w_{0,\alpha}w'_{0,\beta} - w'_{0,\alpha}w_{0,\beta} \bigg|_{r=r_c} = 0,$$

(5.124)

as well as the short distance regularity conditions, Eqs. (5.118) and (5.117). In all we have an overdetermined system with 5 equations and three unknowns $\alpha_{02}$, $\alpha_2$ and $\alpha_0$. Solving the equations in triplets we have checked the numerical compatibility at the 0.01% level for the shortest cut-offs, $r_c \sim 0.02$fm typically used. The values of $\alpha_{02}$ and $\alpha_2$ are not so well known although they have been determined from potential models in Ref. [192].

In Fig. 5.12 we show the dependence of the low energy parameters of the leading $N_c$ contributions to the OBE ($\sigma + \pi + \rho + \omega$) potential as a function of $f_{\rho NN}$ for several values of the the effective coupling constant $g_{\omega NN}$ being $g_{\sigma NN}$ and $m_\sigma$ always readjusted to fit the $^1S_0$ phase shift. Similarly to the deuteron case we observe stronger dependence on $f_{\rho NN}$ and a relative insensitivity on the effective coupling $g_{\omega NN}^*$. We remind that along any of these curves the $^1S_0$ phase shift is well reproduced with an acceptable $\chi^2$/DOF < 1. As we see, the values $f_{\rho NN} = 17.0$ and $g_{\omega NN} = 0$ reproduce quite well the low energy parameters, corresponding to the reasonable $g_{\omega NN} = 10.4$.

Numerical results for the low energy parameters are shown in Table 5.2. Again, the inclusion of $\sigma$ provides some overall improvement while $\rho$ and $\omega$ yield a better description of the deuteron for the choice $f_{\rho NN} = 15.5$ and $g_{\omega NN} = 9.0$. There is nonetheless a small mismatch to the experimental or recommended potential values when the zero energy wave functions are obtained from the orthogonality relations to the deuteron, Eqs. (5.122) and (5.123). As one can see further improvement is obtained when

<table>
<thead>
<tr>
<th>$\gamma$ (fm$^{-1}$)</th>
<th>$\eta$</th>
<th>$A_0$ (fm$^{-2}$)</th>
<th>$r_0$ (fm)</th>
<th>$Q_0$ (fm$^2$)</th>
<th>$P_0$ (fm$^3$)</th>
<th>$\alpha_0$ (fm)</th>
<th>$\alpha_{02}$ (fm$^3$)</th>
<th>$\alpha_2$ (fm$^3$)</th>
<th>$r_0$ (fm)</th>
</tr>
</thead>
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<tr>
<td>$^3S_1$</td>
<td>$\pi$</td>
<td>Input</td>
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<td>0.8681</td>
<td>1.9354</td>
<td>0.2762</td>
<td>0.8886</td>
<td>5.335</td>
<td>1.673</td>
</tr>
<tr>
<td>$^3S_1$</td>
<td>$\pi\sigma\rho\omega$</td>
<td>Input</td>
<td>0.02597</td>
<td>0.9054</td>
<td>2.0098</td>
<td>0.2910</td>
<td>0.633%</td>
<td>0.432</td>
<td>5.335</td>
</tr>
<tr>
<td>$^3S_1$</td>
<td>$\pi\sigma\rho\omega^*$</td>
<td>Input</td>
<td>0.02625</td>
<td>0.8846</td>
<td>1.9659</td>
<td>0.2821</td>
<td>0.623%</td>
<td>0.497</td>
<td>5.415</td>
</tr>
<tr>
<td>$\Sigma NN$</td>
<td>Input</td>
<td>0.02521</td>
<td>0.8843(9)</td>
<td>1.9675</td>
<td>0.2707</td>
<td>0.633%</td>
<td>0.4302</td>
<td>5.415</td>
<td>1.745</td>
</tr>
<tr>
<td>$\Sigma NN$</td>
<td>Input</td>
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<td>0.8845(9)</td>
<td>1.9686</td>
<td>0.2703</td>
<td>0.699%</td>
<td>0.4315</td>
<td>5.422</td>
<td>1.745</td>
</tr>
<tr>
<td>Exp.</td>
<td>$\Sigma NN$</td>
<td>Input</td>
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<td>0.0266(4)</td>
<td>0.8846(9)</td>
<td>0.2859(4)</td>
<td>0.641(7)</td>
<td>5.419(7)</td>
<td>1.745(5)</td>
</tr>
</tbody>
</table>

Table 5.2: Deuteron properties and low energy parameters in the $^3S_1 - ^3D_1$ channel for OBE potentials including $\pi$, $\pi + \sigma$, $\pi + \sigma + \rho + \omega$. We use the non-relativistic relation $\gamma = \sqrt{2\mu_{NN} B}$ with $B = 2.224575(9)$ and take $m = 138.03$MeV, and $g_{\sigma NN} = 13.1083$ [264]. From a fit to the $^1S_0$ channel we have $m_\sigma = 501$MeV and $g_{\rho NN} = 9.1$. The simplifying relation $g_{\omega NN} = f_{\rho NN} m_\rho / \sqrt{2M_N}$ is used throughout. $\pi\sigma\rho\omega$ corresponds to take $f_{\rho NN} = 15.5$ and $g_{\omega NN} = 9.857$ while $\pi\sigma\rho\omega^*$ corresponds to take $f_{\rho NN} = 17.0$ and $g_{\omega NN} = 10.147$.
Chapter 5. Renormalization of One Boson Exchange potentials

5.32 - 5.58

5.6.4 Phase shifts

Finally, in the case of positive energy we consider Eq. (5.90) with

\[ E = \frac{p^2}{M}, \]

with \( p \) the corresponding c.m. momentum. We solve Eq. (5.90) for the \( \alpha \) and \( \beta \) positive energy scattering states and choose the asymptotic normalization

\[ u_{k,\alpha}(r) \rightarrow j_0(kr) \cot \delta_{\alpha} - \tilde{y}_0(kr), \]

\[ w_{k,\alpha}(r) \rightarrow \tan \epsilon \left( j_2(kr) \cot \delta_{\alpha} - \tilde{y}_2(kr) \right), \]

\[ u_{k,\beta}(r) \rightarrow -\tan \epsilon \left( j_0(kr) \cot \delta_{\beta} - \tilde{y}_0(kr) \right), \]

\[ w_{k,\beta}(r) \rightarrow j_2(kr) \cot \delta_{\beta} - \tilde{y}_2(kr), \]
As we have mentioned already, the numerical solution of the problem requires taking care of spurious amplification of the undesired growing exponential at any step of the calculation. The situation is aggravated by the fact that for the phase shifts the maximum momentum $p = 400 \text{MeV}$ explores the region around $0.1 - 0.5 \text{fm}$, so it is important to make sure that we do not see cut-off effects in this region. To provide a handle on the numerical uncertainties we show in Fig. 5.13 the results for the phase shifts $\delta_1$, $\delta_2$ and $\epsilon$ as a function of the cut-off radius, $r_c$, and for several fixed c.m. $np$ momenta, $p = 100, 200, 300, 400 \text{MeV}$. As we see, there appear clear plateaus between $0.1 - 0.2 \text{fm}$ which somewhat steadily shrink when the momentum is increased. Note that these values of the short distance cut-off translates into a c.m. momentum space cut-off range $\Lambda = \pi/(2r_c) = 1.5 - 3 \text{GeV}$.

![Figure 5.13: Convergence of the np spin triplet eigen phase shifts for the total angular momentum $j = 1$ as a function of the short distance cut-off radius $r_c$ (in fm) for several fixed values of the c.m. momentum $p = 100, 200, 300$ and $400 \text{MeV}$.](image)

where $\hat{j}_i(x) = x j_i(x)$ and $\hat{y}_i(x) = x y_i(x)$ are the reduced spherical Bessel functions, $\delta_\alpha$ and $\delta_\beta$ are the eigen-phases in the $^3S_1$ and $^3D_1$ channels, and $\epsilon$ is the mixing angle $E_1$.

In the low energy limit the eigen phase shifts behave as,

$$\delta_\alpha \rightarrow -\alpha_0 k, \quad (5.130)$$

$$\delta_\beta \rightarrow -\left(\alpha_2 - \frac{\alpha_2^2}{\alpha_0}\right) k^5, \quad (5.131)$$

$$\epsilon \rightarrow \frac{\alpha_2}{\alpha_0} k^2. \quad (5.132)$$

and the zero energy solutions discussed in Sec. 5.6.3 are reproduced. The use of the orthogonality constraints to the deuteron wave yields

$$u_\gamma^r u^r_{k,\alpha} - w_\gamma^r w^r_{k,\alpha} + w_\gamma^r w^r_{k,\alpha} \bigg|_{r=r_c} = 0, \quad (5.133)$$

$$w_\gamma^r w^r_{k,\beta} - w_\gamma^r w^r_{k,\beta} + w_\gamma^r w^r_{k,\beta} \bigg|_{r=r_c} = 0, \quad (5.134)$$

which together with the short distance regularity conditions, Eqs. (5.118) and (5.117) allow us to deduce the corresponding $^3S_1 - ^3D_1$ phase-shifts. A further condition is the $\alpha - \beta$ orthogonality

$$u^r_{k,\alpha} u^*_{k,\beta} - w^r_{k,\alpha} w^*_{k,\beta} \bigg|_{r=r_c} = 0. \quad (5.135)$$

In all we have again an over-determined system with 5 equations and three unknowns. We have checked that almost any choice yields equivalent results with an accuracy of $0.001^\circ$ for the highest c.m. momenta and the shortest cut-off, $r_c \sim 0.02 \text{fm}$.

As we have mentioned already, the numerical solution of the problem requires taking care of spurious amplification of the undesired growing exponential at any step of the calculation. The situation is aggravated by the fact that for the phase shifts the maximum momentum $p = 400 \text{MeV}$ explores the region around $0.1 - 0.5 \text{fm}$, so it is important to make sure that we do not see cut-off effects in this region. To provide a handle on the numerical uncertainties we show in Fig. 5.13 the results for the phase shifts $\delta_1$, $\delta_2$ and $\epsilon$ as a function of the cut-off radius, $r_c$, and for several fixed c.m. $np$ momenta, $p = 100, 200, 300, 400 \text{MeV}$. As we see, there appear clear plateaus between $0.1 - 0.2 \text{fm}$ which somewhat steadily shrink when the momentum is increased. Note that these values of the short distance cut-off translates into a c.m. momentum space cut-off range $\Lambda = \pi/(2r_c) = 1.5 - 3 \text{GeV}$. 

![Figure 5.13: Convergence of the np spin triplet eigen phase shifts for the total angular momentum $j = 1$ as a function of the short distance cut-off radius $r_c$ (in fm) for several fixed values of the c.m. momentum $p = 100, 200, 300$ and $400 \text{MeV}$.](image)
respectively. On a first sight we show the phase shift in the
renormalization. An equivalent way of posing the question is to deter-
mine whether finite nucleon size effects can be disentangled from meson exchange effects explicitly in NN scattering in the elastic region.

The results for the $^3S_1 - ^3D_1$ phase shifts as a function of the c.m. momentum are depicted in Fig. 5.14 for $\pi$, $\pi + \sigma$ and $\pi + \sigma + \rho + \omega$ and compared to the Nijmegen analysis [56, 57]. We use $g_{\rho NN} = 9.1$, $m_{\sigma} = 501\text{MeV}$ and when vector mesons are included we take $f_{\rho NN} = 15.5$ and $g_{\omega NN} = 9$ or $f_{\rho NN} = 17.0$ and $g_{\omega NN} = 10.147$ corresponding to Sets $\pi\sigma\rho\omega$ and $\pi\sigma\rho\omega^*$ in Table 5.2 respectively. On a first sight we see an obvious improvement in both the $^3S_1$ and $^3D_1$ phases and not so much in the mixing angle $E_1$ as compared to the simple OPE case. One should note, however, that besides describing by construction the single phase shift $^3S_0$ (see Fig. 3.5) we also improve on the deuteron (see Table 5.2). Obviously, it would be possible to provide a better description of triplet phase shifts, however, at the expense of worsening the deuteron properties and the singlet channel. Clearly, there is room for improvement, and our results call for consideration of sub-leading large $N_c$ corrections in the OBE potential. This would incorporate, the relative to leading $1/N^2_c$ relativistic corrections, spin-orbit effects, finite meson widths, non-localities, etc.

### 5.7 Effect of form factors under renormalization

Given the reasonable phenomenological success of the renormalization approach one may naturally wonder what would be the effect of the form factors in our calculation. Here we discuss the influence of strong form factors in the calculated properties and whether they lead to observable physical effects after renormalization. An equivalent way of posing the question is to determine whether finite nucleon size effects can be disentangled from meson exchange effects explicitly in NN scattering in the elastic region.

To analyze this important issue in detail, in Fig. 5.15 we show the phase shift in the $^1S_0$ channel for fixed LAB energy values as a function of the short distance cut-off radius $r_c$ when the scattering length is fixed to its experimental value, $a_0 = -23.74\text{fm}$ as we explained in Sec. 5.5. We use the same parameters as for the renormalized solution without vertex function, for several fixed values of the LAB energy and for the cut-off values $\Lambda_{\sigma NN} = 1300\text{MeV}$ and $\Lambda_{\rho NN} = 2000\text{MeV}$, all others fixed to $\Lambda_{\pi NN} = \Lambda_{\omega NN} = 2\text{GeV}$. As one clearly sees, strong form factors are invisible for $r_c > 0.3\text{fm}$. For lower values of the short distance cut-off $r_c$ both monopole and exponential form factors agree with each other but deviate strongly from the Nijmegen database. Note that the lines should be supplemented with estimates of theoretical errors, not shown to avoid cluttering of the plot. When those errors are included the Nijmegen data are basically compatible with the theoretical curves in the flat

---

**Figure 5.14:** $np$ spin triplet eigen phase shifts for the total angular momentum $j = 1$ as a function of the c.m. momentum. We show $\pi$, $\pi + \sigma$ and $\pi + \sigma + \rho + \omega$ compared to an average of the Nijmegen partial wave analysis and high quality potential models [56, 57]. We take $(f_{\rho NN}, g_{\omega NN}) = (15.5, 9.857)$, $(f_{\rho NN}, g_{\omega NN}) = (17.0, 10.147)$.
the influence of the vertex functions is analyzed for some of the computed deuteron
factors with the equally renormalized solution including the form factors in the potential in the region
around $r_c \sim 0.3 - 0.6$fm. The deviation below 0.3 fm signals the onset of the irregular $D$-wave solution,
Figure 5.16: Short distance cut-off $r_c$ dependence of deuteron properties for the $\pi\sigma\rho\omega$ case (see table 5.2). We compare the purely renormalized calculation with the cases for both exponential, Eq. (5.34), and monopole Eq. (5.33) form factors taking $\Lambda_{\pi NN} = 1300\text{MeV}$, all other cut-offs being kept to $\Lambda_{\sigma NN} = \Lambda_{\rho NN} = \Lambda_{\omega NN} = 2000\text{MeV}$. We show the dependence of the asymptotic D/S normalization $\eta$ (upper left panel), the S-wave normalization $A_S$ (in fm$^{-1/2}$, upper right panel), the matter radius $r_m$ (in fm, middle right panel), the quadrupole moment $Q_d$ (in fm$^2$, middle left panel), the $D$-state probability (lower left panel) and the inverse radius $\langle r^{-1}\rangle$ (in fm$^{-1}$ lower right panel).

Experimental or recommended values can be traced from Ref. [169].

which behaves as $w(r) \sim r^{-2}$ at small distances and hence yields eventually a divergent result. Note that in order to have a smooth behaviour at short distances when renormalization is over-imposed to the potential with form factors we should choose the regular D-wave solution $w(r) \sim r^3$ but then the potential parameters, either couplings or form factor cut-off parameters should also be fine-tuned.

While it is fairly clear that vertex functions do exist and are of fundamental importance, it is also true that they start playing a role as soon as the probing wavelength resolves the finite nucleon size. Our calculations suggest on a quantitative level that provided the NN scattering data are properly described with form factors, they will be effectively irrelevant under the renormalization process, and for c.m. momenta below 400MeV, vertex functions are expected to play a marginal role.
Chapter 5. Renormalization of One Boson Exchange potentials

5.8 The exceptional non-singular case

As already mentioned in Sec. 5.2.1 there is an exceptional situation $f_{\rho NN} = g_{\pi NN}$ where the OBE potential is not singular, Eq. (5.20), and the use of form factors would not be necessary. If we keep $g_{\pi NN} = 13.1$ that means $f_{\rho NN} = 13.1$, a not completely unrealistic value lying in between the single vector meson dominance estimate and the usual OBE value (see Sec. B.5 in Appendix B), so it is worth analyzing this case separately. Since the singularity affects mainly the coupled spin triplet channel, one may wonder what would be the consequences for the deuteron. We will show that our conclusions are not ruled out by this exceptional case.

Note that within the renormalization approach this particular situation has been scanned through in Fig. 5.11 where nothing particularly noticeable happens. Actually, at short distances we have a coupled channel Coulomb problem where the short distance behaviour can generally be written as a linear admixture of regular and irregular solutions,

\[ u(r) \sim a_1 r + a_2, \]
\[ w(r) \sim b_1 r^3 + b_2 r^{-2}. \]

In order to get a normalizable wave function we must impose the regular solution for the $D$-wave, meaning $b_2 = 0$. The renormalized solution corresponds then to fix the deuteron binding energy as explained in detail in Sec. 5.6 and integrate in with the result that the $S$-wave may have an admixture of the irregular solution. The regular solution takes the value $a_2 = 0$. The bound state properties are now predicted completely from the potential.

In practice we deal with arbitrarily small but finite cut-offs, $r_c \to 0$. In this situation it is simplest to use the superposition principle of boundary conditions given by

\[ u(r) = u_S(r) + \eta u_D, \]
\[ w(r) = w_S(r) + \eta w_D, \]

for a given energy or $\gamma$. From the regularity condition of the $D$-wave we get

\[ r_c \frac{u'(r_c)}{u(r_c)} = 3, \quad \text{(regular D-wave)}, \]

which yields the asymptotic $D/S$-ratio

\[ \eta(r_c) = \frac{-3w_S(r_c) + r_c w'_S(r_c)}{3w_D(r_c) - r_c w'_D(r_c)}. \]

This provides a relation between $\gamma$ and $\eta$. The renormalized condition yields an arbitrary value of $u$ at the origin, so the energy may be fixed arbitrarily, and thus

\[ r_c \frac{u'(r_c)}{u(r_c)} \neq 1, \quad \text{(irregular S-wave)}. \]

The regular solution corresponds to

\[ r_c \frac{u'(r_c)}{u(r_c)} = 1, \quad \text{(regular S-wave)}, \]
Table 5.4: Deuteron properties for the exceptional case $f_{\rho NN} = g_{\pi NN}$ of non-singular large $N_c$ OBE potentials. In all cases we take $r_c = 0.001\text{fm}$. We compare renormalized vs. regular solutions for similar choices of parameters. We use $\gamma = \sqrt{2}\gamma_{\rho NN}B_d$ with $B_d = 2.224575(9)$ and take $g_{\pi NN} = 13.1083$, $m_\omega = 138.03\text{MeV}$, $m_\rho = m_\omega = 782\text{MeV}$. The fit to the $^1S_0$ phase shift gives $m_\omega = 501\text{MeV}$ and $g_{\pi NN} = 9.1$. Experimental or recommended values can be traced from Ref. [169].

\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
                  & $g_{\rho NN}$ & $r_\rho$ & $r_\rho 2^+/3^-$ & $r_\rho 3^+/4^-$ & $\gamma$(fm$^{-1}$) & $g_\gamma$ & $A_\gamma$(fm$^{-1/2}$) & $r_{\text{min}}$(fm) & $Q_d$(fm$^2$) & $P_\rho$ \((r^{-1})\) \\
\hline
Renormalized     & 0              & -0.1274   & 3                         & 0.6615           & 0.02567               & 0.8986      & 1.9594            & 0.2830       & 5.87%     & 0.470 \\
Regular          & 0              & 1           & 3                         & Input            & 0.9925               & 2.2523      & 0.1215            & 10.77%      & 0.851     \\
Renorm. = Reg.   & 3.74           & 1           & 3                         & Input            & 0.02567               & 0.8979      & 1.9933            & 0.2827      & 5.88%     & 0.491 \\
Renorm.          & 2×3.74         & 0.0297      & 3                         & Input            & 0.02569               & 0.8957      & 1.9890            & 0.2817      & 5.92%     & 0.517 \\
\hline
NijmII [57]      & -              & -           & -                         & Input            & 0.02521               & 0.8845(8)   & 1.9675            & 0.2707      & 5.635%    & 0.4502 \\
Reid93 [57]      & -              & -           & -                         & Input            & 0.02514               & 0.8845(8)   & 1.9668            & 0.2703      & 5.699%    & 0.4515 \\
Exp. [169]       & -              & -           & -                         & 0.231665        & 0.0256(4)             & 0.8846(9)   & 1.9754(9)        & 0.2893(3)   & 5.67(4)  \\
\hline
\end{tabular}

Figure 5.17: Normalized Deuteron wave functions, $u$ (left) and $w$ (right), as a function of the distance (in fm) in the OBE for the exceptional non-singular case $f_{\rho NN} = g_{\pi NN}$. We show $\pi + \sigma + \rho + \omega$ both renormalized and the regular solution with the same parameters $g_{\rho NN} = 0$. We compare to the Nijmegen II wave functions [57](see table 5.4).

which in general will not be satisfied by the physical deuteron binding energy. Thus, for the regular solution we will have either a wrong value of the energy or the potential parameters must be readjusted. A value of $r_c = 0.001\text{fm}$ proves more than enough.

Numerical results for a fixed parameter choice with $g_{\rho NN} = 0$ are presented in table 5.4. As we see the regular solution generates a bound state with $E_B \sim -16\text{MeV}$ which is clearly off the deuteron with equally bad properties. In order to achieve the correct deuteron binding energy we just increase the coupling to $g_{\rho NN}^* = 3.75$ in the regular solution case. In this case both renormalized and regular solution would coincide accidentally. However, if we increase to twice this value $g_{\rho NN}^* = 2 \times 3.75$ we observe tiny changes in the deuteron properties as compared to the $g_{\rho NN} = 0$ case when the renormalized solution is considered whereas the regular solution becomes unbound. These results illustrate further the sharp distinction between regular and renormalized solutions where one chooses between fine-tuning and short distance insensitivity respectively. The corresponding wave functions to both the renormalized and regular solutions with the same meson parameters are depicted in Fig. 5.17. In both cases inner nodes of the wave functions exhibit the existence of deeply bound states, as dictated by the oscillation theorem.

Finally, we might try to analyze the consequences of taking $V_{3S_1}(r) = V_{1S_0}(r)$ in the exceptional case $f_{\rho NN} = g_{\pi NN} = 13.1$ and other parameters from the case with no form factor, $\Gamma = 1$, of table 5.1 for the $^3S_0$ channel. Let us remind that two possible scenarios arise in such a case, one with no bound state and another one with a spurious deeply bound state. For the $^3S_1 - ^3D_1$ channel, this complies to the standard picture that the deuteron becomes bound due to the additional binding introduced by the
small tensor force mixing with the $D$-wave, basically shifting the $S$-wave potential to an effective one $V_{S_1}^2(r) \sim V_{S_0}^1(r) + W_T(r)^2/V_{D_1}^1(r)$. While in the case with no spurious bound state for the $1^S_0$ we do not get any deuteron bound state, in the case with the spurious bound state the binding energy is $E_B \sim -50$MeV. This is another manifestation of the fine-tuning discussed at length in Sec. 5.3.2.

In summary, although the $1/r^3$ singularity makes renormalization process mandatory to implement the physical requirement of short distance insensitivity, the important aspect here is that this requirement remains equally valid even if there are no singularities at all.

### 5.9 Effect of the axial-vector meson

As a final task we would like to see how the axial-vector meson $a_1$ affects our results. As it is well known the $a_1$-meson appears in chiral symmetry as the chiral partner of $\rho$-meson in the same way that the $\sigma$-meson is the chiral partner of the $\pi$. The importance of the $a_1$ exchange in NN has been traditionally associated to $\pi\rho$ exchange effects. Obviously these effects take place at very short distances even shorter than vector meson effects but following the spirit of the large-$N_c$ expansion it is worthy studying them. The axial-vector meson cancels part of the $\rho$ vector short distance $1/r^3$ divergence favouring the regularity of the potential. The leading-$N_c$ potential is (see Appendix B),

$$ V_{a_1}(r) = \frac{g_{a_1NN}^2}{4\pi} \frac{m_{a_1}}{3} \left[ -2Y(m_{a_1}r) \vec{\sigma}_1 \cdot \vec{\sigma}_2 + T(m_{a_1}r) S_{12}(\vec{r}) \right] \vec{\tau}_1 \cdot \vec{\tau}_2, $$

and the cancellation of the $1/r^3$ divergence takes place at the particular value,

$$ g_{a_1NN}^2 = \frac{m_{a_1}^2}{4M_N^2} \left( f_{\rho NN}^2 g_{\pi NN}^2 \right), $$

assuming $f_{\rho NN} > g_{\pi NN}$. As a reference we will take the Schwinger relation for the coupling constant $268$ $g_{a_1NN} = \frac{m_{a_1}}{m_{\rho}} f_{\pi NN}$ and two different values for its mass, namely, the one coming from axial-vector meson dominance (AVMD) $269$ $m_{a_1} = \sqrt{2} m_\rho \simeq 1107$MeV and the PDG value $m_{a_1} = 1230$MeV. The deuteron properties and low energy parameters are shown in table 5.5. As we see, except for the inverse momentum ($r^{-1}$) which is clearly above the Nijmegen potential values, the rest of properties are slightly improved. The phase shift are plotted in Fig. 5.18. In general we observe a little improvement in the phase shifts when the PDG value for $m_{a_1}$ is taken. It should be stressed however that the choice of the coupling constant has been a very particular one.

<table>
<thead>
<tr>
<th>$\gamma$ (fm$^{-1}$)</th>
<th>$\eta$</th>
<th>$A_0$ (fm$^{-1}$)</th>
<th>$A_0$ (fm$^{-1}$)</th>
<th>$B_0$ (fm$^{-1}$)</th>
<th>$P_0$</th>
<th>($r^{-1}$)</th>
<th>$a_{02}$ (fm$^{-1}$)</th>
<th>$a_{03}$ (fm$^{-1}$)</th>
<th>$r_0$ (fm)</th>
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<td>0.639</td>
<td>5.467</td>
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<tr>
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<td>5.477</td>
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<td>0.8951</td>
<td>1.9876</td>
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<td>6.01%</td>
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<td>5.470</td>
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<tr>
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<td>6.61(4)</td>
<td>5.419(1)</td>
<td>1.755(8)</td>
<td></td>
</tr>
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</table>

Table 5.5: Deuteron properties and low energy parameters in the $^3S_1 - ^3D_1$ channel for OBE potentials including the axial-vector meson $a_1$. We use the same numbers and notation as in table 5.2. Here AVMD means taking $m_{a_1} = \sqrt{2} m_\rho \simeq 1107$MeV and PDG taking $m_{a_1} = 1230$MeV.
5.10 Conclusions

In this chapter we have analyzed the OBE potential from a renormalization point of view. The OBE potential presents short distance divergences which make the solution of the corresponding Schrödinger equation ambiguous. The traditional remedy for this problem has been the inclusion of phenomenological form factors which parameterize the vertex functions and hence the finite nucleon size within the meson exchange picture. This, however, generates spurious deeply bound states for natural values of the coupling constants. The price to remove those is to fine-tune the potential at all distances, and in particular at short distances.

Renormalization is a practical and feasible way of minimizing short distance ambiguities, by imposing conditions which are fixed by low energy data independently on the potential assuming from the start our inability to pin down the short distance physics below the smallest de Broglie wave length probed in NN scattering. The central scattering waves and the deuteron can be described reasonably well with natural values of the meson-nucleon couplings. Within the standard approach this could only be achieved by fine-tuning meson parameters or postulating the meson exchange picture to even shorter ranges than 0.5fm. In our case the inclusion of shorter range mesons induces moderate changes, due to the expected short distance insensitivity embodied by renormalization, despite the short distance singularity and without introducing strong meson-nucleon-nucleon vertex functions. If phenomenological vertex functions are added minor effects are observed under renormalization.

However, the renormalization process introduces spurious deeply bound states regardless on whether or not the potential is regular or singular. This is a commonplace in EFTs. We have checked that the role played by these spurious states is completely irrelevant. It is noteworthy that in the standard approach with form factors those spurious bound states also take place when natural values of the coupling constants are taken.

Our potential is based in the spin-flavor structure of the NN interaction in the large-$N_c$ limit. The advantage is that its structure simplifies tremendously yielding a non-relativistic and uniquely defined local and energy independent potential. Relativistic effects, spin-orbit, non-localities as well as meson widths or other mesons enter as sub-leading corrections to the potential with a relative order $1/N_c^2$. However, it consists of an infinite tower of multi-meson exchanged states. A truncation of the infinite
number and range of exchanged mesons is based on the assumption that the hardly accessible high mass states are irrelevant for NN energies below the inelastic pion production threshold when a proper renormalization scheme making this short distance insensitivity manifest is applied. In fact, we have implemented a boundary condition regularization and carried out the necessary renormalization. This allows, within the large-$N_c$ OBE potential to keep only $\pi$, $\sigma$, $\rho$ and $\omega$ mesons, which appear on equal footing, and neglect effectively higher mass effects.

The value of the $\sigma$ mass was fixed by a fit to the $^1S_0$ phase shift yielding $m_\sigma = 501(25)$MeV. The value obtained for the coupling constants reproducing the $^1S_0$ and $^3S_1 - ^3D_1$ channels are very reasonable, $g_{\sigma NN} = 9(1)$, $g_{\omega NN} = 9.5(5)$ and $f_{\rho NN} = 16.3(7)$; the range is compatible with the putative 10% accuracy of the $1/N_c^2$ corrections. For the accepted value $g_{\rho NN} = 2.9(1)$ this yields $g_{\omega NN}/g_{\rho NN} = 3.27(17)$ a value in between the $SU(3)$ prediction $g_{\omega NN}/g_{\rho NN}|_{SU(3)} = 3$ and the one from the $e^+e^- \to \rho$ and $e^+e^- \to \omega$ decay ratios, $g_{\omega NN}/g_{\rho NN}|_{e^+e^-} = 3.5$. We also get $f_{\rho NN}/g_{\rho NN} = \kappa_\rho = 5.6(3)$; a value in agreement from tensor coupling studies. It is noteworthy that the repulsion triggered by the $\omega$ meson is not as strong and important as required in the conventional OBE approach where usually a strong violation of the $SU(3)$ relation is observed as well. The reason is that, unlike the traditional approach, the renormalization viewpoint stresses the irrelevance of small distances. This is done by the introduction of counterterms which are fixed by threshold scattering parameters at any given short distance cut-off scale $r_c$. For the minimal de Broglie wavelength probed in NN scattering below pion production threshold, $1/p \sim 0.5$fm, a stable result is obtained generally when $r_c = 0.1-0.2$fm. Any mismatch to the observables can then be attributed to missing physical effects. While the present calculations are encouraging there is of course room for improvement.
Chapter 6

Nucleon-Nucleon interaction, charge symmetry breaking and renormalization

6.1 Introduction

The understanding of Charge Dependence of strong interactions has been a crucial issue in Nuclear Physics (for reviews see e.g. [174, 270, 271]). In fact, the simplest place where this issue can be studied is for the NN interaction. As it is well known, isospin invariance is not an exact symmetry of strong interactions and as a consequence nuclear forces have a small but net charge-dependent component. By definition, charge independence means invariance under any rotation in isospin space. A violation of this symmetry is referred to as charge dependence or charge independence breaking (CIB) and it means in particular that, in the isospin $T = 1$ state, the proton-proton ($T_3 = +1$), neutron-proton ($T_3 = 0$), or neutron-neutron ($T_3 = -1$) strong interactions are different. A particular case, known as charge symmetry breaking (CSB), only considers the difference between proton-proton ($pp$) and neutron-neutron ($nn$) interactions.

But, what is the scale of charge symmetry breaking?. Actually, the effects are important in the s-wave channel where an unnaturally large scattering length, due to a virtual state $^1$ in that partial wave, triggers a high short distance sensitivity. This, of course amplifies effects related to variations in the short distance parameters precisely in the region where the interaction and hence the charge symmetry breaking effects may be less reliable. The current understanding is that CIB, and in particular CSB, are due to a mass difference between the up and down quarks and electromagnetic interactions. On the hadronic level and in a One-Boson-Exchange (OBE) based picture, major causes of CIB and CSB are effects explicitly related to

- Different proton and neutron masses.
- Electromagnetic effects (mainly Coulomb interaction).

$^1$That means a pole in the second Riemann sheet in the negative energy axis.
Mass splitting of isovector mesons $\pi$ and $\rho$ and different coupling constants.

Mass splitting between different $\Delta$-isobar charge states.

Unknown short distance effects which are usually described by models.

Traditionally it is believed that the difference between the charge and neutral pion masses in the One-Pion-Exchange (OPE) potential accounts for a big part of CIB while the difference between the masses of neutron and proton represents the most basic cause for CSB. Pion mass differences were shown to account for a 80% of the $nn$-$pp$ scattering length difference [272]. The nucleon mass splitting also generates a difference in the kinetic energies. Some recent OBE models only consider the differences coming from nucleon mass splitting and kinematical effects [57, 58]. However, these effects can only explain about a 15% of the empirical CSB. As a consequence some models leave CSB unexplained [57] while others simply introduce a term ad hoc to explain the remaining contribution [58]. In Ref. [273] $2\pi$-exchange contributions, $\pi\rho$ diagrams and other multi-meson exchanges including the $\Delta$-isobar as intermediate states were considered to explain the empirical CSB value accurately. In Ref. [274] $2\pi$-exchange contributions with $\Delta$ were found to be noticeable to explain the empirical CIB value being $3\pi$- and $4\pi$-exchanges negligible. The difficulties arising in multi-meson exchange diagrams, in particular the energy dependence that they create, were avoided in the Bonn potential [53] by introducing two effective scalar-isoscalar $\sigma$-mesons simulating $2\pi + \pi\rho$ exchanges. In the highly successful CD-Bonn potential [55] CSB was included at the simplest one-boson-exchange diagrams with the same philosophy as its predecessor [53].

Many authors have also proposed the $\rho - \omega$ mixing as a key ingredient to understand CSB [275, 276]. In Ref. [275] the $\rho - \omega$ mixing is identified as the major source of CSB while proton and nucleon mass differences are identified to produce a minor effect. It should be noted that such a calculation is hampered by the fact that the $g_{\omega NN}$ coupling constant occurring in the CS part is about 40% larger than expected from $SU(3)$ and also from the actual value taken for the CSB potential. The $\eta - \pi^0$ has been shown to be of some relevance as well [277].

The purpose of the present chapter is to approach the problem from a renormalization point of view. While we consider long distance physics to be known and describable by non-relativistic potentials we use a physical low energy parameter such as the scattering length to encode the unknown short distance physics. As we will show in much detail this poses a problem of finiteness in physical observables when connecting different channels such as $np$, $nn$ and $pp$ (strong or Coulomb). We propose a short distance renormalization condition featuring charge independence which guarantees finiteness although an ambiguity arises. However, a natural choice of the renormalization condition works quite well when compared to measured or recommended values. While the traditional point of view tried to compute the scattering lengths, the EFT approach assumes that these scattering lengths are completely unrelated [278–282]. We feel the necessity of studying the possible connection between them from a new perspective which actually is in between, combining both points of view. We assume one scattering length to be known and exploit the concept of short distance insensitivity to determine all other scattering lengths and phase shifts from the requirement of finiteness of the scattering amplitude.
6.2 The charge dependent potential

The scattering lengths for $np$, $pp$ and $nn$ scattering in the $^1S_0$ channel are the best evidence for charge dependence of nuclear forces. Although the $np$ and $pp$ scattering lengths are well established, the $nn$ is not accurately known due basically to a lack of direct measurements using neutron-neutron collisions being the current values extracted from indirect reactions. Despite these difficulties admitted values for low energy parameters are the following $[55, 273, 274, 283]$

\[
\begin{align*}
\alpha_{pp}^C &= -7.8149(29) \text{ fm}, & r_{pp}^C &= 2.769(14) \text{ fm}, \\
\alpha_{pp}^S &= -17.3(4) \text{ fm}, & r_{pp}^S &= 2.85(4) \text{ fm}, \\
\alpha_{nn}^S &= -18.9(4) \text{ fm}, & r_{nn}^S &= 2.75(11) \text{ fm}, \\
\alpha_{np}^S &= -23.74(2) \text{ fm}, & r_{np}^S &= 2.77(5) \text{ fm},
\end{align*}
\]

where the superscripts $C$ and $S$ mean Coulomb and strong part of the interaction respectively. To quantify the effect of the CSB it is customary to define the following differences,

\[
\Delta \alpha_{CSB} \equiv \alpha_{pp}^S - \alpha_{nn}^S = 1.6(6) \text{ fm},
\]

\[
\Delta r_{CSB} \equiv r_{pp}^S - r_{nn}^S = 0.10(12) \text{ fm},
\]

In the case of CIB the following averaged are defined,

\[
\bar{\alpha} \equiv \frac{1}{2} (\alpha_{pp}^S + \alpha_{nn}^S) = -18.1(6) \text{ fm},
\]

\[
\bar{r} \equiv \frac{1}{2} (r_{pp}^S + r_{nn}^S) = 2.80(12) \text{ fm},
\]

and the the following differences arise,

\[
\Delta \alpha_{CIB} \equiv \bar{\alpha} - \alpha_{np}^S = 5.64(60) \text{ fm},
\]

\[
\Delta r_{CIB} \equiv \bar{r} - r_{np}^S = 0.03(13) \text{ fm}.
\]

As can be seen, the CIB/CSB effect is much larger for the scattering length than the effective range and this is due in part to the unnaturally large value of the NN scattering length.

Now, to take into account the various physical effects which generate charge symmetry breaking in our calculations, we consider the neutron-proton and charged-neutral pion mass differences in the OPE potentials, i.e. we take,

\[
\begin{align*}
U_{pp}^{1\pi}(r) &= -M_p \frac{f_{\pi NN}^2}{4\pi} \left( \frac{m_{\pi^0}}{m_{\pi}} \right)^2 e^{-m_{\pi^0}r} / r, \\
U_{nn}^{1\pi}(r) &= -M_n \frac{f_{\pi NN}^2}{4\pi} \left( \frac{m_{\pi^0}}{m_{\pi}} \right)^2 e^{-m_{\pi^0}r} / r, \\
U_{np}^{1\pi}(r) &= -M_{np} \frac{f_{\pi NN}^2}{4\pi} \left[ 2 \left( \frac{m_{\pi^0}}{m_{\pi}} \right)^2 e^{-m_{\pi^0}r} / r - \left( \frac{m_{\pi^0}}{m_{\pi}} \right) \frac{e^{-m_{\pi^0}r}}{r} \right].
\end{align*}
\]

with $m_{\pi^0} = 134.97 \text{ MeV}$, $m_{\pi^+} = 139.57 \text{ MeV}$ and the averaged pion mass $m_{\pi} = 138.036 \text{ MeV}$. $M_{np}$ is twice the $np$ reduced mass, $2m_{np} = 2M_nM_p/(M_n + M_p)$. The pseudo-vector pion coupling constant is
defined by chiral symmetry in terms of the \( m_\pi \), \( g_A \) and \( f_\pi \) as,

\[
\frac{f_{\pi NN}}{m_\pi} = \frac{g_A}{2f_\pi} \]

so the multiplicative factors of pion masses in the potentials put the proper pion mass in the definition. Therefore, for the OBE potential we have,

\[
V_{np}(r) = V_{np}^{1\pi}(r) + V_{np}^{1\sigma}(r) + V_{np}^{1\rho}(r) + V_{np}^{1\omega}(r) + \ldots, \tag{6.14}
\]

\[
V_{nn}(r) = V_{nn}^{1\pi}(r) + V_{nn}^{1\sigma}(r) + V_{nn}^{1\rho}(r) + V_{nn}^{1\omega}(r) + \ldots, \tag{6.15}
\]

\[
V_{pp}(r) = V_{pp}^{1\pi}(r) + V_{pp}^{1\sigma}(r) + V_{pp}^{1\rho}(r) + V_{pp}^{1\omega}(r) + \ldots. \tag{6.16}
\]

Clearly, the potentials in the different channels are not very different from one to another quantitatively. Actually, the \( \sigma \) and \( \omega \) exchange contributions coincide identically. On the other hand, the \( \pi \) and \( \rho \) take into account the different charged mesons which are exchanged. Obviously, one expects the symmetry breaking effects coming from \( \pi \) exchange to be more important than \( \rho \) exchange. Theoretical computations seem to support the previous expectations, giving (see Ref. [274]) \( \Delta \alpha_{CIB,\pi} = 3.24 \text{ fm} \) and \( \Delta \alpha_{CIB,\rho} = -0.29 \text{ fm} \). For that reason, we will not consider symmetry breaking effects coming from the \( \rho \)-meson in what follows.

### 6.3 The regular and irregular solutions

In the previous chapters it has been analyzed at length the OBE potential. We have compared results between the more traditional viewpoint of regulating the singular meson-exchange potentials by means of the introduction of vertex form factors and renormalization in coordinate space. The crucial distinction lies in the sensitivity to short-distance details: from the renormalization point of view we expect complete insensitivity to these details. In the traditional approach everything is obtained from a potential which is assumed to be valid for \( 0 \leq r < \infty \). In practice, strong form factors are included mimicking the finite nucleon size and reducing the short-distance repulsion of the potential, but the regular boundary condition is always kept. As was already noticed one of the problems with this point of view has to do with the fact that the \( {}^1S_0 \) scattering length is unnaturally large \( a_{np} = -23.74(2) \text{ fm} \), while the effective range is natural, \( r_{np} = 2.77(4) \text{ fm} \) (approximately twice the pion Compton wave length, \( \sim 2/m_\pi \)). This has dramatic consequences regarding the short-distance sensitivity and in particular the potential parameters, i.e., a problem of fine-tuning of vector coupling constants arises, as was discussed in Chapter 5. This result appears to be unnatural given the fact that vector mesons have \( 1/m_\omega = 0.25 \text{ fm} \ll 1/p_{\text{max}} = 0.5 \text{ fm} \) so they should not be crucial at least for c.m. momenta \( p \ll p_{\text{max}} \). Thus, despite the undeniable success in fitting the data, this sensitivity to short distances looks counterintuitive.

This unnatural previous results are in conflict with the renormalization viewpoint, in which we expect an insensitivity of low-energy physical observables with respect to the specific details of the potential in the short-distance region. The way to proceed is to impose renormalization conditions which eliminate the short-range sensitivity at the expense of treating low energy parameters as independent variables from the potential. An example of a renormalization condition (RC) is to fix the scattering length to avoid the fine-tuning problem. In other words, we trade the explicit dependence of the results on the short-range parameters of the potential for low-energy observables. The values of the later are usually well known by other means.
In principle there are several ways in which one can impose renormalization conditions, one popular example being counterterms. They correspond to the coupling constants of a short distance contact potential which is expanded in terms of $\delta$ functions and its derivatives

$$V_S(x) = C_0 \delta(x) + C_2 \{\nabla^2, \delta(x)\} + \ldots,$$

where the dots represent terms involving higher derivatives of the $\delta$ function. This potential is added to the usual long range potential $V_L$, and then the corresponding Schrödinger equation for $V_S + V_L$ is solved. The resulting potential is strongly singular and needs to be regularized by introducing a cut-off $r_c$, a length scale which is used to smear the $\delta$ functions. The different coupling constants $C_0(r_c)$, $C_2(r_c)$, etc, are set to reproduce the desired low-energy observables.

The disadvantage is that the procedure of using a potential to renormalize quickly runs into problems when one tries to decrease the size of the cut-off. For example, it may be impossible to reproduce certain physical observables, specifically the effective range, if the short distance cut-off is too small unless one accepts complex values for the counterterms $C_0(r_c)$ and $C_2(r_c)$ which violate either causality or off-shell unitarity (see Ref. [133] for a detailed discussion).

For that reason we use a more indirect method to renormalize which is able to avoid some of the previously mentioned complications and which consists of imposing boundary conditions at a given renormalization scale, say $R = r_c$ where $r_c$ is a cut-off radius, and as it has been shown throughout the previous chapters give reliable results. The idea is to substitute the regularity condition of the Schrödinger equation, $u_k(0) = 0$, by an arbitrary boundary condition at the origin

$$L_k(0) = \frac{u_k'(0)}{u_k(0)}.$$

The regularity condition $u_k(0) = 0$ corresponds to taking the limit $L_k(0) \to \infty$ (as $u_k'(0)$ is a constant), but by changing the precise value and energy dependence $L_k(0)$, the values of low energy observables can be fixed.

The previous procedure would in principle involve a fitting strategy, which can be avoided by taking into account the expansion in powers of $k$ of the wave function, i.e.

$$u_k(r) = u_0(r) + k^2 u_2(r) + \ldots,$$

where $u_0$ and $u_2$ obey the following equations

$$-u''_0(r) + M_N V(r) u_0(r) = 0,$$
$$-u''_2(r) + M_N V(r) u_2(r) = u_0(r),$$

asymptotically normalized to

$$u_0(r) \to 1 - r/\alpha_0,$$
$$u_2(r) \to \left(r^3 - 3\alpha_0 r^2 + 3\alpha_0 r \right)/(6\alpha_0),$$

for $r \gg 1/m_\pi$. For example, if we want to fix the scattering length (i.e. renormalization with one counterterm), we solve the corresponding equation (6.20) for the zero-energy wave function $u_0(r)$, with

\footnote{A nice presentation of the previous method is given in Ref. [265].}
the asymptotic \( r \to \infty \) boundary condition of reproducing the scattering length Eq. (6.22) but, instead of solving the previous equation from \( r = 0 \) to \( r \to \infty \), we solve it downwards from infinity to the origin. Then, we assume that \( u_2(r) \), \( u_4(r) \), etc., are subjected to regular boundary conditions at the origin, \( u_2(0) = u_4(0) = 0 \) and \( u_2'(0) = u_4'(0) = 0 \), which means to take

\[
L_k(0) = \frac{u_k'(0)}{u_k(0)}. \tag{6.24}
\]

By doing so we achieve some insensitivity at short distances. Renormalization with more counterterms is implemented by fixing more scattering parameters: one solves downwards the corresponding equations for \( u_0(r) \), \( u_2(r) \), ..., \( u_{2n}(r) \) with the asymptotic conditions of reproducing \( \alpha_0, r_0, \ldots, v_n \), and assumes trivial boundary conditions for \( u_{2n+2}(r) \), \( u_{2n+4}(r) \), etc, resulting the following logarithmic boundary condition

\[
L_k(0) = \frac{u_k'(0) + k^2 u_k'(0) + \cdots + k^{2n} u_k'(0)}{u_0(0) + k^2 u_2(0) + \cdots + k^{2n} u_{2n}(0)}. \tag{6.25}
\]

In practical computations it is convenient to introduce a short distance cut-off, \( r_c \), and then take the limit \( r_c \to 0 \).

If the potential is energy independent (but not necessarily local) as the leading large-\( N_c \) OBE potential, then different energy states are orthogonal,

\[
\int_0^\infty u_k(r)u_p(r)dr = 0, \tag{6.26}
\]

for \( k \neq p \), which requires an energy independent boundary condition at the origin, as a consequence of the next equality

\[
\int_0^\infty u_k(r)u_p(r)dr = u_k'(0)u_p'(0) - u_p u_k'(0) \bigg|_0^\infty, \tag{6.27}
\]

which means that the orthogonality condition Eq. (6.26) can be re-expressed as

\[
\frac{u_k'(0)}{u_k(0)} = \frac{u_p'(0)}{u_p(0)}, \tag{6.28}
\]

or, equivalently, \( L_k(0) = L_p(0) \), implying an energy independent boundary condition. The restriction is that orthogonality implies that we can only fix one scattering parameter, namely the scattering length. However, this has great consequences because universal relations are obtained. In particular we remember here that the effective range and the phase shifts can be expressed as functions of the scattering length as an independent parameter from the potential by\(^3\),

\[
r_0 = A + \frac{B}{\alpha_0} + \frac{C}{\alpha_0^2}, \tag{6.29}
\]

for the effective range and

\[
k \cot \delta_0 = \frac{\alpha_0 A(k) + B(k)}{\alpha_0 C(k) + D(k)}, \tag{6.30}
\]

\(^3\)See the derivation in Chapter 3.
Chapter 6. Nucleon-Nucleon interactions, charge symmetry breaking and renormalization

<table>
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<th>BC</th>
<th>$r_c$(fm)</th>
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<th>$g_{\omega NN}^*$</th>
<th>$\chi^2$/DOF</th>
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Table 6.1: Fits to the $^1S_0$ phase shift of the Nijmegen group [57] using the OBE potential with a charge dependent OPE part. We take $m_{\pi^0} = 134.97$ MeV, $m_{\pi^+} = 139.57$ MeV, $g_A = 1.29$ and $f_\pi = 92.4$ MeV [264]. We neglect the CSB coming from the $\rho$-meson and take $m_\rho = m_\omega = 770$ MeV fitting $m_\sigma$, $g_{\sigma NN}$ and $g_{\omega NN}^*$. We use the value $\alpha_{np} = -23.74$ MeV as an input when renormalizing.

Table 6.1: Fits to the $^1S_0$ phase shift of the Nijmegen group [57] using the OBE potential with a charge dependent OPE part. We take $m_{\pi^0} = 134.97$ MeV, $m_{\pi^+} = 139.57$ MeV, $g_A = 1.29$ and $f_\pi = 92.4$ MeV [264]. We neglect the CSB coming from the $\rho$-meson and take $m_\rho = m_\omega = 770$ MeV fitting $m_\sigma$, $g_{\sigma NN}$ and $g_{\omega NN}^*$. We use the value $\alpha_{np} = -23.74$ MeV as an input when renormalizing.

for the phase shifts. This formulas allow, for instance, to determine the $^1S_0$ phase shifts for $np$, $nn$, $pp(s)$ and $pp(c)$ from their corresponding scattering lengths $\alpha_{np}$, $\alpha_{nn}$, $\alpha_{pp}^s$ and $\alpha_{pp}^c$ respectively. The previous computation can be compared with the experimental values for these quantities in order to test the renormalization procedure. In particular the $np$ $^1S_0$ channel will be used, as in previous chapters, to fix the OBE parameters $m_\sigma$, $g_{\sigma NN}$ and $g_{\omega NN}^*$. The $np$ charge symmetry breaking OBE potential in the $^1S_0$ channel is then,

$$V_{^1S_0}^{CSB}(r) = V_{np}^{1\pi} - \frac{g_{\sigma NN}^2}{4\pi} \frac{e^{-m_\sigma r}}{r} + \frac{g_{\omega NN}^*}{4\pi} \frac{e^{-m_\omega r}}{r},$$

(6.31)

where CSB is implemented in $V_{np}^{1\pi}$ by Eq. (6.13). The results of the fits to the Nijmegen $np$ phase shifts in the $^1S_0$ channel using the different approaches are shown in Table 6.1 for charge dependent OPE potential. In the case of regular boundary condition we also fit the scattering length to the experimental value $\alpha_{np} = -23.74$ fm. Due to the strong correlations that appear between the parameters (see Fig. 5.4) an error analysis is difficult to carry out. We define the parameter error as the projection of the 68% confidence level ellipse over the given parameter axis. As we can see again a fine-tuning problem appears in the case of the regular boundary condition while bigger errors are achieved in the case of renormalization. We can see the large uncertainty on the value of $g_{\omega NN}^*$, which shows that there is a greater insensitivity to shorter distances after renormalization.

The spurious bound state problem has been discussed in Chapter 5 and will be omitted here for simplicity. However it is worth mentioning that the three different scenarios correspond to selecting a potential possessing spurious bound states or not. This is shown in Fig 6.1 where we represent the zero energy wave function for the three cases. In the regular case, the OBE potential with a big $g_{\omega NN}^*$ is free of spurious bound states. However if a small $g_{\omega NN}^*$ is chosen, then one has to deal with a spurious bound state which is very close to the first spurious bound state in the renormalization case.

The short distance sensitivity can be vividly seen in Fig. 6.2, where the regular (parabola like curve) as well as the renormalized (flat curve) effective range for the OBE potential are shown as a function of $g_{\omega NN}^*$. For simplicity only the solution with the small $g_{\omega NN}^*$ (Regular solution I) is represented.

We can say that contrary to common wisdom, but according to our naive expectations, no strong short range repulsion is essential if the inaccessible short distance physics is encoded through the scattering length. The moral is that building $\alpha_0$ from the potential is equivalent to absolute knowledge at short distances, and in the $^1S_0$ channel a strong fine-tuning is needed.
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Figure 6.1: Zero energy wave function for the singlet np $^1S_0$ channel as a function of distance (in fm) and for the different scenarios with large and small $\omega$—couplings. This wave function goes asymptotically to $u_0(r) \to 1 - r/\alpha_0$ with $\alpha_0 = -23.74$ fm the scattering length in this channel. The zero at about $r = 0.5$ fm signals the existence of a spurious bound state.

Figure 6.2: Dependence of the effective range with respect to $g^*_\omega NN$ in the regular case with a small coupling constant and in the renormalized one.

6.4 Renormalization with Coulomb interactions

In this section we generalize the previously discussed renormalization approach to the case of proton-proton scattering, where the infinite range of the Coulomb interaction will pose some problems. The corresponding s-wave reduced Schrödinger equation is

\[-u_k^{C''} + M_p \left( V_{pp}(r) + \frac{\alpha}{r} \right) u_k^C = k^2 u_k^C, \tag{6.32} \]

where $k$ is the c.m. momentum, $M_p$ the proton mass, $V_{pp}(r)$ the strong proton-proton potential, and $\alpha \simeq 1/137$ is the fine structure constant. Actually, the current discussion is tightly linked to the corresponding one for the two potential formula presented in [285].
6.4.1 Coulomb scattering at zero energy

The longest range piece of the proton-proton interaction is the Coulomb repulsion between the protons. Ignoring any strong effects, zero energy proton-proton scattering in s–waves can be described by the reduced Schrödinger equation

\[ -v''_0 + \frac{2}{a_B} v_0(r) = 0, \]  

(6.33)

where \( a_B \) is the proton Bohr radius, which is defined as \( a_B = 2/M_p \alpha = 56.62 \text{ fm} \). The previous equation has the following two linearly independent solutions

\[ v_{0,\text{reg}}(r) = \frac{a_B}{2} \sqrt{x} I_1(2 \sqrt{x}), \]  

(6.34)

\[ v_{0,\text{irr}}(r) = 2 \sqrt{x} K_1(2 \sqrt{x}), \]  

(6.35)

where \( x = 2r/a_B \) and \( K_1(x) \) and \( I_1(x) \) are modified Bessel functions of the first and second kind respectively (see for example [286]). At short distances these solutions behave as

\[ v_{0,\text{reg}}(r) \to r + \frac{r^2}{a_B} + \frac{r^3}{3a_B} + \mathcal{O}(r^4), \]  

(6.36)

\[ v_{0,\text{irr}}(r) \to 1 + \frac{2r}{a_B} \left\{ \log \frac{2r}{a_B} + 2\gamma_E - 1 \right\} + \left( \frac{2r}{a_B} \right)^2 \left( \frac{1}{2} \log \frac{2r}{a_B} + \gamma_E - \frac{5}{4} \right) + \mathcal{O}(r^3), \]  

(6.37)

where \( \gamma_E = 0.57722 \) is the Euler-Mascheroni constant. The previous means in particular that \( v_{0,\text{reg}} \) is the short distance regular solution and \( v_{0,\text{irr}} \) the short distance irregular solution. In principle, in the absence of any strong potential, \( v_{0,\text{reg}} \) would be the zero energy solution for the repulsive Coulomb potential. The presence of the strong interaction between the protons means that the zero-energy asymptotic solution for \( r \to \infty \) will be in general a linear combination of \( v_{0,\text{reg}} \) and \( v_{0,\text{irr}} \).

For the proton-proton system the Coulomb scattering length is related with the asymptotic behaviour of the zero energy wavefunction at large enough distances

\[ v_0^C(r) = v_{0,\text{irr}}^C(r) - \frac{v_{0,\text{reg}}^C(r)}{\alpha_{0,C}}, \]  

(6.38)

where \( v_{0,\text{reg}} \) and \( v_{0,\text{irr}} \) are the previously defined regular and irregular zero energy wave functions, and \( \alpha_{0,C} \) is the s–wave Coulomb scattering length. If the Coulomb interaction is switched off by taking \( a_B \to \infty \), the previous wave function reduces to the corresponding one for finite range forces, \( v_0(r) = 1 - r/\alpha_0 \).

6.4.2 Effective range

The Coulomb effective range is given by the following formula

\[ r_{0,C} = 2 \int_0^\infty dr \left[ v_0^C(r)^2 - u_0^C(r)^2 \right], \]  

(6.39)
where \( v_C^0(r) \) is the Coulomb zero energy solution given by Eq. (6.38), and \( u_C^0(r) \) is the full zero energy solution to the Schrödinger equation

\[
- u''_0^C + \left( M_p V_{pp}(r) + \frac{2}{a_B r} \right) u_C^0(r) = 0, \tag{6.40}
\]

subjected to the asymptotic boundary condition

\[
u_0^C(r) \to v_C^0(r), \quad \text{for } r \to \infty. \tag{6.41}\]

By making use of the superposition principle, the solution \( u_0^C \) can be decomposed as

\[
u_0^C(r) = u_{0,\text{irr}}^C(r) - \frac{u_{0,\text{reg}}^C(r)}{\alpha_{0,C}}, \tag{6.42}\]

where \( u_{0,\text{irr}}^C \) and \( u_{0,\text{reg}}^C \) are solutions of the zero energy Schrödinger equation, Eq. (6.40), behaving asymptotically \( (r \to \infty) \) as

\[
u_{0,\text{irr}}^C(r) \to v_{0,\text{irr}}^C(r), \tag{6.43}\]

\[
u_{0,\text{reg}}^C(r) \to v_{0,\text{reg}}^C(r). \tag{6.44}\]

The subscripts \( \text{reg} \) and \( \text{irr} \) do not refer to the regularity of the solutions at the origin, but with the long range behaviour of the full solutions.

By plugging the decomposition of the full and purely Coulomb wave functions, Eqs. (6.42) and (6.38), into the integral representation of the Coulomb effective range, Eq. (6.39), we obtain the following correlation between the Coulomb scattering length and effective range

\[
r_{0,C} = A_{0}^C + \frac{B_{0}^C}{\alpha_{0,C}} + \frac{C_{0}^C}{\alpha_{0,C}^2}, \tag{6.45}\]

which is a direct generalization of Eq. (6.29) for the non-Coulomb case. The Coulomb correlation functions \( A_{0}^C, B_{0}^C \) and \( C_{0}^C \) are given by the integral expressions below

\[
A_{0}^C = 2 \int_0^\infty dr (v_{0,\text{irr}}^C(r)^2 - u_{0,\text{irr}}^C(r)^2), \tag{6.46}\]

\[
B_{0}^C = -4 \int_0^\infty dr (v_{0,\text{irr}}^C(r) v_{0,\text{reg}}^C(r) - u_{0,\text{irr}}^C(r) u_{0,\text{reg}}^C(r)), \tag{6.47}\]

\[
C_{0}^C = 2 \int_0^\infty dr (v_{0,\text{reg}}^C(r)^2 - u_{0,\text{reg}}^C(r)^2). \tag{6.48}\]

### 6.4.3 Coulomb scattering at finite energy and Coulomb effective range expansion

The definition of the phase shifts in the presence of the Coulomb potential is related with the behaviour of the wave function at long distances, which is given by

\[
u_K^C(r) \to \cot \delta_0^C(k) F_0(\eta, \rho) + G_0(\eta, \rho), \tag{6.49}\]
where $\delta_C^0(k)$ is the Coulomb-modified proton-proton phase shift, $k$ the c.m. momentum and $F_0(\eta, \rho)$ and $G_0(\eta, \rho)$, with $\eta = 1/(ka_B)$ and $\rho = kr$, are the $s$–wave Coulomb wave functions [286]. The $u_k^C$ wave function is the solution to the reduced Schrödinger equation Eq. (6.32). The $F_0(\eta, \rho)$ and $G_0(\eta, \rho)$ wave functions behave asymptotically ($r \to \infty$) as

$$F_0 \to \sin (kr - \eta \log(2kr) + \sigma_0),$$  

$$G_0 \to \cos (kr - \eta \log(2kr) + \sigma_0),$$

with $\sigma_0$ the pure Coulomb phase shift, which is defined as

$$e^{2i\sigma_0} = \frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}.$$  

The phase shift in presence of the infinite-ranged Coulomb force does not obey the usual effective range expansion, valid for short-ranged potentials, but a Coulomb-modified effective range expansion, given by

$$k \cot \delta_C^0 C^2(\eta) + \frac{2}{a_B} h(\eta) = -\frac{1}{a_{0,C}} + \frac{1}{2} r_{0,C} k^2 + \sum_{n=2}^{\infty} v_{n,C} k^{2n},$$

with $C(\eta)$ and $h(\eta)$ defined as

$$C^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta} - 1},$$  

$$h(\eta) = \eta^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2 + \eta^2)} - \log \eta - \gamma_E.$$  

For obtaining the Coulomb extension of Eq. (6.30) we use again the superposition principle to write $u_k^C$ in the following way

$$C(\eta) u_k^C(r) = \left( k \cot \delta_C^0 C^2(\eta) + \frac{2}{a_B} h(\eta) \right) u_{k,\text{reg}}^C(r) - u_{k,\text{irr}}^C(r),$$

where $u_{k,\text{reg}}^C$ and $u_{k,\text{irr}}^C$ are solutions of Eq. (6.32), which obey the asymptotic boundary conditions

$$u_{k,\text{reg}}^C(r) \to \frac{F_0(\eta, \rho)}{kC(\eta)},$$  

$$u_{k,\text{irr}}^C(r) \to -C(\eta) G_0(\eta, \rho) + \frac{2\eta h(\eta)}{C(\eta)} F_0(\eta, \rho),$$

for $r \to \infty$. These two solutions have been normalized in such a way that for $k \to 0$ they coincide with the previously defined zero-energy wave functions $u_{0,\text{reg}}^C$ and $u_{0,\text{irr}}^C$ (see Eqs. (6.43) and (6.44)).

Once these definitions have been made, it is straightforward to obtain the correlation

$$k \cot \delta_C^0 C^2(\eta) + \frac{2}{a_B} h(\eta) = \frac{\alpha_{0,C} A^C(k) + B^C(k)}{\alpha_{0,C} C^0(k) + D^C(k)},$$

$$\sigma_0$$

The phase shift in presence of the infinite-ranged Coulomb force does not obey the usual effective range expansion, valid for short-ranged potentials, but a Coulomb-modified effective range expansion, given by

$$k \cot \delta_C^0 C^2(\eta) + \frac{2}{a_B} h(\eta) = -\frac{1}{a_{0,C}} + \frac{1}{2} r_{0,C} k^2 + \sum_{n=2}^{\infty} v_{n,C} k^{2n},$$

with $C(\eta)$ and $h(\eta)$ defined as

$$C^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta} - 1},$$  

$$h(\eta) = \eta^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2 + \eta^2)} - \log \eta - \gamma_E.$$  

For obtaining the Coulomb extension of Eq. (6.30) we use again the superposition principle to write $u_k^C$ in the following way

$$C(\eta) u_k^C(r) = \left( k \cot \delta_C^0 C^2(\eta) + \frac{2}{a_B} h(\eta) \right) u_{k,\text{reg}}^C(r) - u_{k,\text{irr}}^C(r),$$

where $u_{k,\text{reg}}^C$ and $u_{k,\text{irr}}^C$ are solutions of Eq. (6.32), which obey the asymptotic boundary conditions

$$u_{k,\text{reg}}^C(r) \to \frac{F_0(\eta, \rho)}{kC(\eta)},$$  

$$u_{k,\text{irr}}^C(r) \to -C(\eta) G_0(\eta, \rho) + \frac{2\eta h(\eta)}{C(\eta)} F_0(\eta, \rho),$$

for $r \to \infty$. These two solutions have been normalized in such a way that for $k \to 0$ they coincide with the previously defined zero-energy wave functions $u_{0,\text{reg}}^C$ and $u_{0,\text{irr}}^C$ (see Eqs. (6.43) and (6.44)).

Once these definitions have been made, it is straightforward to obtain the correlation

$$k \cot \delta_C^0 C^2(\eta) + \frac{2}{a_B} h(\eta) = \frac{\alpha_{0,C} A^C(k) + B^C(k)}{\alpha_{0,C} C^0(k) + D^C(k)},$$
where \( \mathcal{A}(k), \mathcal{B}(k), \mathcal{C}(k) \) and \( \mathcal{D}(k) \) are defined as

\[
\mathcal{A}(k) = \lim_{r_c \to 0} \left[ u_{0,\text{irr}}(r_c) u_{k,\text{irr}}'(r_c) - u_{0,\text{reg}}(r_c) u_{k,\text{irr}}'(r_c) \right], \\
\mathcal{B}(k) = \lim_{r_c \to 0} \left[ u_{0,\text{reg}}(r_c) u_{k,\text{irr}}'(r_c) - u_{0,\text{reg}}(r_c) u_{k,\text{irr}}'(r_c) \right], \\
\mathcal{C}(k) = \lim_{r_c \to 0} \left[ u_{0,\text{irr}}(r_c) u_{k,\text{reg}}'(r_c) - u_{0,\text{irr}}(r_c) u_{k,\text{reg}}'(r_c) \right], \\
\mathcal{D}(k) = \lim_{r_c \to 0} \left[ u_{0,\text{reg}}(r_c) u_{k,\text{reg}}'(r_c) - u_{0,\text{reg}}(r_c) u_{k,\text{reg}}'(r_c) \right].
\]

The long distance correlation between the scattering length and effective range looks as

\[
r_{0,np} = A_{np} + \frac{B_{np}}{\alpha_{0,np}} + \frac{C_{np}}{\alpha_{0,pp}^2},
\]

\[
r_{0,pp} = A_{pp} + \frac{B_{pp}}{\alpha_{0,pp}} + \frac{C_{np}}{\alpha_{0,pp}^2},
\]

\[
r_{0,nn} = A_{nn} + \frac{B_{nn}}{\alpha_{0,nn}} + \frac{C_{pp}}{\alpha_{0,pp}^2},
\]

\[
r_{0,\text{C,pp}} = A_{pp} + \frac{B_{pp}}{\alpha_{0,pp}^2} + \frac{C_{pp}}{\alpha_{0,pp}^2},
\]

while the phase shifts are given by

\[
k \cot \delta_{0,np} = \frac{\alpha_{0,np} A_{np}(k) + B_{np}(k)}{\alpha_{0,pp} C_{np}(k) + D_{np}(k)},
\]

\[
k \cot \delta_{0,nn} = \frac{\alpha_{0,nn} A_{nn}(k) + B_{nn}(k)}{\alpha_{0,nn} C_{nn}(k) + D_{nn}(k)},
\]

\[
k \cot \delta_{0,pp} = \frac{\alpha_{0,pp} A_{pp}(k) + B_{pp}(k)}{\alpha_{0,pp} C_{pp}(k) + D_{pp}(k)},
\]

\[
C^2(\eta) k \cot \delta_{0,pp} + \frac{2}{\alpha_B} h(\eta) = \frac{\alpha_{0,pp} A_{pp}(k) + B_{pp}(k)}{\alpha_{0,pp} C_{pp}(k) + D_{pp}(k)},
\]

We remind that the scattering lengths are independent of the potentials.

In Fig. 6.3 we show the universal functions \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) and \( \mathcal{D} \) for the four cases considered. As can be seen, for \( nn, np \) and \( pp(s) \) they coincide even though the potentials are different, which means in particular that most of the CIB and CSB low energy effects come from the difference in the scattering length. It is also interesting to see that the Coulomb corrections to the pp(c) universal functions differ increasingly for higher energies.

### 6.5 The short distance connection

As it is well known, at high energies the \( nn, np \) and \( pp \) phase-shifts start to resemble each other, which means that charge invariance is respected for large enough momenta. Most of the charge invariance and charge symmetry breaking effects only affect the low energy behaviour, specifically the scattering lengths, where one finds \( \Delta \alpha_{\text{CIB}} = 5.64(60) \text{ fm} \) and \( \Delta \alpha_{\text{CSB}} = 1.6(6) \text{ fm} \). When one considers the effective range,
the symmetry breaking effects are already ten times smaller than in the scattering length case, being of the order of the tenth of a fermi. The problem is how to explain these differences.

In the traditional approach all the CIB and CSB effects are explained via the OBE potential. The Schrödinger equation is integrated from the origin to infinity with regular boundary conditions and all the difference between scattering observables must come from the potential. In the renormalization approach things get more involved: there are explicit contributions coming from short distance operators which are used to weaken the short distance sensitivity. The problem is how to implement either charge independence or its breaking within this approach in a regulator independent way. If we assume that at lowest order all the charge independence breaking comes from the finite range potential, one is tempted to identify short distance charge independence with identical logarithmic boundary conditions. For example, if we relate the $nn$ and $np$ problems with

$$
\frac{u'_nn(r_c)}{u_{nn}(r_c)} = \frac{u'_{np}(r_c)}{u_{np}(r_c)},
$$

we will find that this relation produces log-divergent results for the OBE potential in the limit $r_c \to 0$.

Another option is to regulate with a short distance delta potential

$$
V_C(r; r_c) = \frac{C_0(r_c)}{4\pi r_c^2} \delta(r - r_c),
$$

which corresponds to a specific regularization of the $\delta$ function potential, and assume that charge independence at short distance is equivalent to the identity $C_{0,nn}(r_c) = C_{0,np}(r_c) = C_{0,pp}(r_c) = C_0(r_c)$. This
choice leads to the following logarithmic boundary condition between \( nn \) and \( np \)

\[
\frac{1}{M_n} \left( \frac{u'_{nn}(r_c)}{u_{nn}(r_c)} - \frac{1}{r_c} \right) = \frac{1}{M_{np}} \left( \frac{u'_{np}(r_c)}{u_{np}(r_c)} - \frac{1}{r_c} \right),
\]

(6.74)

where \( M_n \) is the neutron mass and \( M_{np} \) is twice the reduced \( np \) mass. The counterterm conditions also run into the same cut-off dependence problems than the logarithmic boundary condition. This means in particular that the two previous proposals are regulator dependent, and hence model dependent, and pose a serious problem on what is meant by charge independence of short distance operators. We will show that by using the hypothesis of charge independence at short distances together with finiteness, a relation between them can be established which works rather satisfactorily.

At short distances all the \( pp \) (strong/Coulomb), \( np \) and \( nn \) potentials have an attractive Coulomb like behaviour

\[
2\mu_{NN} V_{NN}(r) \rightarrow -\frac{1}{R},
\]

(6.75)

where \( NN \) either refers to \( pp \) (strong/Coulomb), \( np \) or \( nn \), and \( \mu_{NN} \) and \( V_{NN} \) are the corresponding reduced mass and potential. The constant \( R \) depends on the problem; for the OBE potential with the additional simplification of taking \( m_\omega = m_\rho \) and defining \( g_{\omega NN} \), we get the scales

\[
\begin{align*}
\frac{1}{R_{np}} &= M_{np} \left( 2f_{\pi NN}^2 + 2g_{\sigma NN}^2 - g_{\omega NN}^2 \right), \\
\frac{1}{R_{nn}} &= M_n \left( 2f_{\pi NN}^2 + 2g_{\sigma NN}^2 - g_{\omega NN}^2 \right), \\
\frac{1}{R_{pp}^S} &= M_p \left( 2f_{\pi NN}^2 + 2g_{\sigma NN}^2 - g_{\omega NN}^2 \right), \\
\frac{1}{R_{pp}^C} &= M_p \left( 2f_{\pi NN}^2 + 2g_{\sigma NN}^2 - g_{\omega NN}^2 - \alpha \right),
\end{align*}
\]

(6.76)

(6.77)

(6.78)

(6.79)

with \( M_{np} \) twice the reduced \( np \) mass, \( M_{np} = 2\mu_{np} \) and where CSB is implicit in \( f_{\pi NN} \). As a consequence of the short distance Coulomb singularity, the wave function at short distances approximately behaves as linear combinations of attractive Coulomb wavefunctions

\[
\begin{align*}
u_{k,NN}(r) &\rightarrow -\pi \sqrt{x} Y_1(2\sqrt{x}) \frac{R}{\alpha_0} \sqrt{x} J_1(2\sqrt{x}) + \mathcal{O}(m\gamma, mR, k^2 r^2, r/R),
\end{align*}
\]

(6.80)

where \( x = r/R, R \) can either be \( R_{nn}, R_{np}, \) and \( R_{pp} \) (strong/Coulomb), \( \alpha_0 \) the scattering length of the problem and \( m \) generically denotes the mass of any of the exchanged bosons. The expected \( mR \) contributions will only shift the irregular solutions by a constant.

The previous behaviour can be quite problematic at short distances as we can see if we consider the log-derivative of the wave function at small enough cut-off radii, which behaves as

\[
R \frac{u'_{k,NN}(r_c)}{u_{k,NN}(r_c)} \rightarrow -2\gamma - \frac{R}{\alpha_0} - \log \frac{r_c}{R} + \ldots,
\]

(6.81)

The solution of \(-y''(r) - \frac{y'(r)}{r} y(r) = 0\) is \( y(r) = c_1 \sqrt{x} J_1(2\sqrt{x}) + 2ic_2 \sqrt{x} Y_1(2\sqrt{x}) \) with \( x = r/R \) and \( c_1 \) and \( c_2 \) two constants that determine the linear combination of solutions. We fix these constants to fulfill the asymptotic behaviour \( y(r \rightarrow \infty) \rightarrow 1 - \frac{r_c}{\alpha_0} \) with \( \alpha_0 \) the scattering length of the given problem.
where the dots refer to higher order terms, like $mr_c$ or $k^2r_c^2$ corrections. With this behaviour, we can see that naively identifying the log-derivative at the cut-off radius in order to obtain correlations between observables of the different two nucleon systems will yield divergent results. For example, relating $np$ and $nn$

$$
\frac{u'_{k,nn}(r_c)}{u_{k,nn}(r_c)} = \frac{u'_{k,np}(r_c)}{u_{k,np}(r_c)},
$$

(6.82)
generates the singularity

$$
\frac{1}{R_{np}} \log \left( \frac{r_c}{R_{np}} \right) - \frac{1}{R_{nn}} \log \left( \frac{r_c}{R_{nn}} \right),
$$

(6.83)

This singularity is indeed mild, as it can only be seen at very short distances (depending on how small is the difference between $1/R_{nn} - 1/R_{np}$), but sooner or later will ruin our results.

Under these circumstances there is a quantity that can be constructed from the log-derivative at short distance that is finite in the $r_c \to 0$ limit. This quantity is the following

$$
S = R \frac{u'(r_c)}{u(r_c)} + \log \left( \frac{r_c}{R} \right), \quad r_c \ll R,
$$

(6.84)

which is cut-off and energy independent. This suggests that different scattering problems, having different short distance constants but the same logarithmic scale dependence, can be connected in such a way that the scale dependence is eliminated. This is done by equating the corresponding $S$'s

$$
S_1 = S_2 ,
$$

(6.85)

where 1 and 2 refer to two different $NN = nn, np, pp(s), pp(c)$ cases. We can give here two examples of the adequacy of the short distance connection. The first one is to obtain the strong $pp$ scattering length from the experimental Coulomb one, $\alpha_{C,pp}^{S} = -7.8149 \text{ fm}$ yielding $\alpha_{0,pp}^{S} = -18.46 \text{ fm}$, a not unreasonable results (to be compared with the extraction $\alpha_{0,pp}^{S} = -17.3(4) \text{ fm}$, see Ref. [270], where the error comes from model-dependence). The CD-Bonn potential gives a value of $\alpha_{0,pp}^{S} = -17.46 \text{ fm}$. The extracted effective ranges are $r_{0,pp}^{C} = 2.735 \text{ fm}$ and $r_{0,pp}^{S} = 2.789 \text{ fm}$. As a second example, by taking the $np$ scattering length as input, $\alpha_{0,np} = -23.74 \text{ fm}$, we can obtain all the NN scattering lengths, giving $\alpha_{0,nn} = -19.626 \text{ fm}, \alpha_{0,pp}^{S} = -17.806 \text{ fm}$ and $\alpha_{0,pp}^{C} = -7.706 \text{ fm}$ for the scattering lengths and $r_{0,np} = 2.672 \text{ fm}, r_{0,nn} = 2.771 \text{ fm}, r_{0,pp}^{S} = 2.802 \text{ fm}$ and $r_{0,pp}^{C} = 2.747 \text{ fm}$ for the effective ranges. A remarkable aspect of the previous computation is that one obtains

$$
\Delta \alpha_{CIB} = 5.024 \text{ fm}, \quad \Delta r_{CIB} = 0.115 \text{ fm},
$$

(6.86)

$$
\Delta \alpha_{CSB} = 1.82 \text{ fm}, \quad \Delta r_{CSB} = 0.031 \text{ fm},
$$

(6.87)

which agree within error estimations with the expected values for these two quantities [270] (see also Eqs. (6.5)- (6.10)). In Table 6.2 we summarize the results obtained with the short distance connection (renormalized) and the one obtained integrating upward with a regular boundary condition (regular). We can see that in the case of a big $g_{NN}^2$ the regular solution does a poor job in calculation the low
energy parameters (LEP) in other channels. The CD-Bonn potential \cite{55} corresponds with this scenario, i.e., a big SU(3) breaking coupling constant but with any spurious bound state. Looking at this table one can understand why in this model a different mass for a ficticious $\sigma$-meson is used in each NN channel. The strong fine-tuning that appears in this situation hinders the relation between different NN problems.

Note, however, that the previous relation is not the only possible covariant short distance connection, as we could have defined

$$ S' = R \frac{u'(r_c)}{u(r_c)} + \log \left( \frac{\lambda r_c}{R} \right), \quad r_c \ll R, \quad (6.90) $$

with $\lambda$ some arbitrary constant, which depends on the specific NN problem which is being considered. A natural choice is to take $\lambda$ of order unity, which does not make much difference between different choices of $C$ due to the weak logarithmic behaviour. It must be stressed though that the results are not unique: arbitrary $\lambda$’s can be introduced to better connect the different two nucleon systems. As the hypothesis of charge dependence of short distance operator cannot be implemented in a completely model independent way, we will choose to take $\lambda_{nn} = \lambda_{np} = \lambda_{pp}$ at first order. We have already seen that this simple condition generates quite accurate results, meaning that corrections due to $\Delta \lambda$ are indeed small, and can be effectively considered as higher order effects, confirming thus our expectations. As an example of what values of $\Delta \lambda$ to expect, if we try to correlate the strong and Coulomb scattering lengths, we will get $\lambda_{pp}^s - \lambda_{pp}^C = 0.0321 - 0.0471$, where the range given is a consequence of the uncertainty in $\alpha_{pp}^s = -17.3(4)$ fm.

To clarify the implications of the short distance connection, let us consider two different problems 1 and 2, which have the associated short distance Coulomb length scales $R_1$ and $R_2$. In other words, we have the differential equations

$$ - u''_1 + 2\mu_1 V_1(r)u_1(r) = k^2 u_1(r), \quad (6.91) $$
$$ - u''_2 + 2\mu_2 V_2(r)u_2(r) = k^2 u_2(r), \quad (6.92) $$

where the reduced potentials behave as $1/r$ at short distances

$$ 2\mu_1 V_1(r) \rightarrow -\frac{1}{R_1 r}, \quad (6.93) $$
$$ 2\mu_2 V_2(r) \rightarrow -\frac{1}{R_2 r}. \quad (6.94) $$

These two problems are related at short distances through the boundary condition corresponding to the short distance connection $S_1 = S_2$

$$ R_2 \frac{u'_2(r_c)}{u_2(r_c)} = \log \frac{R_1}{R_2} + R_1 \frac{u'_1(r_c)}{u_1(r_c)}. \quad (6.95) $$

If we have only fixed the scattering length, the above condition becomes energy independent when the cut-off is small enough, which means that it can be evaluated with the zero-energy wave functions of the two-body systems 1 and 2. Using the superposition principle, the previous zero energy wave functions can be written as

$$ u_1(r) = v_1(r) - \frac{1}{\alpha_1} w_1(r), \quad (6.96) $$
$$ u_2(r) = v_2(r) - \frac{1}{\alpha_2} w_2(r). \quad (6.97) $$
These wave functions can be included in Eq. (6.95), yielding the following relation between the scattering lengths $\alpha_1$ and $\alpha_2$ of the two different problems

$$\frac{A}{\alpha_1} = \frac{B}{\alpha_2} + C + \frac{D}{\alpha_1 \alpha_2}. \quad (6.98)$$

Therefore, if we make the hypothesis of charge independence at short distances \(^5\)

$$S_{np} = S_{nn} = S_{pp}^S = S_{pp}^C, \quad (6.99)$$

and by making use of the superposition principle, we can write

$$u_{0, np}(r) = v_{0, np}(r) - \frac{1}{\alpha_{0, np}} w_{0, np}(r), \quad (6.100)$$

$$u_{0, nn}(r) = v_{0, nn}(r) - \frac{1}{\alpha_{0, nn}} w_{0, nn}(r), \quad (6.101)$$

$$u_{0, pp}^S(r) = v_{0, pp}^S(r) - \frac{1}{\alpha_{0, pp}^S} w_{0, pp}^S(r), \quad (6.102)$$

$$u_{0, pp}^C(r) = v_{0, pp}^C(r) - \frac{1}{\alpha_{0, pp}^C} w_{0, pp}^C(r), \quad (6.103)$$

so we get the bilinear relations between all scattering lengths

$$\frac{A_{nn}}{\alpha_{nn}} = \frac{B_{nn}}{\alpha_{np}} + C_{nn} + \frac{D_{nn}}{\alpha_{nn} \alpha_{np}}, \quad (6.104)$$

$$\frac{A_{pp}^S}{\alpha_{pp}^S} = \frac{B_{pp}^S}{\alpha_{np}} + C_{pp}^S + \frac{D_{pp}^S}{\alpha_{pp}^S \alpha_{np}}, \quad (6.105)$$

$$\frac{A_{pp}^C}{\alpha_{pp}^C} = \frac{B_{pp}^C}{\alpha_{np}} + C_{pp}^C + \frac{D_{pp}^C}{\alpha_{pp}^C \alpha_{np}}, \quad (6.106)$$

etc. In Fig. 6.4 we show the dependence of the scattering lengths as obtained from the $np$ scattering length and the previous correlations. As can be seen, the correlations work rather well, confirming the idea that finiteness is a good criterion to implement charge independence of short distance operators.

---

\(^5\)Generally one might expect

$$S_{pp}^C = S_{pp}^S + \alpha S_{pp}^{(1)} + \ldots$$
Chapter 6. Nucleon-Nucleon interactions, charge symmetry breaking and renormalization

Table 6.2: NN low-energy parameters in the different scenarios. Renormalization only needs the np scattering length as an input parameter, all the other are calculated without ambiguities. The OBEP parameters have been fitted in the np case and kept the same in the other cases. Here (c) means Coulomb interaction is switch on.

<table>
<thead>
<tr>
<th>NN</th>
<th>LEP</th>
<th>Renormalization</th>
<th>Regular solution-I</th>
<th>Regular solution-II</th>
<th>CD-Bonn [55]</th>
<th>Experimental [55]</th>
</tr>
</thead>
<tbody>
<tr>
<td>np</td>
<td>α₀ [fm]</td>
<td>input</td>
<td>-23.737</td>
<td>-23.738</td>
<td>-23.738</td>
<td>-23.74(2)</td>
</tr>
<tr>
<td></td>
<td>r₀ [fm]</td>
<td>2.672</td>
<td>2.678</td>
<td>2.677</td>
<td>2.671</td>
<td>2.77(5)</td>
</tr>
<tr>
<td>pp</td>
<td>α₀ [fm]</td>
<td>-17.806</td>
<td>-18.350</td>
<td>-20.088</td>
<td>-17.46</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>r₀ [fm]</td>
<td>2.802</td>
<td>2.799</td>
<td>2.768</td>
<td>2.845</td>
<td>-</td>
</tr>
<tr>
<td>pp(c)</td>
<td>α₀ [fm]</td>
<td>-7.706</td>
<td>-7.824</td>
<td>-8.265</td>
<td>-7.815</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>r₀ [fm]</td>
<td>2.747</td>
<td>2.641</td>
<td>2.693</td>
<td>2.733</td>
<td>2.769(14)</td>
</tr>
<tr>
<td></td>
<td>r₀ [fm]</td>
<td>2.771</td>
<td>2.780</td>
<td>2.763</td>
<td>2.819</td>
<td>2.75(11)</td>
</tr>
</tbody>
</table>

It is interesting to see how the short distance connection works at finite energy as well. In particular we will see that if a specific NN channel is given one can predict the phase shifts for the remaining ones. In Fig. 6.5 we plot the extracted nn and pp(c) phase shifts when the OBE parameters have been fixed from the 1S₀ Nijmegen np phase shifts. We have computed these phase shifts renormalizing in the np channel, i.e., fixing α₀,np as input and integrating inward the Schrödinger equation, then using Eq. (6.99) we connect with the other channels. As we can see the short distance connection can be used to predict the 1S₀ phase shifts for the rest of the channel with a high degree of accuracy.

6.6 Application to proton-proton fusion

Finally we would like to analyze further consequences of the short distance connection assumed by Eq. (6.99). An interesting process is the proton-proton fusion reaction pp → d e⁺νₑ which is of central importance to stellar physics and neutrino astro-physics. In fact, it is the dominant solar neutrino source. The temperature in the Sun core is around Tₑ = 15 × 10⁶ K which means that we have protons of momentum p ∼ (2mₑTₑ)¹/² ∼ 1.1 MeV. At these low energies, the reaction is dominated by the 1S₀ → d nuclear transition. The Gamow-Teller (GT) matrix element for this process (without MECs) is given by,

\[ A_S M_{GT} = \int_0^\infty dr \ u_γ(r) u_{0,pp}(r) \]  

(6.107)

where \( u_{0,pp} \) is the zero energy reduced wave function for the pp(c) system which can be related with the np problem by Eq. (6.99). Then taking α₀,np as input and integrating in we can calculate \( u_{0,pp} \). For deuteron we take as a first approximation the normalized bound state,

\[ u_γ(r) \rightarrow A_S e^{-γ_d r} \]  

(6.108)

with \( γ_d = 0.2316 \text{fm}^{-1} \) and integrate inward the Schrödinger equation with negative energy \( E = -γ_d^2/M_{np} \). We obtain a value \( M_{GT} = 5.189 \text{ fm} \) to be compared to a more sophisticated one [287] using Argonne v18 wave functions \( M_{GT}|_{AV18} = 4.859 \text{ fm} \).

In Fig. 6.6 we show the GT matrix element correlation with the np scattering length compared with the AV18 calculation. Of course we have not include the tensor force which mixed S and D waves in the calculation of the deuteron. However we can appreciate that our numbers are not very far from much more elaborate calculations [287].
Chapter 6. Nucleon-Nucleon interactions, charge symmetry breaking and renormalization

In this chapter we have analyzed the charge dependence and charge symmetry breaking of the NN interaction. We have used the OBE model with exchange of $\pi$, $\sigma$, $\omega$ and $\rho$ mesons and we have implemented CSB by means of pion mass splitting in the OPE potential and different nucleon masses. Due to the strong correlation between coupling constants that appear in this channel when $m_\omega = m_\rho$ we have defined the effective coupling $g^*_{\omega NN}$ for vectors mesons. In particular, and as in previous chapters, we have selected the $^1S_0$ np channel to fit $m_\sigma$, $g_\sigma$ and $g^*_{\omega NN}$. We have used the Nijmegen group phase shifts [56] to perform the corresponding fit. Again a fine-tuning problem arises when we use the usual regular boundary condition at the origin $u(0) = 0$ but short distance insensitivity is achieved when we approach the problem from a renormalization viewpoint where the fine-tuning disappears and in particular vector mesons become irrelevant.

We have use a short distance connection to relate the renormalized np channel with the others $nn$, $pp$ and $pp(c)$. This short distance connection is so far an assumption based on finiteness but we have seen that reasonable results are obtained for low energy parameters and phase shifts. Our predictions for $(\Delta_{CIB}, \Delta_{CIB})$ and $(\Delta_{CSB}, \Delta_{CSB})$ given by Eqs. (6.86)- (6.89) are compatible with the empirical one within the error estimation. This is in fact a remarkable result because traditionally a great amount

6.7 Conclusions

Figure 6.5: Renormalized phase shifts for the OBE potential with CSB OPE + $\sigma$ as a function of the c.m. momentum in the singlet $^1S_0$ channel. In the upper panel we show the fitted np phase-shift to the Nijmegen results [56]. In the lower panel the predicted pp(c) and nn are depicted and compared to the CD-Bonn result [55].
of effects such as multi-meson exchanges have been essential to explain these differences [55, 273, 274]. Obviously, further investigations should be done in particular concerning the role played by $\rho - \omega$ and $\pi - \eta$ mixing.

Finally it is worth mentioning that it could be possible to extend the previous ideas on charge independence to strangeness, i.e. assume that the short-distance piece of the baryon-baryon interaction is independent with respect to strangeness. With the previous hypothesis, supplemented with the necessary changes in the finite-range piece of the potential, we can for example correlate the $\Lambda N$ or $\Lambda\Lambda$ interaction with the two-nucleon one. This could in particular provide a framework to test $SU(3)$ symmetry but some work has still to be done in that respect.
Chapter 7

Gauge invariance and renormalization in OBE currents

7.1 Introduction

Hadron electromagnetic (EM) form factors provide valuable information about their internal structure. They can be studied by means of electron-nucleus scattering. In the case of atomic nuclei where the interaction between nucleons is due to the exchange of mesons [53] the virtual photon exchanged between the electron and the nucleus can couple to the virtual in-flight meson exchanged between nucleons giving a net contribution to the form factors. This effect due to the exchange of mesons between nucleons generates the so-called Meson Exchange Currents (MECs) which should be added to the nucleonic currents to build the total electromagnetic current of the nucleus. As a result, in both the static properties of nuclei (such as their magnetic moments) and in EM interactions of nuclei with radiation, an important portion of the observed phenomena is due to these MECs.

Indeed, MECs have been very successful in accounting for a number of long-standing discrepancies between experiments and calculations based on the impulse approximation (IA) where only nucleonic currents are considered. Their relevance was soon established since 1972 when the missing contribution between the calculated and measured cross sections for radiative neutron capture was explained in terms of the pion exchange current mechanisms and the general framework for calculating MECs was set up [288].

The importance of constructing MECs consistent with the way we describe the NN interaction is becoming clear now. Very recent calculations [289] compute MECs between the high precision but unrelated AV18 NN-potential wave functions. This is a common practice in many nuclear calculations, which strictly violates gauge invariance. From a quantitative point of view the nuclear wave function, the NN interaction and the electromagnetic operators must be built in a consistent way. In fact, the exact significance of MECs depends on the particular wave function we consider for the relative two-nucleon system, i.e., a close relationship between the potential with which we describe the exchange of mesons and the MECs which arise when coupling photons must be preserved.
However, it is a fact that up to date, we do not have a detailed knowledge of the nuclear interaction below 1fm. This is clearly seen by looking at different phenomenological potentials such as Paris, Argonne v18, Nijmegen and (CD)Bonn for example. They all describe the available scattering NN data with formidable precision but differ considerably in shape and form (local, nonlocal, energy dependent, momentum dependent) at short distances. One learns thus that these details are not so crucial for low-energy properties of the interaction. Obviously, this lack of knowledge of short distances should naturally extend also to electroweak matrix elements of low energy nuclear reactions as we do not expect to know the current any better than the interaction. In the more modern versions of these potentials one starts from an effective lagrangean in such a way that it leads, in the non-relativistic limit, to the NN interaction potential. Clearly and due to the almost zero mass at the nucleon mass scale, the pion is expected to play a dominant role mostly at long distances. In practice, however, heavier mesons are included to account for shorter distances. In the model lagrangean the only parameters are the meson-nucleon coupling constants and meson masses. The latter are usually taken from the free values while the former are adjustable parameters if not known experimentally. The potential derived in this way presents short distance singularities and hadronic form factors mimicking nucleon size are used in order to regularize them. Although these could in principle be calculated, they are introduced by hand and taken as free parameters. All these parameters are adjusted to get a reasonable NN phase shifts over a wide range of energies.

As one can conspicuously recognize the possibility of making a good phenomenology while replacing strong form factors in the NN potential for renormalization conditions has further and important benefits when dealing with MECs. In particular it makes the discussion of gauge invariance much simpler, as we are effectively dealing with local theories with no cut-off, i.e. point couplings. Under this circumstance the cumbersome gauging procedures involving path-dependent link operators and which becomes necessary in order to minimally implement gauge invariance would not be needed.

Renormalization techniques have been applied in Ref. [290] to evaluate the deuteron charge, quadrupole, and magnetic form factors using wave functions obtained from chiral effective field theory (χEFT) when the potential includes one-pion exchange, chiral two-pion exchange, and genuine contact interactions and a extraordinarily good description of the deuteron EM form factors has been achieved. In Ref. [291] we have evaluated EM deuteron form factors in the IA using the renormalized leading-\textit{Nc} OBE potential studied along this thesis, with a reasonable momentum transfer dependent behaviour up to about $q \sim 800\text{MeV}$ and definitely improving over OPE.

Actually, these form factors as well as some of the presently computed deuteron properties are expected to have significant corrections from MECs. As MECs are a genuine consequence of the Meson Exchange picture in the NN interaction, they also require constructing exact NN wave functions from the corresponding Hamiltonian, as we have done in the present Thesis. We have shown that renormalization for the OBE potential is not only feasible as a previous and theoretically appealing step to evaluate wave functions and phase-shifts but also and perhaps surprisingly yields a sound phenomenologically. It also helps in reducing the impact of the hardly accessible short distance region of the NN interaction, thereby reducing standard and much debated ambiguities.

In this chapter we try to go further analyzing whether this holds true also for low energy electroweak reactions where the meson exchange picture is traditionally expected to work.
7.2 Gauge invariance, continuity equation and MECs

Let us consider a nuclear system composed by nucleons and mesons. The lagrangean density for such a system is given by,

\[ L = L_0 + L_{\text{int}} + L_{\text{nl}}, \tag{7.1} \]

where \( L_0 \) is the free fiels lagrangean, \( L_{\text{int}} \) is the meson-nucleon interaction lagrangean and \( L_{\text{nl}} \) stands for a non-linear lagrangean involving couplings between mesons which we consider as subdominant. If this system is now placed in presence of a photon field \( A_\mu = (\phi(x), A(x)) \), then a new term appears in the lagrangean when taken minimal substitution \( p_\mu \rightarrow p_\mu - e A_\mu \),

\[ L_{EM} = -J_\mu A_\mu, \tag{7.2} \]

where \( J_\mu = (\rho(x), J(x)) \) with \( \rho \) and \( J \) the charge and vector current densities. Gauge invariance of the theory is achieved by the continuity equation

\[ \partial_\mu J_\mu = 0, \tag{7.3} \]

where in the Heisenberg picture we have \( \partial_\mu \rho = i [H, \rho] \). Now, the exchange of charge between two nucleons corresponds to isospin \( I = 1 \) exchange and generates a \( \tau_1 \cdot \tau_2 \) term in the two body interaction potential,

\[ V_{NN} = V_0 + V_{ex} \tau_1 \cdot \tau_2. \tag{7.4} \]

This exchange term modifies the EM properties of the system and leads to the appearance of MECs which are constrained to preserve gauge invariance, i.e., to satisfy the continuity equation (7.3). To see that, let us consider a system of two point-like nucleons described by the non-relativistic Hamiltonian,

\[ H = T + V_{NN} \tag{7.5} \]

with \( T \) the kinetic energy and the interaction potential \( V_{NN} \) given by Eq. (7.4). If \( V_{ex} = 0 \), then the charge and current densities are,

\[ \rho_{IA}(x) = \rho_1(x) + \rho_2(x), \tag{7.6} \]
\[ J_{IA}(x) = J_1(x) + J_2(x), \tag{7.7} \]

but the presence of \( V_{ex} \) leads in addition to an exchange current \( J_{ex}^\mu = (\rho_{ex}, J_{ex}) \) such that the total charge and current densities are now,

\[ \rho(x) = \rho_{IA}(x) + \rho_{ex}(x; r_1, r_2), \tag{7.8} \]
\[ J(x) = J_{IA}(x) + J_{ex}(x; r_1, r_2), \tag{7.9} \]

where \( \rho_{ex} \) and \( J_{ex} \) depend on the positions \( r_1 \) and \( r_2 \) of the two nucleons. From the continuity equation (7.3) and taking into account that the total charge is conserved one has,

\[ \nabla \cdot J = \nabla \cdot (J_{IA} + J_{ex}) = -\frac{\partial \rho}{\partial t} = -i [H, \rho] = -i [H, \rho_{IA} + \rho_{ex}], \tag{7.10} \]
so that one can separate the one- and two-body currents,

\[
\nabla \cdot J_{BA} = -i [H_0, \rho_{BA}], \quad (7.11) \\
\nabla \cdot J_{ex} = -i [V_{ex} \tau_1 \cdot \tau_2, \rho_{BA}] - i [H, \rho_{ex}], \quad (7.12)
\]

where \( H_0 = T + V_0 \). The one-body current is defined in such a way that Eq. (7.11) is verified directly. However Eq. (7.12) is actually the condition which define the two-body current operator. In fact, the first term on the right-hand side of Eq. (7.12) describes the flow of charge between two unperturbed nucleons due to the exchange interaction. The second term represents higher-order corrections and in the static limit can be neglected at first order.

Now, for small momenta the one-body EM current is \(^4\),

\[
J^\mu_{em} = \frac{1 + \tau^2}{2} \gamma^\mu + \frac{\kappa_s + \kappa_v \tau^2}{2} i \sigma_{\mu\nu} q^\nu \frac{q_c}{2M_N}, \quad (7.13)
\]

with,

\[
\kappa_s = \mu_p + \mu_n - 1 = -0.12, \quad \kappa_v = \mu_p - \mu_n - 1 = 3.706, \quad (7.14)
\]

Inserting this vector current operator between nucleon states one obtains, discarding relativistic \((q/M_N)^2\) corrections, the nucleonic (IA) charge and current densities for one nucleon, \((\rho_{1B}, J_{1B}) = \langle N(p')| (j^\mu_{em}, J^\mu_{em}) |N(p)\rangle\),

\[
\rho_{1B} = \frac{1 + \tau^2}{2}, \quad (7.15) \\
J_{1B}(q) = \frac{1 + \tau^2}{2} \frac{p' + p}{2M_N} + \left( \frac{1 + \tau^2}{2} + \frac{\kappa_s + \kappa_v \tau^2}{2} \right) \frac{\sigma \times q}{2M_N}, \quad (7.16)
\]

with \( q = p' - p \) the momentum transfer to the nucleon. For two nucleons the IA charge density is then,

\[
\rho_{BA}(x) = \frac{1}{2} \left[ (1 + \tau_1^3) \delta^{(3)}(x - r_1) + (1 + \tau_2^3) \delta^{(3)}(x - r_2) \right], \quad (7.17)
\]

Using this charge density in Eq. (7.12) we arrive at,

\[
\nabla \cdot J_{ex}(x; r_1, r_2) = [\tau_1 \times \tau_2]_z V_{ex}(r_1 - r_2) (\delta^{(3)}(x - r_1) - \delta^{(3)}(x - r_2)), \quad (7.18)
\]

This expression can be translate into momentum space easily if we note that translation invariance requires that \( J_{ex} \) depends only on \( x - r_1 \) and \( x - r_2 \). We write,

\[
J_{ex}(x; r_1, r_2) = \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \langle \tau_1 \times \tau_2 \rangle_\tau e^{i k_1 \cdot (x - r_1)} e^{i k_2 \cdot (x - r_2)} J_{ex}(k_1, k_2), \quad (7.19)
\]

which leads to the equation,

\[
i(k_1 + k_2) \nabla \cdot J_{ex}(k_1, k_2) = [\tau_1 \times \tau_2]_\tau (V_{ex}(k_1) - V_{ex}(k_2)), \quad (7.20)
\]

with,

\[
V_{ex}(k) = \int d^3r V_{ex}(r) e^{-i k \cdot r}. \quad (7.21)
\]

\(^4\)In what follows we take \( N_c = 3 \) and \( \Lambda_N = M_N \).
Eq. (7.20) or equivalently Eq. (7.18) will be called the continuity equation for the MECs.

7.3 The $\pi$- and $\rho$-meson exchange currents

Within a meson exchange framework the simplest model for an isospin-dependent interaction involves exchange of one single pion and rho meson which on the other hand, give the dominant contributions to the MECs. The corresponding interaction and the exchange current operator, which satisfy the continuity equation (7.20), are well known. These are known as isovector exchange currents because they are associated with the isospin dependent NN interaction and are model independent as they are constrained only by the interaction. To construct the $\pi$- and $\rho$-meson exchange currents one starts from the interaction lagrangeans $\mathcal{L}_{\text{int}}$ involving mesons and nucleons,

\[
\begin{align*}
\mathcal{L}_{\pi NN} &= -\frac{f_{\pi NN}}{m_\pi} \overline{N} \gamma_5 \gamma_\mu \tau \cdot \partial^\mu \pi N, \\
\mathcal{L}_{\rho NN} &= -g_{\rho NN} \overline{N} \left( \gamma_\mu + \frac{\kappa_\rho}{2M_N} \sigma_{\mu\nu} \partial^\nu \right) \tau \cdot \rho^\nu N,
\end{align*}
\] (7.22, 7.23)

with $g_{\pi NN}/2M_N = f_{\pi NN}/m_\pi$ and $f_{\rho NN} = \kappa_\rho g_{\rho NN}$. Then one couples photons by applying minimal substitution $\partial_\mu \rightarrow \partial_\mu - iA_\mu$ getting the following new interaction lagrangeans,

\[
\begin{align*}
\mathcal{L}_{\gamma\pi NN} &= -\frac{f_{\pi NN}}{m_\pi} \overline{N} \gamma_5 \gamma_\mu A^\mu (\tau \times \pi)_z N, \\
\mathcal{L}_{\gamma\rho NN} &= -A_\mu (\tau \times \partial^\mu \pi)_z, \\
\mathcal{L}_{\gamma\rho NN} &= -(1 + \kappa_\rho)g_{\rho NN} \overline{N} \sigma_{\mu\nu} A^\nu (\rho_\mu \times \tau)_z N, \\
\mathcal{L}_{\gamma\rho NN} &= -A_\mu (G_{\mu\nu} \times \rho_\nu)_z,
\end{align*}
\] (7.24, 7.25, 7.26, 7.27)

with $G_{\mu\nu} = \partial_\nu \rho_\mu - \partial_\mu \rho_\nu$. The new vertices generated for such a lagrangeans are shown in Fig. 7.1.

\[\text{Figure 7.1: Feynman diagrams representation of the components of the isovector exchange current operators which are associated with the } \pi \text{- and } \rho \text{-meson exchange interactions. The seagull contributions correspond to diagrams (a) and (b) and the mesonic one to diagram (c).}\]
From these lagrangean one can obtain in the non-relativistic limit the following currents for $\pi$- and $\rho$-meson exchanges [292],

\[
\mathbf{J}^{\text{sea}}_{\pi}(k_1, k_2) = -i \frac{g^2_{\pi NN}}{m^2_{\pi}} (\tau_1 \times \tau_2) \left\{ \frac{\sigma_1 (\sigma_2 \cdot k_2)}{m^2_{\pi} + k_2^2} - \frac{\sigma_2 (\sigma_1 \cdot k_1)}{m^2_{\pi} + k_1^2} \right\},
\]

(7.28)

\[
\mathbf{J}^{\text{mes}}_{\pi}(k_1, k_2) = i \frac{g^2_{\pi NN}}{m^2_{\pi}} (\tau_1 \times \tau_2) \left\{ \frac{\sigma_1 \cdot k_1 \sigma_2 \cdot k_2}{(m^2_{\pi} + k_1^2)(m^2_{\pi} + k_2^2)} \right\},
\]

(7.29)

\[
\mathbf{J}^{\text{sea}}_{\rho}(k_1, k_2) = -i \frac{g^2_{\rho NN}(1 + \kappa_\rho)^2}{4M^2_N} (\tau_1 \times \tau_2) \left\{ \frac{\sigma_2 \times (\sigma_1 \times k_1)}{m^2_{\rho} + k_1^2} - \frac{\sigma_1 \times (\sigma_2 \times k_2)}{m^2_{\rho} + k_2^2} \right\},
\]

(7.30)

\[
\mathbf{J}^{\text{mes}}_{\rho}(k_1, k_2) = i \frac{g^2_{\rho NN}(1 + \kappa_\rho)^2}{4M^2_N} (\tau_1 \times \tau_2) \left[ \frac{((k_1 - k_2) \cdot (\sigma_1 \times k_1)) \cdot (\sigma_2 \times k_2)}{(m^2_{\rho} + k_1^2)(m^2_{\rho} + k_2^2)} \right. \\
\left. + (\sigma_1 \times k_1) \sigma_2 \cdot (k_1 \times k_2) + (\sigma_2 \times k_2) \sigma_1 \cdot (k_1 \times k_2) \right],
\]

(7.31)

where sea means seagull contributions (Fig. 7.1(a,b)) and mes means mesonic contributions (Fig. 7.1(c)).

It is however very useful in what follows writing these currents in an alternative model-independent way due to Riska [293]. This allows us to relate the NN potential and the current operator in a very simple way. Suppose a generic isospin-dependent NN potential in momentum space,

\[
V_{NN}(k) = \left[ W_C(k) + W_S(k) \sigma_1 \cdot \sigma_2 + W_T(k) S_{12}(\hat{k}) \right] \tau_1 \cdot \tau_2,
\]

(7.32)

and consider only the leading-$N_c$ one $\pi$- and one $\rho$-meson exchange potentials. We have,

\[
V_{NN}(k) = -\frac{1}{3} \frac{g^2_{\pi NN}}{4M^2_N} \frac{1}{m^2_{\pi} + k^2} \left[ k^2 \sigma_1 \cdot \sigma_2 - S_{12}(\hat{k}) \right] \tau_1 \cdot \tau_2 \\
-\frac{1}{4} \frac{f^2_{\rho}}{3M^2_N} \frac{1}{m^2_{\rho} + k^2} \left[ 2 k^2 \sigma_1 \cdot \sigma_2 + S_{12}(\hat{k}) \right] \tau_1 \cdot \tau_2,
\]

(7.33)

where $S_{12}(\hat{k}) = \sigma_1 \cdot \sigma_2 k^2 - 3 \sigma_1 \cdot \hat{k} \sigma_2 \cdot \hat{k}$. Following Riska’s prescription [293] we write

\[
V_{NN}(k) = [v_{SS}^\pi(k) + v_{SS}^\rho(k)] k^2 \sigma_1 \cdot \sigma_2 + [v_{TT}^\pi(k) + v_{TT}^\rho(k)] S_{12}(\hat{k}),
\]

(7.34)

since in the case of $\pi$ and $\rho$-exchange we have $W_C(k) = 0$, $W_S(k) = k^2 \left[ v_{SS}^\pi(k) + v_{SS}^\rho(k) \right]$ and $W_T(k) = v_{SS}^\pi(k) + v_{SS}^\rho(k)$ with the following potentials,

\[
v_{SS}^\pi(k) = -\frac{1}{3} \frac{g^2_{\pi NN}}{4M^2_N} \frac{1}{m^2_{\pi} + k^2},
\]

(7.35)

\[
v_{TT}^\pi(k) = +\frac{1}{3} \frac{g^2_{\pi NN}}{4M^2_N} \frac{1}{m^2_{\pi} + k^2},
\]

(7.36)

\[
v_{SS}^\rho(k) = -\frac{1}{4} \frac{f^2_{\rho}}{3M^2_N} \frac{1}{m^2_{\rho} + k^2},
\]

(7.37)

\[
v_{TT}^\rho(k) = -\frac{1}{4} \frac{f^2_{\rho}}{3M^2_N} \frac{1}{m^2_{\rho} + k^2},
\]

(7.38)
having the relations,

\begin{align}
  v_{SS}^\tau(k) &= -v_T^\tau(k), \\
  v_{SS}^\rho(k) &= 2v_T^\rho(k).
\end{align}

With this interaction potentials, the currents satisfying the continuity equation (7.3) are [292, 293],

\begin{align}
  J_\tau(k_1, k_2) &= -3i (\tau_1 \times \tau_2) \{ v_T^\tau(k_2) \sigma_1 (\sigma_2 \cdot k_2) - v_T^\tau(k_1) \sigma_2 (\sigma_1 \cdot k_1) \\
  &\quad - \frac{k_1 - k_2}{k_1^2 - k_2^2} \sigma_1 \cdot k_1 \sigma_2 \cdot k_2 [v_T^\tau(k_2) - v_T^\tau(k_1)] \}, \\
  J_\rho(k_1, k_2) &= -3i (\tau_1 \times \tau_2) \{ v_T^\rho(k_2) \sigma_1 \times (\sigma_2 \times k_2) - v_T^\rho(k_1) \sigma_2 \times (\sigma_1 \times k_1) \\
  &\quad + \frac{v_T^\rho(k_2) - v_T^\rho(k_1)}{k_1^2 - k_2^2} [(k_1 - k_2) (\sigma_1 \times k_1) \cdot (\sigma_2 \times k_2) \\
  &\quad + (\sigma_1 \times k_1) \sigma_2 \cdot (k_1 \times k_2) + (\sigma_2 \times k_2) \sigma_1 \cdot (k_1 \times k_2)] \}
\end{align}

In the calculation of these currents the seagull diagrams (Fig. 7.1(a,b)) correspond to the first lines in Eqs. (7.41) and (7.42), and the mesonic diagrams (Fig. 7.1(c)) correspond to the rest of Eqs. (7.41) and (7.42). The functions \( v_T^\tau \) and \( v_T^\rho \) are given by Eqs. (7.36) and (7.38) respectively. However one can also replace these single meson exchange potentials in the current operators by generalized one pseudoscalar and one pseudovector potentials to obtain generalized current satisfying the continuity equation. This is the usual way to proceed with phenomenological potentials.

So far, we have focused in \textit{model-independent} MECs which are explicitly constructed to satisfy current conservation for a given interaction potential. There are however transverse currents (\( k \cdot J = 0 \)) which cannot be related to the NN potential through the continuity equation (7.3), or which involve new parameters that cannot be fixed from this equation. These currents are then \textit{model-dependent}. The first type of model-dependent currents are those including nucleon excitations as the \( \Delta \)-isobar and the Roper resonance \( N^\ast \) (see Fig. 7.2 (a)). We will not consider here these excitations. A second type of model-dependent contributions are the ones coming from electromagnetic decays of the exchanged meson, i.e., \( \rho \to \gamma \pi \) in our case (see Fig. 7.2 (b)). The latter diagram can be calculated with the lagrangian,

\begin{equation}
  \mathcal{L}_{\rho\pi\gamma} = -\frac{g_{\rho\pi\gamma}}{m_\rho} \epsilon_{\mu\nu\lambda\delta} \partial^\mu A^\nu \rho^\lambda \cdot \partial^\delta \pi,
\end{equation}

However, in the spirit of VMD \(^2\) this vertex is associated with the \( \omega \to \rho \pi \) decay via the diagram shown in Fig. 7.2 (c). By G-parity arguments the \( \rho - \pi \) vertex couples to an isoscalar vector current \( J_\rho^\mu \). Assuming \( \omega \)-meson dominance we write,

\begin{equation}
  J_\rho^\mu = \frac{m_\rho^2}{2 g_{\omega NN}} \omega^\mu,
\end{equation}

and the lagrangean for the \( \rho\omega\pi \)-vertex as,

\begin{equation}
  \mathcal{L}_{\omega\rho\pi} = -g_{\omega\rho\pi} \epsilon_{\mu\nu\lambda\delta} \partial^\mu \omega^\nu \rho^\lambda \cdot \partial^\delta \pi,
\end{equation}

\(^2\)See also Appendix C
Using this lagrangian density together with the isoscalar current Eq. (7.44) to evaluate the diagram of Fig. 7.2 (c) one obtains the current,

\[
J_{\rho\pi\gamma}(k_1, k_2) = i \frac{f_{\pi NN} g_{\rho NN} g_{\rho\pi\gamma}}{m_\rho m_\pi} \tau_1 \cdot \tau_2 \frac{k_1 \times k_2}{(m_\rho^2 + k_1^2)(m_\pi^2 + k_2^2)} \left\{ \frac{\sigma_1 \cdot k_1}{m_\rho^2 + k_1^2} - \frac{\sigma_2 \cdot k_2}{m_\pi^2 + k_2^2} \right\} .
\]

(7.46)

Using again VMD one can model the radiative decay \( \rho \to \pi\gamma \) as being mediated through the \( \omega \)-meson, so the radiative decay width is [267],

\[
\Gamma(\rho \to \pi\gamma) = \frac{1}{24} \frac{\alpha}{\alpha} \left( \frac{g_{\rho\omega\pi}}{2 g_{\omega NN}} \right)^2 \left[ 1 - \frac{m_\pi^2}{m_\rho^2} \right]^3 ,
\]

(7.47)

with \( \alpha = e^2/\hbar c \), the fine structure constant and \( g_{\rho\pi\gamma} = g_{\rho\omega\pi}/(2 g_{\omega NN}) \). From the experimental value [294] \( \Gamma = 71 \pm 7 \text{keV} \) one gets \( g_{\rho\pi\gamma} = 0.578 \pm 0.028 \).

### 7.4 Note on hadronic and electromagnetic form factors

The choice of nucleon form factors is a key ingredient in the calculation of the current. They are specially important for a good description of electron scattering experiments at high momentum transfers (\( |q| \) of the order of 1 GeV) where the structure of nucleons starts playing an important role. While electromagnetic form factors parameterize the electromagnetic structure of nucleons, hadronic form factors take into account their finite size as composite particles. The former can be parameterized very accurately while the latter are rather difficult to calculate, besides being extremely model dependent (see e.g. the discussion in Ref. [257]). This, in particular, clouds the gauge invariance process of constructing MECs. In fact, the inclusion of EM form factors in the continuity equation is a rather clean process while hadronic form factors are more connected with a particular choice of the NN potential at short distances.

#### 7.4.1 Nucleon electromagnetic form factors

The exchange current operators can be multiplied by an electromagnetic form factor to account for the electromagnetic size of the nucleons for high momentum transfers without loosing the gauge invariance.
of the theory. For that purpose, the nucleonic current Eq. (7.13) is written as,

$$j_{IA}^\mu = F_1(q^2)\gamma^\mu + F_2(q^2)\gamma^\mu$$

(7.48)

where $F_1$ and $F_2$ are the Dirac and Pauli form factors. In the non-relativistic limit and dropping all terms $O(q^3/M^3)$ and higher, the spacial part of the current can be written as

$$J_{IA} = i G_M(q^2)\sigma \times q + F_1(q^2)\frac{p + p'}{2M_N}$$

(7.49)

where the Sachs electric and magnetic form factors are defined as

$$G_E(q^2) = F_1(q^2) - \frac{q^2}{4M_N^2}F_2(q^2)$$

(7.50)

$$G_M(q^2) = F_1(q^2) + F_2(q^2)$$

(7.51)

and where in the last step we have taken into account the fact that we drop terms $O(q^3/M^3)$. It is important to notice here that from a theoretical point of view the elementary form factors are the Dirac and Pauli form factors $F_1$ and $F_2$, while the electric and magnetic form factors $G_E$ and $G_M$ are defined as a convenient parameterization in describing experimental scattering cross sections.

It is appropriate to separate the isoscalar and isovector contributions to the form factors

$$G_{E,M}(q^2) = \frac{1}{2} \left[ G^{S}_{E,M}(q^2) + G^{V}_{E,M}(q^2) \tau^z \right]$$

(7.52)

$$F_{1,2}(q^2) = \frac{1}{2} \left[ F^{S}_{1,2}(q^2) + F^{V}_{1,2}(q^2) \tau^z \right]$$

(7.53)

with

$$F^{S}_1(0) = F^{V}_1(0) = 1$$

(7.54)

$$F^{S}_2(0) = \kappa_s$$

(7.55)

$$F^{V}_2(0) = \kappa_v$$

(7.56)

$$G^{S}_E(0) = G^{V}_E(0) = 1$$

(7.57)

$$G^{S}_M(0) = 1 + \kappa_s = \mu_p + \mu_n = 0.880$$

(7.58)

$$G^{V}_M(0) = 1 + \kappa_v = \mu_p - \mu_n = 4.706$$

(7.59)

EM form factors acquire a clear interpretation within the VMD model. The photon couples to the nucleon via the exchange of a vector meson, i.e., either the $\rho$-meson for the isovector part of the current, or the $\omega$-meson for its isoscalar part. Based on this model one can use a dipole function to parameterize accurately the EM form factor. Here we use,

$$G^{S,V}_E(q^2) = \frac{1}{(1 + q^2/m_0^2)^2}$$

(7.60)

$$G^{S,V}_M(q^2) = \frac{G^{S,V}_M(0)}{(1 + q^2/m_0^2)^2}$$

(7.61)
with $m_2^2 = 0.71 \text{ GeV}^2$. Equally, VMD can be used to determine the EM form factor at the $\rho \pi \gamma$ vertex having $[295–297]$,

$$G_{\rho \pi \gamma}(q^2) = \frac{g_{\rho \pi \gamma}}{1 + q^2/m_2^2}$$

In Ref. [298] the $\rho \pi \gamma$ electromagnetic form factor is calculated from quark loop diagrams. They found that it differs considerably from the usual monopole form derived from VMD. This fact has several consequences for the elastic electromagnetic form factors of the deuteron specially at high momentum transfers where a strong cancellation between $\rho \pi \gamma$ and $\omega \pi \gamma$ MECs has to take place if the monopole forms are used.

### 7.4.2 Nucleon hadronic form factors

As we have discussed at length, the OBE potential for the NN interaction presents short distance singularities which makes the nuclear problem ambiguous. The way out to deal with these is the introduction of hadronic form factors which corresponds to the replacement,

$$V_{mNN}(q) \rightarrow V_{mNN}(q) [\Gamma_{mNN}(q^2)]^2$$

with $\Gamma_{mNN}(q^2)$ a regulator function depending on a cut-off $\Lambda$. The modification of the OBE potential by hadronic form factors and the corresponding change in MEC operators is represented diagrammatically in Fig. 7.3 in the static limit case. One assumes the exchange of a new fictitious heavy particle of effective mass $\Lambda$, being $\Lambda$ the cut-off energy.

**Figure 7.3:** Representation of the OBE potential when we introduce hadronic form factors at each meson-nucleon vertex in the static limit.

If one wants to construct MECs satisfying the continuity equation with this modified OBE potential then the seagull diagram is just modified by adding a corresponding factor $\Gamma_{mNN}(q^2)^2$ on top of the expressions (see Fig. 7.3 (a)). However, the mesonic diagram acquires new contributions: in order to preserve gauge invariance one has to include new diagrams where the photon couples to the fictitious heavy particle (see Fig. 7.3 (b,c,d)). In general, the direct introduction of hadronic form factors into the interaction Lagrangian and, therefore, into the Born amplitudes leads to a violation of gauge invariance. To restore gauge invariance different methods have been developed. In relativistic theories Gross and Riska found in Ref. [299] that strong form factor can be introduced without violating current conservation but they have to be reinterpreted as phenomenological self-energies, i.e., they must depend only on the
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7.5 Renormalization of the MECs in NN scattering

We now apply our renormalization method to study several important processes where the MECs take an important role. These are deuteron backward electro-disintegration near threshold, radiative neutron capture and elastic electron-deuteron scattering.

7.5.1 The NN potential and renormalized wave functions

For simplicity, the NN potential we shall consider here is the leading $N_c$ OBE potential with $\pi$, $\sigma$, $\omega$- and $\rho$- exchange,

$$V_C(r) = -\frac{g_{\pi NN}^2}{4\pi} \frac{e^{-m_\pi r}}{r} + \frac{g_{\rho NN}^2}{4\pi} \frac{e^{-m_\rho r}}{r},$$

$$W_S(r) = \frac{1}{12} \frac{g_{\rho NN}^2}{4\pi} \frac{m_\rho^2}{M_N^2} \frac{e^{-m_\rho r}}{r} + \frac{f_{\rho NN}^2}{6} \frac{m_\rho^2}{4\pi M_N^2} \frac{e^{-m_\rho r}}{r},$$

$$W_T(r) = \frac{1}{12} \frac{g_{\rho NN}^2}{4\pi} \frac{m_\rho^2}{M_N^2} \frac{e^{-m_\rho r}}{r} \left[ 1 + \frac{3}{m_\rho r} + \frac{3}{(m_\rho r)^2} \right]$$

with $M_N = 939\text{MeV}$, $m_\pi = 138\text{MeV}$, $m_\sigma = 501\text{MeV}$, $m_\rho = 770\text{MeV}$, $m_\omega = 783\text{MeV}$, $g_{\pi NN} = 13.1$ and $g_{\sigma NN} = 9.1$. We take two different values for $f_{\rho NN} = 15.5$ and 17 and $g_{\rho NN} = f_{\rho NN} m_\rho / \sqrt{2} M_N$. The
\[ U_{1S_0} = U_{3S_1} = M_N (V_C - 3W_S), \]  
\[ U_{E_1} = -6\sqrt{2}M_NW_T, \]  
\[ U_{3D_1} = M_N (V_C - 3W_S + 6W_T). \]

We write the wavefunction of the \(^1S_0\) \(np\) state in the c.m. system as
\[ \Psi(x) = \frac{1}{\sqrt{4\pi r}} u_{1S_0}(r) \lambda_{np}^s \phi_{np}^{\tau m_t}, \]  
with the total spin \(s = 0\) and \(m_s = 0\) and total isospin \(t = 1\) and \(m_t = 0\). The function \(u_{1S_0}(r)\) is the reduced S-wave function, satisfying
\[ -u''_p(r) + U_{1S_0}(r)u_p(r) = p^2u_p(r) \]  
with the asymptotic normalization
\[ u_p(r) \to \frac{\sin(pr + \delta_0(p))}{\sin\delta_0(p)}, \]  
\[ u_0(r) \to 1 - \frac{r}{\alpha_0}, \]  
with \(\alpha_0\) the \(^1S_0\) scattering length. The renormalized \(^1S_0\) wave function \(u_p(r)\) is obtained by integrating in the Schrödinger Eq. (7.71) with the zero energy wave function Eq. (7.73) fixing \(\alpha_0\) as an input. Then we impose the energy independent boundary condition at \(r = r_c\),
\[ \frac{u'_p(r_c)}{u_p(r_c)} = \frac{u'_0(r_c)}{u_0(r_c)}, \]
and integrate out Eq. (7.71). For the deuteron state in the c.m. system we write
\[ \Psi(x) = \frac{1}{\sqrt{4\pi r}} \left[ u(r) + \frac{w(r)}{\sqrt{8}}S_{12} \right] \lambda_{np}^s \phi_{np}^{\tau m_t}, \]
with the total spin \(s = 1\) and \(m_s = 0, \pm 1\) and total isospin \(t = 0\) and \(m_t = 0\). The functions \(u(r)\) and \(w(r)\) are the reduced S- and D-wave components of the relative wave function respectively. They satisfy the coupled set of equations in the \(^3S_1 - ^3D_1\) channel
\[ -u''(r) + U_{3S_1}(r)u(r) + U_{E_1}(r)w(r) = MEu(r), \]
\[ -w''(r) + U_{E_1}(r)u(r) + \left[ U_{3D_1}(r) + \frac{6}{r^2} \right] w(r) = MEw(r), \]
with \(U_{3S_1}(r), U_{E_1}(r)\) and \(U_{3D_1}(r)\) the corresponding matrix elements of the coupled channel potential and \(u(r)\) and \(w(r)\) satisfying the asymptotic conditions,
\[ u(r) \to A_S e^{-\gamma r}, \]  
\[ w(r) \to A_S \eta e^{-\gamma r} \left( 1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right), \]
with \( E = -\gamma^2/M \), \( A_S \) the asymptotic wave function normalization and \( \eta \) the asymptotic D/S ratio. Then we fix the deuteron binding energy \( \gamma = \sqrt{2m_{np} B} \) with \( B = 2.224575(9) \) and integrate in the coupled channel Eqs. (7.75). The regularity condition is imposed at \( r = r_c \) to get \( \eta(r_c) \) and the deuteron wave functions.

### 7.5.2 Deuteron electro-disintegration

The simplest observable for investigating the effects of the isovector exchange current is the backward electro-disintegration of the deuteron \([300, 301]\). Near threshold this observable is dominated by the transition between the \( ^3S_1 \rightarrow ^3D_1 \) deuteron ground state and the \( ^1S_0 \) \( np \) scattering state, i.e., \( e^- + d \ (^3S_1 \rightarrow ^3D_1) \rightarrow e^- + np \ (^1S_0) \). The IA prediction for the cross section is very much smaller than the empirical results. This is due to a destructive interference between the \( ^3S_1 \rightarrow ^1S_0 \) and \( ^3D_1 \rightarrow ^1S_0 \) amplitudes in the IA which predicts a sharp minimum in the cross section not seen experimentally. As we will see most of this discrepancy can be removed by taking into account the simplest meson exchange current processes of pionic range.

The contribution to the backward \((180^0)\) differential cross section in the laboratory frame from the final \(^1S_0\) state is given by \([293, 301, 302]\)

\[
\frac{d\sigma}{dE_d d\Omega}(180^0) = \frac{\alpha^2}{4\pi} \frac{pq^2}{E_f^2 M_N} [g(q) + h(q)]^2 \tag{7.78}
\]

Here \( p \) is the c.m. momentum of the final \( np \) system, \( E_f \) is the final energy of the scattered electron, \( E_i \) is the incident electron energy, \( M_N \) is the averaged nucleon mass, \( q \) is the momentum transfer and \( \alpha \) is the fine structure constant. Since we have relativistic electrons then \( E_f \approx k', E_i \approx k \) and multiplying the cross section by \( k^2 \) to get rid of the dependence of the incident electron energy we can write

\[
k^2 \frac{d\sigma}{dk d\Omega}(180^0) = \frac{\alpha^2}{4\pi} \frac{pq^2}{M_N} [g(q) + h(q)]^2 \tag{7.79}
\]

The coefficient functions \( g \) and \( h \) are given in the IA by \([301]\)

\[
g(q) = -G_M^V(q) J_S(q) \tag{7.80}
\]

\[
h(q) = G_M^V(q) [H_S(q) - J_S(q)] \tag{7.81}
\]

where \( G_M^V(q^2) \) is the isovector nucleon magnetic form factor defined by Eq. (7.61) and the structure function \( H_S \) and \( J_S \) are defined as

\[
H_S(q) = (-\alpha_0) \int_0^\infty dr \ u_{1s_0}(pr) j_0 \left( \frac{qr}{2} \right) u(r) \tag{7.82}
\]

\[
J_S(q) = \frac{-\alpha_0}{\sqrt{8}} \int_0^\infty dr \ u_{1s_0}(pr) j_2 \left( \frac{qr}{2} \right) w(r) \tag{7.83}
\]

The IA structure functions \( H_S \) and \( J_S \) when we include \( \pi-, \sigma-, \omega- \) and \( \rho \)-exchange are plotted in Fig. 7.5 (left panel) for the case \( f_{\rho NN} = 15.5 \) and a final c.m. energy \( E_{c.m.} = 1.5 \) MeV. The differential cross section as a function of \( q^2 \) is displayed in Fig. 7.6. As we can see the IA differential cross section exhibits a minimum at \( q^2 \approx 12 \text{ fm}^{-2} \) and a maximum at \( q^2 \approx 20 \text{ fm}^{-2} \) due to the destructive interference between the \( ^3S_1 \rightarrow ^1S_0 \) and \( ^3D_1 \rightarrow ^1S_0 \) transitions. Both of these features are eliminated when we include the simplest pion exchange current contribution.
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Figure 7.5: Structure functions corresponding to a final np energy $E_{c.m.} = 1.5$ MeV using our renormalized wave functions when the exchange of the $\pi$, $\sigma$, $\omega$ and $\rho$ are considered and Riska's expressions. In this figures we have taken the case $f_{\rho NN} = 15.5$. Left panel: IA, middle panel: $\pi$-MEC, right panel: $\rho$-MEC.

The differential cross section for the $\pi$-exchange current is given by Eq. (7.79) but with new coefficient functions

$$g_\pi(q) = G_E^V(q) \left[ H_\pi(q) + J_\pi(q) \right]$$  \hspace{1cm} (7.84)$$

$$h_\pi(q) = G_E^V(q) J_\pi(q)$$  \hspace{1cm} (7.85)
where $G_V(q^2)$ is the isovector nucleon electric form factor defined by Eq. (7.60) and the structure function $H_\pi$ and $J_\pi$ are defined as [293]

$$H_\pi(q) = (-\alpha_0) g_{2NN}^2 \frac{m_\pi}{12\pi} M_N \int_0^\infty dr \, w_{1,S_0}(pr) \, u(r) \left\{ (1 + m_\pi r) Y_0(m_\pi r) \left[ j_0 \left( \frac{qr}{2} \right) + j_2 \left( \frac{qr}{2} \right) \right] \right\}$$

$$- \int_0^1 dx \, \frac{\Lambda(x)}{m_\pi} \left[ j_0 \left( \frac{xqr}{2} \right) (2 - \Lambda(x)r) - j_2 \left( \frac{xqr}{2} \right) (1 + \Lambda(x)r) \right] Y_0(\Lambda(x)r) \right\}$$

$$J_\pi(q) = (-\alpha_0) g_{2NN}^2 \frac{m_\pi}{12\sqrt{8\pi} M_N} \int_0^\infty dr \, w_{1,S_0}(pr) \, w(r) \left\{ (1 + m_\pi r) Y_0(m_\pi r) \left[ j_0 \left( \frac{qr}{2} \right) + j_2 \left( \frac{qr}{2} \right) \right] \right\}$$

$$+ \int_0^1 dx \, \frac{\Lambda(x)}{m_\pi} \left[ j_0 \left( \frac{xqr}{2} \right) (1 + \Lambda(x)r) - j_2 \left( \frac{xqr}{2} \right) (2 - \Lambda(x)r) \right] Y_0(\Lambda(x)r) \right\}$$

with $\Lambda(x) = \sqrt{m_\pi^2 + \frac{q^2}{4}(1 - x^2)}$. The structure functions $H_\pi$ and $J_\pi$ for the $\pi$ contribution are plotted in Fig. 7.5 (middle panel) and the corresponding contribution to the differential cross section as a function of $q^2$ in Fig. 7.6. It is evident from this figure that the pion exchange contribution explains the experimental data up to $q^2 \sim 20 \text{fm}^{-2}$.

In the $\rho$-exchange case the coefficient functions are

$$g_\rho(q) = G_V^V(q) \left[ H_\rho(q) + J_\rho(q) \right]$$

$$h_\rho(q) = G_V^V(q) \left[ J_\rho(q) \right]$$

with $H_\rho$ and $J_\rho$ defined as [293]

$$H_\rho(q) = (-\alpha_0) g_{2NN}^2 \frac{m_\rho}{12\pi} M_N \int_0^\infty dr \, w_{1,S_0}(pr) \, u(r) \left\{ (1 + m_\rho r) Y_0(m_\rho r) \left[ j_0 \left( \frac{qr}{2} \right) + j_2 \left( \frac{qr}{2} \right) \right] \right\}$$

$$- 2 \int_0^1 dx \, \frac{\Lambda(x)}{m_\rho} \left[ j_0 \left( \frac{xqr}{2} \right) (2 - \Lambda(x)r) - j_2 \left( \frac{xqr}{2} \right) (1 + \Lambda(x)r) \right] Y_0(\Lambda(x)r) \right\}$$

$$J_\rho(q) = (-\alpha_0) g_{2NN}^2 \frac{m_\rho}{12\sqrt{8\pi} M_N} \int_0^\infty dr \, w_{1,S_0}(pr) \, w(r) \left\{ (1 + m_\rho r) Y_0(m_\rho r) \left[ j_0 \left( \frac{qr}{2} \right) + j_2 \left( \frac{qr}{2} \right) \right] \right\}$$

$$- \int_0^1 dx \, \frac{\Lambda(x)}{m_\rho} \left[ j_0 \left( \frac{xqr}{2} \right) (1 + \Lambda(x)r) - j_2 \left( \frac{xqr}{2} \right) (2 - \Lambda(x)r) \right] Y_0(\Lambda(x)r) \right\}$$

with $\Lambda(x) = \sqrt{m_\rho^2 + \frac{q^2}{4}(1 - x^2)}$. The structure functions $H_\rho$ and $J_\rho$ for the $\rho$ contribution are plotted in Fig. 7.5 and the corresponding contribution to the differential cross section as a function of $q^2$ in Fig. 7.6. Here we find a huge discrepancy with the experimental data when the vector mesons are included. This need to be analyzed further in future works.

### 7.5.3 Neutron capture

The neutron radiative capture $n + p \rightarrow d + \gamma$ at threshold is one of the main reactions which provides clear evidence for the existence of MECs. The experimental cross-section measured with thermal neutrons ($v_n = 2200 \text{ m/s}$) is known with very high accuracy and has the value [305]

$$\sigma(np \rightarrow d\gamma) = (334.2 \pm 0.5) \text{ mb}$$

whereas the theoretical value calculated using single-particle magnetic moment operators is [306, 307] $\sigma^{IA} = 302.5 \pm 4.0 \text{ mb}$. As we seen the 10% discrepancy between experiment and one-body contribution
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is significant. This cannot be accounted for by any detail in the nuclear wave function due to short distance physics because the most important uncertainty in the theoretical calculation comes from the asymptotic behaviour of the wave function since the momentum transfer carried by the photon is almost zero. The first explanation for such a discrepancy was due to Riska and Brown [288] who shown that the simplest pion meson exchange current accounts for a 10% of the differential cross section. In the present section we want to test our renormalized wave functions in explaining these differences.

To study the effect of the MECs we can express the neutron capture cross section in the following form [293, 302]

\[ \sigma = \frac{\pi\alpha\omega^3}{2pM_N} [g(0) + h(0)]^2 \]  

(7.93)

where \( \omega \simeq B_d \) is the momentum of the photon and \( p \) that of the neutron in the c.m. system\(^3\). Here \( g \) and \( h \) are the coefficient function evaluated at zero momentum transfer.

In the IA we have

\[ g(0) = 0, \]  

(7.94)

\[ h(0) = -\alpha_0\mu_N \int_0^\infty dr u_{1\Sigma_0}(pr) u(r). \]  

(7.95)

The \( \pi \)-exchange contribution is

\[ g_\pi(0) = [H_\pi(0) + J_\pi(0)], \]  

(7.96)

\[ h_\pi(0) = J_\pi(0), \]  

(7.97)

with

\[ H_\pi(0) = (-\alpha_0)\frac{g_{\pi NN}^2}{12\pi} \int_0^\infty dr u_{1\Sigma_0}(pr) u(r)(2m_\pi r - 1)Y_0(m_\pi r), \]  

(7.98)

\[ J_\pi(0) = (-\alpha_0)\frac{g_{\pi NN}^2 m_\pi}{6\sqrt{8\pi} M_N} \int_0^\infty dr u_{1\Sigma_0}(pr) w(r)(1 + m_\pi r)Y_0(m_\pi r), \]  

(7.99)

and the \( \rho \)-exchange contribution is

\[ g_\rho(0) = [H_\rho(0) + J_\rho(0)], \]  

(7.100)

\[ h_\rho(0) = J_\rho(0). \]  

(7.101)

with

\[ H_\rho(0) = (-\alpha_0)\frac{f_{\rho NN}^2}{4\pi} \int_0^\infty dr u_{1\Sigma_0}(pr) u(r)(m_\rho r - 1)Y_0(m_\rho r), \]  

(7.102)

\[ J_\rho(0) = 0. \]  

(7.103)

The results using our renormalized wave functions for different contributions are shown in Table 7.1. It is worthy to study the cut-off dependence of the neutron capture cross section. This is shown in Fig. 7.7 (left panel). In addition in Fig. 7.7 (right panel) we also shown how the sum \( g(0) + h(0) \) depends with the cut-off when we include the MECs. The result seems to indicate that removing the cut-off plays an important role when we consider more effects than the simplest OPE.

\(^3\)Note that in the c.m. system \( p = \frac{1}{2} M_N v_n = 3.4451 \times 10^{-3} \) MeV
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### Table 7.1: Neutron capture cross section for thermal neutrons ($v_n = 2200$ m/s) and $\gamma$-ray of energy 2.2 MeV. In the IA we include $\pi + \sigma + p + \omega$ exchanges. The notation $\pi\sigma\rho\omega$ corresponds to take $f_{\rho NN} = 15.5$ and $g_{\omega NN} = 9.857$ while $\pi\sigma\rho\omega^*$ corresponds to take $f_{\rho NN} = 17.0$ and $g_{\omega NN} = 10.147$.

<table>
<thead>
<tr>
<th>Contribution</th>
<th>$\sigma(np \rightarrow d\gamma)$ [mb]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$-IA</td>
<td>307.7</td>
</tr>
<tr>
<td>$\pi$-IA + $\pi$-MEC</td>
<td>323.1</td>
</tr>
<tr>
<td>$\pi\sigma\rho\omega$-IA</td>
<td>302.7</td>
</tr>
<tr>
<td>$\pi\sigma\rho\omega$-IA + $\pi\rho$-MEC</td>
<td>317</td>
</tr>
<tr>
<td>$\pi\sigma\rho\omega^*$-IA</td>
<td>297.3</td>
</tr>
<tr>
<td>$\pi\sigma\rho\omega^*$-IA + $\pi\rho$-MEC</td>
<td>312.5</td>
</tr>
</tbody>
</table>

For elastic electron-deuteron scattering we can write the differential cross section in the lab-frame [308] as

$$\frac{d\sigma}{d\Omega_e}(q^2, \theta_e) = \left( \frac{d\sigma}{d\Omega_e} \right)_{\text{Mott}} \left[ A(q^2) + B(q^2) \tan^2\left(\frac{\theta_e}{2}\right) \right].$$

(7.104)

Here $\theta_e$ is the electron scattering angle and $q$ is the four-momentum transfer. The functions $A$ and $B$ are the deuteron structure functions and are given by:

$$A(q^2) = G_C^2(q^2) + \frac{2}{3} \eta G_M^2(q^2) + \frac{8}{9} \eta^2 G_Q^2(q^2),$$

(7.105)

$$B(q^2) = \frac{4}{3} \eta (1 + \eta) G_M^2(q^2),$$

(7.106)

where we have defined $\eta = q^2/4M_d^2$ being $M_d$ the deuteron mass, $M_N$ is the nucleon mass and $G_C$, $G_M$, $G_Q$ the deuteron monopole, dipole and quadrupole form factors which are determined by the deuteron one-body (IA) and two-body (MEC) currents. The form factors are defined with the following normalization

$$G_C(0) = 1,$$

(7.107)

$$G_Q(0) = Q_d,$$

(7.108)

$$G_M(0) = \frac{M_d}{M_N} \mu_d,$$

(7.109)
Figure 7.8: Deuteron EM form factors in the IA when we renormalize the OBE potential. Left panel: charge form factor $G_C$, middle panel: magnetic form factor $G_M$ and right panel: quadrupole form factor $G_Q$. The experimental data are taken from the compilation of Ref. [309] in the case of $G_C$ and $G_Q$ and from Refs. [310–313] in the case of $G_M$.

with $Q_d$ and $\mu_d$ the deuteron quadrupole and magnetic moment in units of nuclear magneton.

In the IA the form factors are given by:

\[
G_C^{(\text{IA})}(q^2) = G_S^E(q^2) \int dr \left[u^2(r) + w^2(r)\right] j_0\left(\frac{qr}{2}\right),
\]

\[
G_M^{(\text{IA})}(q^2) = G_S^E(q^2) \frac{6\sqrt{2}}{q^2} \int dr \left[u(r)w(r) - \frac{w^2(r)}{\sqrt{8}}\right] j_2\left(\frac{qr}{2}\right),
\]

\[
\frac{2M_N}{M_d} G_M^{(\text{IA})}(q^2) = G_S^E(q^2) \frac{3}{2} \int dr \left[u^2(r)\left[j_0\left(\frac{qr}{2}\right) + j_2\left(\frac{qr}{2}\right)\right]\right] + G_M^S(q^2) 2 \int dr \left[u^2(r)j_0\left(\frac{qr}{2}\right)\right] + G_M^S(q^2) \left\{\sqrt{2} \int dr \left[u(r)w(r)j_2\left(\frac{qr}{2}\right)\right] - \int dr \left[u^2(r)\left[j_0\left(\frac{qr}{2}\right) - j_2\left(\frac{qr}{2}\right)\right]\right]\right\},
\]

where $j_0$ and $j_2$ are spherical Bessel functions.
deuteron in the IA
\[ \mu_{dA}^{I} = P_S(\mu_n + \mu_p) + P_D \left[ -\frac{1}{2}(\mu_n + \mu_p) + \frac{3}{4} \right]. \] 
(7.113)

The resulting values for the different meson exchanges are shown in the second column of Table 7.2.

Now we want to study the effect of the MECs. To do so we directly use the formulas by Sitarski et al. in Ref. [308] where the mesonic currents arising from pair terms due to \( \pi-, \rho- \) and \( \omega- \) exchange together with the model dependent \( \rho\pi\gamma \) exchange current are considered to construct the deuteron magnetic form factor. The result are the following expressions,

\[
\frac{M_N}{M_d}G_M^\pi(q^2) = G_M^S(q^2) \frac{g_{\pi NN}^2}{2\pi} \frac{m_\pi^3}{(2M)^3} \times \int \text{d}r \left\{ A(r) \left[ j_0 \left( \frac{qr}{2} \right) Y_0(m_\pi r) + j_2 \left( \frac{qr}{2} \right) Y_2(m_\pi r) \right] + B(r) \left[ j_0 \left( \frac{qr}{2} \right) Y_2(m_\pi r) + \frac{1}{2} (Y_2(m_\pi r) + Y_2(m_\pi r)) j_2 \left( \frac{qr}{2} \right) \right] \right\},
\]

(7.114)

\[
\frac{M_N}{M_d}G_M^\rho(q^2) = \frac{1}{3} F_1^S(q^2) \frac{g_{\rho NN}^2}{2\pi} \frac{m_\rho}{2M} \times \int \text{d}r \left\{ (1 + \kappa_\rho) \left[ A(r) - \frac{1}{2} B(r) \right] (1 + m_\rho r) Y_0(m_\rho r) \right\} \left[ j_0 \left( \frac{qr}{2} \right) + j_2 \left( \frac{qr}{2} \right) \right],
\]

(7.115)

\[
\frac{M_N}{M_d}G_M^\omega(q^2) = \frac{1}{3} F_1^S(q^2) \frac{g_{\omega NN}^2}{2\pi} \frac{m_\omega}{2M} \times \int \text{d}r \left\{ (1 + \kappa_\omega) \left[ A(r) - \frac{1}{2} B(r) \right] (1 + m_\omega r) Y_0(m_\omega r) \right\} \left[ j_0 \left( \frac{qr}{2} \right) + j_2 \left( \frac{qr}{2} \right) \right],
\]

(7.116)

\[
\frac{M_N}{M_d}G_M^{\rho\pi\gamma}(q^2) = G_{\rho\pi\gamma}^S(q^2) \frac{g_{\pi NN}g_{\rho NN}g_{\pi\gamma NN}}{4\pi m_\rho} \times \int \text{d}r \int_0^1 \text{d}z \left\{ L(1 + Lr) Y_0(Lr) [A(r)j_2(y) + B(r)j_1(y)] \
- (2 - Lr) L Y_0(Lr) A(r) j_0(y) \
+ \frac{1}{2} (2Lr - 1) L Y_0(Lr) B(r) j_2(y) \
- \frac{9}{10} qz Lr Y_0(Lr) [j_1(y) + j_3(y)] B(r) \right\},
\]

(7.117)

where \( j_0, j_1, j_2 \) and \( j_3 \) are spherical Bessel functions, \( A(r) = w^2(r) - \frac{1}{2} w^2(r), B(r) = w^2(r) + \sqrt{2} u(r) w(r), \)
\( L^2 = m_\pi^2 + (m_\rho^2 - m_\pi^2) z + q^2 z (1 - z), y = (1 - 2z)qr/2, Y_0(x) = e^x/x, Y_2(x) = (1 + 3/x + 3/x^2) Y_0(x), \)
\( G_M^S \) and \( F_1^S \) are the isoscalar magnetic and Dirac form factors respectively and \( G_{\rho\pi\gamma} \) is the form factor at the vertex \( \rho\pi\gamma \). We want to emphasize that this is a preliminary calculation in which we test our renormalized wave function by using expressions derived by other authors. Of course a more deep analysis must be carried out in the near future. These MEC contributions to the deuteron magnetic form factor are plotted in Fig. 7.9 where we only include the \( \pi-, \rho- \) and the \( \rho\pi\gamma\)-MECs.

4 Note that In Ref. [314] the \( \pi- \) exchange contribution to the magnetic form factor of the deuteron is calculated and the result is \( 1/2 \) Eq. (7.114). In Ref. [315] the \( \rho\pi\gamma\)-exchange contribution is calculated and the expression is almost the one in Eq. (7.117) but with \( z - 1 \) instead of \( z \) in the last term.
Figure 7.9: Deuteron magnetic form factor including MEC contributions. We only include the $\pi$-, the $\rho$- and the $\rho\pi\gamma$-MECs. In this figure we have used a cut-off $r_c = 0.05$ fm and the bands represent the choice for the rho meson coupling constant $f_{\rho NN} = 15.5$ and $f_{\rho NN} = 17$ when the exchange of $\pi$, $\sigma$, $\omega$ and $\rho$-meson are considered.

Figure 7.10: Deuteron magnetic form factor including MEC contributions for cut-off ranging $r_c = 0.05 - 0.8$ fm in the case of $\pi$-exchange and $r_c = 0.05 - 0.3$ fm in the case of $\pi\sigma\omega$-exchange. Here we keep fix the rho meson coupling constant to $f_{\rho NN} = 15.5$.

for cut-offs large enough without spoiling the deuteron properties. In the case of $\pi$-exchange where the dependence with the cutoff is smooth we can take $r_c = 0.05 - 0.8$ fm but when introducing vector mesons we have to take $r_c = 0.05 - 0.3$ fm. Results are displayed in Fig. 7.10.
where we include the IA + (\pi + \rho + \rho\pi\gamma)-MEC contributions due to the \pi, \pi + \sigma, \pi + \sigma + \rho + \omega using our renormalized wave functions for the OBE potential of sec. 7.5.1. \pi\sigma\rho\omega corresponds to take \( f_{\rho NN} = 15.5 \) and \( g_{\omega NN} = 9.857 \) while \( \pi\sigma\rho\omega^{\ast} \) corresponds to take \( f_{\rho NN} = 17.0 \) and \( g_{\omega NN} = 10.147 \). We take \( \kappa_\pi = -0.12, \kappa_\omega = 3.66, g_{\rho\pi\gamma} = 0.56 \) and \( g_{\rho NN} = 2.9 \) for the MEC contributions. We only include the \pi-, the \rho- and the \rho\pi\gamma-MECs. The experimental value \[ 316 \] is \( \mu_d(\mu_N) = 0.8574382329(92) \).

Table 7.2: Deuteron magnetic moment in Bohr magneton units for OBE potentials including \pi-, \pi + \sigma, \pi + \sigma + \rho + \omega using our renormalized wave functions for the OBE potential of sec. 7.5.1. \pi\sigma\rho\omega corresponds to take \( f_{\rho NN} = 15.5 \) and \( g_{\omega NN} = 9.857 \) while \( \pi\sigma\rho\omega^{\ast} \) corresponds to take \( f_{\rho NN} = 17.0 \) and \( g_{\omega NN} = 10.147 \). We take \( \kappa_\pi = -0.12, \kappa_\omega = 3.66, g_{\rho\pi\gamma} = 0.56 \) and \( g_{\rho NN} = 2.9 \) for the MEC contributions. We only include the \pi-, the \rho- and the \rho\pi\gamma-MECs. The experimental value \[ 316 \] is \( \mu_d(\mu_N) = 0.8574382329(92) \).

The limit \( q \to 0 \) of Eqs. (7.114)-(7.117) gives the MEC contributions to the magnetic moment of the deuteron,

\[
\mu_\pi^{(\text{pair})} = \frac{g_{\pi NN}^2}{2\pi} \frac{m_\pi^2}{(2M)^3} \int dr \left[ A(r)Y_0(m_\pi r) + B(r)Y_2(m_\pi r) \right],
\]

(7.118)

\[
\mu_\rho^{(\text{pair})} = \frac{g_{\rho NN}^2}{2\pi} \frac{m_\rho}{(2M)} \int dr \left\{ (1 + \kappa_\rho) \left[ A(r) - \frac{1}{2} B(r) \right] (1 + m_\rho r)Y_0(m_\rho r) - \frac{9}{4} w^2(r)Y_0(m_\rho r) \right\},
\]

(7.119)

\[
\mu_\omega^{(\text{pair})} = -\frac{1}{3} \frac{g_{\omega NN}^2}{2\pi} \frac{m_\omega}{(2M)} \int dr \left\{ (1 + \kappa_\omega) \left[ A(r) - \frac{1}{2} B(r) \right] (1 + m_\omega r)Y_0(m_\omega r) - \frac{9}{4} w^2(r)Y_0(m_\omega r) \right\},
\]

(7.120)

\[
\mu_{\rho\pi\gamma} = \frac{g_{\rho NN} g_{\pi NN} g_{\rho\pi\gamma}}{2\pi m_\rho \left( m_\rho^2 - m_\omega^2 \right)} \frac{1}{m_\rho \left( m_\rho^2 - m_\pi^2 \right)} \int dr \left\{ A(r) \left[ m_\rho^3 Y_0(m_\pi r) - m_\rho^3 Y_0(m_\rho r) \right] + B(r) \left[ m_\omega^3 Y_2(m_\pi r) - m_\omega^3 Y_2(m_\omega r) \right] \right\},
\]

(7.121)

which result using our wave functions are shown in Table 7.2. As in previous calculations, we only include the contributions due to the \pi-, the \rho- and the \rho\pi\gamma-MECs. The dependence of the deuteron magnetic moment \( \mu_d(\mu_N) \) with the cut-off radius \( r_c \) is shown in Fig. 7.11 where we include the \pi-, the \rho- and the \rho\pi\gamma-MECs. We observe a divergence when the \pi-MEC is included in addition to the \pi-IA but this divergence is removed once one includes the vector mesons. An important observation is that plateaus happen below \( r_c = 0.1 \text{fm} \) in the case of vector meson exchanges.

7.6 Conclusions

In this chapter we have analyzed the MEC effects for several EM processes using our renormalized leading-\( N_c \) OBE potential wave functions. We have seen how gauge invariance relates meson exchange potentials and longitudinal currents through the continuity equation and how these currents can be calculated consistently from the NN potential. This intrinsic relation suggests that one should expect the same degree of accuracy either in the longitudinal currents and in the potentials.
In the traditional OBE potentials, strong form factors are usually added to mimic the finite size of the Nucleon [53], which is about 0.6 fm. In the case of gauge invariance, the inclusion of a form-factor introduced by hand, i.e., not computed consistently within meson theory, implies a kind of non-locality in the interaction which could be made gauge invariant by introducing link operators between two points, thereby generating a path dependence, for which no obvious resolution has been found yet. On the other hand, purely phenomenological potentials [51, 58] not based entirely on the Meson Exchange picture are inherently ambiguous or eventually produce conflicting results with gauge invariance which are difficult to quantify.

Following the philosophy of a recent work [317] we suggest using a renormalization procedure for potentials, wave functions and currents. For practical purposes the finite nucleon size is comparable to the minimal resolution probed in NN scattering, so that we do not expect to see the difference between a point-like nucleon and an extended one at sufficiently low energies. Thus, we propose replacing strong form factors by renormalization conditions on suitable low energy scattering properties. In the lowest partial waves we have shown that after renormalization finite nucleon effects parameterized as strong form factors are indeed marginal. We expect a similar effect for MEC.

In fact we have analyzed deuteron electro-disintegration using the Hockert [301] and Riska’s [293, 302] formulas for the differential cross section. We have seen how the simplest \(\pi\)-MEC is enough to explain the experimental data up to \(q^2 \sim 15\) fm\(^{-2}\). However the inclusion of heavier mesons gives a discrepancy with the data which have to be analyzed further.

Radiative neutron capture results have been also provided. The interesting result here is the dependence of the total cross section with the cut-off radius \(r_c\) which suggests that in fact cut-offs below 0.15 fm should be achieved.

Finally we have analyzed electron-deuteron scattering and the deuteron EM form factors \(G_C, G_Q\) and \(G_M\) in the IA and the MEC corrections to \(G_M\). A good agreement to experimental data is achieved in the description of the IA deuteron EM form-factors using our renormalized wave functions. In the case of the MEC corrections to \(G_M\) a shift respect to the experimental data is observed when the cut-off is removed in the case of the \(\pi\)-MEC. The inclusion of heavier mesons improves the description. The dependence of
this MEC corrections to $G_M$ with $r_c$ has also been analyzed. There is a range $r_c = 0.05 - 0.8$fm where
the $\pi$-MEC is on top of the experimental data.

It should be noted that a more deep study must be carried in what respect the dominant current appearing
in the $1/N_c$ expansion in consistency with our large-$N_c$ OBE potential. However we believe that the
renormalization process described here is a very useful tool in order to avoid inconsistencies between
potentials and currents.
Conclusions

In the present Thesis we have analyzed the OBE potential and the associated currents from a renormalization point of view. As we have shown, the meson-nucleon Lagrangean does not predict the S-matrix beyond perturbation theory. The non-perturbative nature of low partial waves and the deuteron in the NN problem suggests resuming OBE diagrams by extracting the corresponding potential. The OBE potential, however, presents short distance divergences which make the solution of the corresponding Schrödinger equation ambiguous. The traditional remedy for this problem has been the inclusion of phenomenological form factors which parameterize the vertex functions in the meson exchange picture. We have shown that the meson exchange potential with form factors generates spurious deeply bound states for natural values of the coupling constants. The price to remove those is to fine-tune the potential at all distances, and in particular at short distances. Thus, while it is claimed that vertex functions implement the finite nucleon size, it is very difficult to disentangle this from meson dressing and many other effects where the meson theory does not hold.

The implementation of vertex meson-nucleon functions has also notorious side effects, in particular it affects gauge invariance, chiral symmetry and causality via dispersion relations. As it is widely accepted, besides the description of NN scattering and the deuteron, one of the great successes and confirmations of Meson theory has been the prediction of Meson Exchange Currents (MEC’s) for electroweak processes. In the case of gauge invariance, the inclusion of a form-factor introduced by hand, i.e., not computed consistently within meson theory, implies a kind of non-locality in the interaction. This can be made gauge invariant by introducing link operators between two points, thereby generating a path dependence, and thus an ambiguity is introduced. In the limit of weak non-locality the ambiguity is just the standard operator ordering problem, for which no obvious resolution has been found yet. Form factors can also be in open conflict with dispersion relations, particularly if they imply that the interaction does not vanish as a power of the momentum everywhere in the complex plane. We have shown that within the renormalization approach, all singularities fall on the real axes and spurious deeply bound states are shifted to the real negative axis.

The renormalization approach suggests that extracting detailed short distance information may in fact be unnecessary for the purposes of Nuclear Physics and the verification of the meson exchange picture. Contrarily to what one might naively think, renormalization is a practical and feasible way of minimizing short distance ambiguities, by imposing conditions which are fixed by low energy data independently on the potential. We have argued that within this approach we face from the start our inability to pin down the short distance physics below the smallest de Broglie wave length probed in NN scattering. Indeed, the central scattering waves and the deuteron can be described reasonably well and with natural values of the meson-nucleon couplings. Within the standard approach this could only be achieved by fine-tuning meson parameters or postulating the meson exchange picture to even shorter ranges than 0.5fm.
In our case the inclusion of shorter range mesons, such as \(\rho, \omega, a_1\) mesons, induces moderate changes, due to the expected short distance insensitivity embodied by renormalization, despite the short distance singularity and without introducing strong meson-nucleon-nucleon vertex functions. If phenomenological vertex functions are added on top of the renormalized calculations minor effects are observed confirming the naive expectation that finite nucleon size \(\sim 0.5\text{fm}\) need not be explicitly introduced within the OBE calculations for c.m. momenta corresponding to the minimal wavelength \(1/p \sim 0.5\text{fm}\).

One of the problems with potential model calculations is the ambiguity in form of the potential, since it is determined from the on-shell S-matrix in the Born approximation and an off-shell extrapolation becomes absolutely necessary. In the large \(N_c\) limit the spin-isospin and kinematic structure of the NN potential simplifies tremendously yielding a non-relativistic and uniquely defined local and energy independent function. Relativistic effects, spin-orbit, non-localities as well as meson widths or other mesons enter as sub-leading corrections to the potential with a relative order \(1/N_c^2\). However, it consists of an infinite tower of multi-meson exchanged states, which range is given by the Compton wavelength of the total multi-meson mass. One of the advantages of the large \(N_c\) expansion is that it is not particularly restricted for low energies. This is exemplified by several recent calculations of NN potentials using the holographic principle based on the AdS/CFT correspondence [318–320] \(^5\). A truncation of the infinite number and range of exchanged mesons is based on the assumption that the hardly accessible high mass states are irrelevant for NN energies below the inelastic pion production threshold. This need not be the case, unless a proper renormalization scheme makes this short distance insensitivity manifest. Actually, within such a scheme the counterterms include all unknown short distance effects, but enter as free parameters which do not follow from the potential and which must be fixed directly from NN scattering data or deuteron properties. In the present work we have implemented a boundary condition regularization and carried out the necessary renormalization. This allows, within the OBE potential to keep only \(\pi, \sigma, \rho, \omega\) and \(a_1\) mesons and neglect effectively higher mass effects for the lowest central s-waves as well as the deuteron wave function. In many regards we see improvements which come with very natural choices of the couplings, and are compatible with determinations from other sources. From this viewpoint, the leading \(N_c\) contribution to the OBE potential where \(\pi, \sigma, \rho\) and \(\omega\) mesons appear on equal footing, seems superior than the leading chiral contribution which consists just on \(\pi\).

Let us remind that MEC are a genuine consequence of the Meson Exchange picture in the NN interaction, but in fairness also require constructing exact NN wave functions from the corresponding Hamiltonian, as we have done here. The present thesis helps in reducing the impact of the hardly accessible short distance region of the nucleon-nucleon interaction and their currents, thereby reducing standard and much debated ambiguities.

A further interesting insight which we have exploited in the present thesis is related to the usefulness of renormalization as applied to CSB. Actually, this allows us to design a short distance connection based on the assumption of finiteness between apparently disconnected problems. We have been able to successfully predict strong pp, nn and Coulomb pp phase shifts and low energy constants from the np ones. A natural and exciting extension of such an insight would be to equally connect two-baryon systems with different strangeness within a \(SU(3)\) framework.

One serious source of complications and limitations for renormalization in general lies in its difficult marriage with the variational principle. The existence of two-body spurious deeply bound states drives

\(^5\)In this calculations only \(\pi, \rho, \omega\) and \(a_1\) mesons and their radial excitations contribute. Note, however, that the only contribution to the central force \(V_C\) stems from the tower \(\omega, \omega', \omega'', \ldots\) which is generally repulsive.
naturally the energy of the system to its lowest energy state, if allowed to. On the other hand, one should recognize that the existence of a minimum is tightly linked to a subtle balance between kinetic and potential energy, which undoubtedly exists but may well take place beyond the applicability range of the meson exchange picture requiring an artificial fine-tuning. This clearly influences the three, four, etc. body problems if they would be treated in the standard and variational fashion but not necessarily so if the few body problem is consistently renormalized. Our results show that one has to choose between fine-tuning and renormalization. The standard approach has traditionally been sensitive to short distance details and has required fine-tuning meson coupling constants, in particular those corresponding to vector mesons, to unnatural values. In contrast, the renormalization approach is free of fine-tuning, and allows to fix meson constant from other sources to their natural values.

While QCD is entitled to eventually generate all features of Nuclear Physics, one may profit from a semiquantitative analysis based on distinct properties of the fundamental theory. Within a large $N_c$ expansion the leading piece of the NN potential is $O(N_c)$ and has the tensorial structure

$$V_{NN}(r) = V_c(r) + \tau_1 \cdot \tau_2 [\sigma_1 \cdot \sigma_2 W_5(r) + S_{12}(\hat{x}) W_T(r)] + O(1/N_c) \quad (7.122)$$

with corrections (comprising relativistic corrections, spin orbit, meson widths, etc.), suppressed by a relative $1/N_c^2$. This potential only complies to the Wigner symmetry for even partial waves. These large-$N_c$ counting rules are based on quark and gluon dynamics, but for large distances quark-hadron duality allows to saturate them by the standard (multi-)meson exchange picture. The striking thing is that this symmetry pattern emerges in the effective interaction !!. This important point prevents us from using large $N_c$ literally, but rather as a long distance symmetry. This way, the ubiquitous fine-tunings, triggered by unknown short distance physics, are efficiently disentangled from long distance physics with the help of renormalization. We have pursued this large-$N_c$ framework for the low partial waves and the deuteron in the case of One Boson Exchange (OBE) and its Meson Exchange Currents where only $\pi, \sigma, \rho, \omega, a_1$ contribute. The large $N_c$ status of Serber symmetry is a bit more shaky. We have noted that when Wigner symmetry fails, as allowed by large $N_c$ considerations, Serber symmetry holds instead. Our OBE analysis seems consistent with the mended symmetry interpretation of Weinberg [216] or chiral quark model calculations [218], namely the similarity between scalar and vector meson masses, $m_\sigma = m_\rho$, are favoured.

In any case the large $N_c$ form of the NN potential can be retained with relative $1/N_c^2$ accuracy since meson widths enter beyond that accuracy as sub-leading corrections, on equal footing with many other effects (spin-orbit, relativistic and other mesons), independently on how large the $\sigma$ width is in the real $N_c = 3$ world. While we have been using the leading large $N_c$ contributions to the full OBE as a simplifying book-keeping reduction, we do not expect that such an approximation becomes crucial regarding the main conclusions on form factors. However, the most speculative prospective of the present calculation lies in the possibility of promoting it to a model independent large $N_c$ result. One should bear in mind, however, that we have only kept leading $N_c$ OBE contributions. There is, of course, the delicate question on what $2\pi, 3\pi$ and $\Delta$ contributions should be considered, firstly to avoid double counting with the collective $\sigma, \rho$ and $\omega$ states, and second to comply with the large $N_c$ requirements. To our knowledge, the expectations of Ref. [142] of a large $N_c$ consistent multi-meson exchange picture have not been explicitly realized for the chiral potentials without [110] and with [111] $\Delta$-isobar contributions as they do not scale properly with $N_c$; one has $g_A \sim N_c$, $f_\pi \sim \sqrt{N_c}$ and there are terms scaling as $V_{2\pi}^{\text{ChPT}} \sim g_A^2 / f_\pi^2 \sim N_c^2$ and not as $\sim N_c$ as found in Refs. [137, 140]. Our results suggest a scenario where the multi-meson contributions invoked
in Ref. [142] would indeed be small, but this should be checked explicitly. One further complication comes from the fact that in the large $N_c$ limit the nucleon-delta splitting becomes small, and in fact lighter than the pion mass. According to the Regge theory formula $M_\Delta^2 - M_N^2 = m_\rho^2 - m_\pi^2$ [321] and assuming the scaling $M_N = N_c m_\rho / 2$, the crossover between both mass parameters happens at about $N_c \sim 6$. Actually, in the strict limit one should consider not only NN but at least also $N\Delta$ and $\Delta\Delta$ channels as well, as they become degenerate. The calculation of [140, 141] only includes the restriction of the baryon-baryon interaction to the NN sector. In a more elaborate treatment one should include the $\Delta$ as intermediate dynamical states which in the elastic NN region contribute as sub-threshold effects [322] which decouple for large $N\Delta$ splitting but which become degenerated when the $N\Delta$ splitting is driven to zero. In addition, it would also be interesting, still within the OBE framework, to see what is the effect of the relative $1/N_c^2$ corrections, which include in particular relativistic, non-local, finite-size, spin-orbit, finite meson width corrections as well as other mesons. The renormalization approach can be suitably tailored to accommodate such non-local corrections. This requires a separate study of renormalization conditions which hopefully might explain the P-waves.
Appendix A

Large $N_c$ QCD and the Nucleon-Nucleon interaction

In this appendix we review the Large-$N_c$ limit of QCD. This is intended to collect the relevant aspects used in the main text which usually appear scattered in the literature.

A.1 A toy model: the N-dimensional Hydrogen atom

In order to appreciate the idea behind the large-$N_c$ limit of QCD it is instructive the following Witten’s toy model [323]. Let me consider the simple example in quantum mechanic of a non-relativistic hydrogen atom with Hamiltonian ($\alpha = e^2/4\pi$)

$$H = \frac{\not{p}^2}{2m} - \frac{\alpha}{r},$$  \hspace{1cm} (A.1)

Suppose one does not know how to solve the problem exactly and wants to find the energy spectrum by considering the interaction term as a perturbation inspired by the fact that $\alpha = 1/137$ is a small and dimensionless number. This is not possible since we can always perform a scale transformation and define $\bar{r}/t$ and $\bar{p} = pt$ with $t = 1/(ma)$ in such a way that we can rewrite the Hamiltonian as

$$H = (ma^2)\left[\frac{\bar{p}^2}{2} - \frac{1}{\bar{r}}\right].$$ \hspace{1cm} (A.2)

As we can see our expansion parameter disappear because $ma^2$ is an overall energy scale which can be absorbed by the Hamiltonian redefining the temporal scale, i.e.,

$$H \rightarrow \frac{H}{ma^2} = \left[\frac{\bar{p}^2}{2} - \frac{1}{\bar{r}}\right].$$ \hspace{1cm} (A.3)

Therefore, making perturbation theory using $\alpha$ as expansion parameter does not make sense.

Assuming that the physics of atoms and molecules can be described by the Hamiltonian (A.3) what could we use as expansion parameter?. Because there is no evident expansion parameter we have to find a hidden one. This situation remind to what happened in critical phenomena where the community
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was frustrated for decades by the absence of an expansion parameter. Wilson and Fisher [324] suggested to look at the number of spatial dimensions not as a fix number but as a variable parameter. Critical phenomena are simpler in four dimension and in $4 - \epsilon$ dimensions one can have a deep understanding by doing perturbation theory in $\epsilon$ [325]. In analogy instead of considering atomic physics in $N = 3$ dimensions with a $O(3)$ symmetry, let us consider atomic physics in $N$-dimensions with a $O(N)$ symmetry and take the limit $N \to \infty$. For $N$-dimensions with $N$ big, we can write the Schrödinger equation as

$$\left[-\frac{1}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{N}{r}\frac{\partial}{\partial r}\right) - \frac{\alpha}{r}\right]\Psi = E\Psi.$$  \hfill (A.4)

If we re-define the wave function $\Psi = r^{-N/2}\bar{\Psi}$ and re-scale the radial coordinate $r = N^2 R$ we can express the original Hamiltonian as,

$$H = \frac{1}{N^2}\left[-\frac{1}{2mN^2}\frac{\partial^2}{\partial R^2} + \frac{1}{8mR^2} - \frac{\alpha}{R}\right],$$  \hfill (A.5)

with again $1/N^2$ a overall energy scale. The important point is that our new Hamiltonian describes the dynamics of a particle of effective mass $M_{eff} = m N^2$ in an effective potential $V_{eff}(R) = 1/(8mR^2) - \alpha/R$. Obviously in the limit $N \to \infty$ the particle is very heavy $M_{eff} \to \infty$ and will be located near the minimum of $V_{eff}$, so to the lowest order in $1/N$ the ground state energy is $E_0 = V_{eff}(r_0)/N^2$ with $V'_{eff}(r_0) = 0$. The result is $E_0 = -2m\alpha^2/N^2$ which for the real world $N = 3$ gives $E_0 = -2/9m\alpha^2$ in comparison to the exact result $E_0 = -1/2m\alpha^2$. In general, the energy of the ground state can be written as a series in powers of $1/N$ and the excitation spectrum may also be found in this scheme.

In the next section we will study the qualitative nature of the large-$N_c$ limit for mesons, baryons and its interactions.

A.2 Review of Witten’s diagrammatic rules

A.2.1 ’t Hooft double line notation

In the $1/N_c$ expansion, one considers QCD with $N_c$ colors and a $SU(N_c)$ gauge group, in the large-$N_c$ limit. To find the $1/N_c$ order of a given diagram it is farther convenient the use of the double line notation introduced by ’t Hooft in Refs. [136, 185]. This notation comes from the following observation. The quark and anti-quark fields are $q^a(x)$ and $\bar{q}_b(y)$ each having $N_c$ components. The quark propagator is

$$\langle q^a(x)\bar{q}^b(y) \rangle = \delta^{ab}D(x - y),$$  \hfill (A.6)

where $a$, $b$ are indexes in the fundamental representation $a, b = 1, \ldots, N_c$ and is represented diagrammatically by a single line with the color at the beginning of the line the same as at the end, because of the $\delta_{ab}$. The gluon field $G_\mu = G_\mu^AT^A$ transforms under the adjoint representation rather than the fundamental one and its propagator is

$$\langle G^A_\mu(x)G^B_\nu(y) \rangle = \delta^{AB}D_{\mu\nu}(x - y),$$  \hfill (A.7)
where $A, B$ are now indexes in the adjoint representation, $A, B = 1, \ldots, N_c^2 - 1$ and the generators of the group $T^A$ satisfy

$$\text{Tr } T^A T^B = \frac{1}{2} \delta^{AB}. \quad (A.8)$$

Instead of treating a gluon as a field with a single adjoint index, it much more convenient in the large-$N_c$ limit to treat it as a $N_c \times N_c$ traceless matrix $G_{\mu b} = (T^A)_b^a G^A_{\mu} (T^A)_{a}^{c}$ with $N_c^2$ components in the fundamental representations. The gluon propagator can then be re-written as

$$\langle G_{\mu b}(x) \bar{G}^{c}_{\nu d}(y) \rangle = \left( \frac{1}{2} \delta^a_d \delta^c_b - \frac{1}{2N_c} \delta^a_d \delta^c_b \right) D_{\mu \nu} (x - y), \quad (A.9)$$

where the $SU(N_c)$ identity

$$(T^A)_b^a (T^A)^c_d = \frac{1}{2} \delta^a_d \delta^c_b - \frac{1}{2N_c} \delta^a_d \delta^c_b, \quad (A.10)$$

has been used. For large values of $N_c$ we recover the $U(N_c)$ identity without the last term in Eq. (A.10) and the differences between $SU(N_c)$ and $U(N_c)$ become unimportant. In this representation the $U(N_c)$ gluon field has one upper index like the quark field $q^a$ carrying color $a$ and one lower index like the anti-quark $\bar{q}_b$ carrying color $b$. Thus, so far as color is concerned and for keeping track of color quantum numbers, a gluon may be regarded as a quark-anti-quark combination $G_{\mu b} \sim q^a \bar{q}_b$. Thus, one may represent the gluon as a double line in a Feynman diagram, one line for the quark and other for the anti-quark with opposite directions (see Fig. A.1). Similarly we can play the same game with the interaction vertex of Fig. A.2. In the large $N_c$ there are many quark and anti-quark states ($\sim N_c$) but even more gluon states ($\sim N_c^2$) and everything consists on combinatoric analysis. For example, let us consider one of the simplest Feynman diagrams such as the one-loop gluon vacuum polarization. By looking at Fig. A.3 we see that the color index lines at the edge of the diagram are contracted with those of the initial and final states. However, there is a closed color line (having color $k$) that is contracted only with itself. The value of $k$, the color running around this loop, is unspecified even when the initial and final states are given, and the sum over $k$ gives a factor of $N_c$. Then, in the limit $N_c \to \infty$ this diagram is infinite. But on the other hand, there is also a factor of coupling $g$ at each of the two interaction vertices. If we
want the one-loop gluon vacuum polarization to have a smooth limit for large \(N_c\), we must choose the coupling constant to satisfy the 't Hooft limit,

\[
\lim_{N_c \to \infty} g^2 N_c = \text{constant}. \tag{A.11}
\]

Namely, the QCD coupling constant has to scale as \(g \sim 1/\sqrt{N_c}\). In the gluon one-loop vacuum polarization diagram there are two vertices and one color loop, and then this diagram scale as,

\[
\left(\frac{1}{\sqrt{N_c}}\right)^2 \times N_c = O(1) \tag{A.12}
\]

In Fig.\(A.2\) the quark-gluon vertex and three gluon vertex scale as \(O(1/\sqrt{N_c})\) and the four-gluon vertex as \(O(1/N_c)\).

Once this rules have been established it is not difficult to determine the \(1/N_c\) order of a given diagram. For example one can verify that higher order loop corrections to the gluon propagator (see Fig.\(A.4\)) all scale \(O(1)\) regardless of how complicated is the diagram because by adding an extra gluon to a planar diagram in such a way that the diagram remain planar, always creates two extra interaction vertices and one closed loop which contributes as \((1/\sqrt{N_c})^2 \times N_c = O(1)\). However quark loop corrections to the gluon propagator (see Fig.\(A.5\)) are suppressed by a factor of \(1/N_c\) because the quark propagator corresponds to a single color line, not two, and the closed color line is absent, no large combinatoric factor appears, and its only dependence on \(N_c\) comes from factors of \(1/\sqrt{N_c}\) at each of the two vertices. On the other side non-planar diagrams (see Fig.\(A.6\)), i.e., those which are not possible to draw in a plane without line crossing, always vanish at least like \(1/N_c^2\) for large \(N_c\).
Therefore, the leading order Feynman diagrams in the large-$N_c$ limit are planar with a minimum number of quark loops.

### A.2.2 Mesons in the large-$N_c$ limit

Now we shall analyze mesons in the large-$N_c$ limit. At this point we have to make an assumption. So far it is believed that QCD is a confining theory at $N_c = 3$, we will assume that confinement persists also at large-$N_c$. On the other side it should be noted that although planar diagrams are dominant, we do not even know how to sum them. Nevertheless, we will assume that the planar diagrams sum up to give a confining theory. Under these assumptions we can consider large-$N_c$ mesons as a colorless bound state of quarks and anti-quarks. The color singlet meson wave function is,

$$|1\rangle_c = \frac{1}{\sqrt{N_c}} (q_{c_1} \bar{q}_{c_1} + \cdots + q_{c_{N_c}} \bar{q}_{c_{N_c}}), \quad (A.13)$$

with $c_i, i = 1, \ldots, N_c$ the color quantum numbers. In a confining theory $q\bar{q}$ pairs are always bound together into a meson, and the fact that there is always exactly one $q\bar{q}$ pair means that there is always exactly one meson, rather than one meson plus glue states.

Let us consider the operator $J$ which creates a single meson from the vacuum and prove that the intermediate state of the two point function of $J$ acting on the vacuum creates only one single meson, i.e., a $q\bar{q}$ state. We represent a two point function of a quark bilinear in Fig. A.7 with three intermediate glue states in addition to the quark-anti-quark pair. If only one intermediate meson state is required then quark, anti-quark and gluons must be coupled in such a way that a colorless state arises. The color structure at the cut is $\bar{q}_j A^j_k A^k_i q^i$. The four fields are coupled such that all the color indexes are contracted to form a color singlet state, and no smaller combination of them is separately a color singlet. It is impossible to split the combination of these four fields into two or more color singlet pieces. By
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Figure A.7: Intermediate states contributing to $\langle JJ \rangle$.

comparison, for example the structure $\bar{q}_k A^f_l q^l A^m_j A^m_j$ is the product of two color singlet operators $\bar{q}_k A^f_l q^l$ and $A^m_j A^m_j$ which can be interpreted as an intermediate state representing a meson ($\bar{q}_k A^f_l q^l$) and a glue state ($A^m_j A^m_j$). It turns out that this short of structures only appear for non-planar diagrams which are in fact suppressed in the large-$N_c$.

The fact that the intermediate states are one-particle states means that the two-point function of $J$ can be written as

$$\langle J(k)J(-k) \rangle = \sum_n \frac{f_n^2}{k^2 - m_n^2}, \quad (A.14)$$

where $f_n = \langle 0|J|n \rangle$ is the amplitude to create a meson from the vacuum and $n$ runs over meson states. The left hand side of Eq. (A.14) is leading $\mathcal{O}(N_c)$ since it can be represented only by one closed-loop diagram (internal gluon states do not change the order of the diagram but internal quark loops are suppressed $\mathcal{O}(1)$), therefore the meson decay constant is $f_n \sim \mathcal{O}(\sqrt{N_c})$ and meson masses have smooth limits $\mathcal{O}(1)$. They are stable since the one particle poles of Eq. (A.14) must be in the real axis, otherwise it would violate the spectral representation. Finally we can also say that the number of mesons is infinity and the reason is the following; by asymptotic freedom the behavior of the left-hand side is know to be logarithmic for large $k^2$ and this cannot be fulfilled with a finite number of terms at the right-hand side.

We can now determine how narrow the meson states are. We have already seen that mesons are stable and we will see that the amplitudes for a meson two-body decay $A \rightarrow BC$ are $\mathcal{O}(1/\sqrt{N_c})$. Let us consider a three-point function of $J$ such as in Fig. A.8. The three point function is $\mathcal{O}(N_c)$ since is given by the one-loop diagram which contributes with a factor of $N_c$ because the quark loop. Each cross in the diagram represents the creation or annihilation of a meson, which contributes as $f_n \sim \mathcal{O}(\sqrt{N_c})$. The amplitude of the process is of the form $\langle |n|J|0 \rangle^3 \Gamma_{mmm}$ where $\Gamma_{mmm}$ is the three meson vertex which obviously has to be $\mathcal{O}(1/\sqrt{N_c})$ by consistency. As a consequence meson decay amplitudes vanish in the limit $N_c \rightarrow \infty$ and we can say that mesons are very narrow in the large-$N_c$ limit.

The two-body meson scattering amplitudes $AB \rightarrow CD$ is given by the four-point function of $J$ (see Fig. A.8) which can be regarded as a sum of tree diagrams which may involve meson exchange, with local vertices of order $1/\sqrt{N_c}$, or may also be a contact interaction $ABCD$ of order $1/N_c$. Note that in diagrams with one meson exchange one have to sum over all possible mesons compatible with the reaction, for example, for $\pi\pi \rightarrow \pi\pi$ one has to consider a tower of meson exchanges with isospin one.\footnote{This is actually in agreement with the rather successful Regge phenomenology in which the strong interactions are interpreted as an infinite sum of tree diagrams with hadron exchange.}
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In general we can say that as the scattering amplitude vanish at the large-$N_c$ limit, mesons are free and non-interacting.

The generalization to $n$-meson interactions can be easily done. The $n$-point function is represented by a planar quark-loop diagram which is order $O(N_c)$ and each current creating or annihilating a meson contributes $O(\sqrt{N_c})$, so in general the $n$-meson vertex must be order $(\frac{1}{\sqrt{N_c}})^{(n-2)}$.

\[ n\text{-mesons vertex} \approx \sqrt{N_c}^{n-1} \left(\frac{1}{\sqrt{N_c}}\right)^{(n-2)} \tag{A.15} \]

One can do a similar study for glue states. In particular, it turns out that at $N_c \to \infty$ the glue states are free, stable, non-interacting and infinite in number. An important property for us is that to lowest order in $1/N_c$, the glue states are decoupled from mesons. In general, the $N_c$-counting rules imply that one has a weakly interacting theory of mesons and glueballs with a coupling constant $1/N_c$. Because it is weakly interacting one can expand in the coupling constant $1/N_c$. As we see QCD, a strongly interacting theory of quarks and gluons, has been re-written as a weakly interacting theory of hadrons.

A.2.3 Baryons in the large-$N_c$ limit

In the large-$N_c$ limit of QCD baryons are color singlet combinations of $N_c$ quarks, antisymmetric in color because quarks obey Fermi statistic and symmetric in the other quantum numbers such as spin and flavor,

\[ \epsilon_{i_1 \ldots i_{N_c}} q^{i_1} \cdots q^{i_{N_c}}, \tag{A.16} \]

Since a baryon has $N_c$ quarks we will have Feynman diagrams with $N_c$ quark lines. For example, let us have a look at the first three perturbative corrections to the free propagation of $N_c$ quarks (see Fig A.9). At the lowest order we have one-gluon exchange between two of the quarks. The quark-gluon coupling constant is of order $1/\sqrt{N_c}$ and then this diagram has an explicit factor of $1/N_c$. In addition the gluon
may have been exchanged by any two of the $N_c$ quarks in the baryon. There are $\sim N_c(N_c - 1)/2 \sim N_c^2$ different choices. So the total contribution of this diagram is of order,

$$N_c^2 \left( \frac{1}{\sqrt{N_c}} \right)^2 = N_c.$$  \hfill (A.17)

If we now consider the higher order diagram where two gluons are exchanged between the quarks we have four vertices giving a factor $(1/\sqrt{N_c})^4$ and $N_c^4$ different choices for the exchange. The contribution is then

$$N_c^4 \left( \frac{1}{\sqrt{N_c}} \right)^4 = N_c^2.$$  \hfill (A.18)

So the higher is the correction the higher is the divergence. It turns out, however, that a simple large $N_c$ limit exists, but that diagrams are not a very convenient way to study this limit. To understand this apparently divergent behavior of perturbation theory we can consider a simple quark-model picture where the baryon mass is expected to be the sum of the quark masses, the quark kinetic energy and the quark-quark potential energy. The quark masses contribute an amount $N_cm$ to the baryon mass, where $m$ is the quark mass. For the quark kinetic energy, we may guess that the kinetic energy of $N_c$ quarks is $N_c$ times the kinetic energy $t$ of one quark. For the potential energy, we already know that the interaction between two quarks is of order $1/N_c$ as they exchange a gluon. We can say that it is $1/N_c$ times some $v$. Since the total potential energy is a sum of all of the pair interactions, and there are $\sim N_c^2$ pairs we can write the baryon mass as

$$M_B = N_c m + N_c t + N_c^2 \left( \frac{1}{N_c} v \right).$$  \hfill (A.19)

The key point is that in the potential energy the combinatory factors combine to give a contribution $O(N_c)$, the same order as the other terms in the baryon mass. Thus, the baryon mass is $O(N_c)$ in the large-$N_c$ limit and the series we were studying corresponds to an expansion of the amplitude for a free baryon propagation,

$$e^{-iM_B t} = 1 - iM_B t - \frac{1}{2!} M_B^2 t^2 - \frac{i}{3!} M_B^3 t^3 + \cdots.$$  \hfill (A.20)

Thus the bad behavior of perturbation theory does not mean that the large $N_c$ limit for baryons does not exist, but only that the baryon mass is of order $N_c$.

Another important point is the size and shape of the baryon. In the large-$N_c$ limit where the baryon is very heavy we can describe the dynamics of the $N_c$ quarks bound into a baryon by a non-relativistic Schrödinger equation where the interaction between any given pair of quarks is negligible, i.e., order $1/N_c$. But the potential felt by any one quark is order one since it interacts with the $N_c$ other quarks.
with a strength $1/N_c$. We may then regard the potential experienced by one quark as a background. In such a situation the total wave function for the baryon can be written as a product of single particle wave functions of this averaged potential as in the Hartree approximation,

$$\psi(x_1, \ldots, x_{N_c}) = \prod_{i=1}^{N_c} \phi(x_i),$$

(A.21)

where $\phi$ is the one particle wave function, i.e., the ground state of the averaged potential. It turns out that the Hartree equation which determines the large-$N_c$ limit of the baryon wave function is independent of $N_c$,

$$h \phi(x) = t \phi(x) + \phi(x) \int d^3y \, \phi^*(y) v(x, y) \phi(y) = \epsilon \phi(x),$$

(A.22)

where $h$ the Hamiltonian for one quark in the baryon, $t$ its kinetic energy and $v$ the averaged non-local potential that the quark feel inside the baryon. Note that the Hamiltonian for the baryon is $H_B = N_c h$.

Therefore, although the mass of the baryon is of order $N_c$, its structure is determined by solving the non-linear Hartree equations where $N_c$ has scaled out having as a consequence that its size and shape have smooth limits as $N_c \to \infty$ since $\phi(x)$ does not depend on $N_c$. As a conclusion, baryons are heavy with a mass $O(N_c)$ but with size and shape $O(1)$.

We should note that at all times we are working in a particular kinematic regime with baryon momenta $p \sim O(N_c)$ and velocities $v \sim O(1)$.

### A.2.4 Meson-baryon and baryon-baryon scattering in the large-$N_c$

In first place let us study the large-$N_c$ dependence of the meson-baryon coupling. We can see that mesons are strongly coupled to baryons for large-$N_c$ by considering the two typical diagrams of Fig. A.10 where the baryon is coupled to a single meson. The meson can be inserted into a single quark line in the baryon Fig. A.10 (a) without affecting the color structure of the quark lines. There are $N_c$ possibilities of choosing a quark line in the baryon and a factor $1/\sqrt{N_c}$ coming from the meson insertion (because the normalization of the wave function Eq. (A.13)), so the overall diagram is $O(\sqrt{N_c})$. If the quark lines of the meson are inserted into different positions in the baryon Fig. A.10 (b), the color rearrangement is mediated by the exchange of a gluon. There are $N_c^2$ possible choices for the pair of quark lines, the gluon exchange contributes with $1/N_c$ and the insertion of the meson with $1/\sqrt{N_c}$, so the overall diagram is also $O(\sqrt{N_c})$. Therefore, the meson-baryon coupling constant is $O(\sqrt{N_c})$.

![Figure A.10: Quark-gluon diagrams for the meson-baryon coupling.](image)
With the same idea we can analyze meson-baryon scattering. In Fig. A.11 we represent the scattering process baryon - meson → baryon - meson with quark-gluon diagrams. If both mesons are inserted into the same quark line of the baryon Fig. A.11 (a) the color structure of the quark lines is preserved and we have a factor $N_c$ from choosing a quark line and a factor $1/\sqrt{N_c}$ from each meson insertion. The overall process is $O(1)$. Alternatively, the two mesons can be inserted into different quark lines Fig. A.11 (b). In this case a gluon must be exchanged to transfer energy between the incoming and outgoing meson and to preserve color. We have a factor $N_c^2$ from choosing two quark lines, a factor $1/\sqrt{N_c}$ from each meson insertion and a factor $1/N_c$ from the gluon exchange which gives an overall factor $O(1)$.

So, the meson-baryon scattering amplitude is $O(1)$. This means that, meson-baryon interaction is negligible compared to the baryonic kinetic energy, which is $O(N_c)$ because the baryon mass is $O(N_c)$ and its velocity $O(1)$, being too small as to affects the motion of the baryon. The baryon propagates freely as if the meson were not present at all. In fact, if we regard the interaction at the hadronic level, the momentum of the intermediate baryon can be written as,

$$ P^\mu = M v^\mu + k^\mu \equiv M v'^\mu, \quad (A.23) $$

where $v^\mu$ is the velocity of the initial baryon and $k^\mu$ is the momentum transfer. Because $M v^\mu \sim O(N_c)$ and the momentum transfer is $O(1)$, as it comes from the meson, the intermediate baryon four-velocity

$$ v'^\mu = v^\mu + O(1/N_c), \quad (A.24) $$

is equal to the initial baryon four-velocity in the large-$N_c$ limit and there is no recoil in the baryon. However, the meson mass is $O(1)$ and then the meson kinetic energy is of the same order than the meson-baryon interaction. This means that, the meson is scattered by the baryon because the interaction is large enough as to influence the meson motion.

We now analyze the baryon-baryon interaction. The dominant baryon-baryon interaction comes from the exchange of a pair of constituents. Typical diagrams of this short are represented in Figs. A.12. In these diagrams one quark "jump" from a baryon to the other. Since these quarks always carry color and we cannot fix its quantum numbers by hand, to preserve color neutrality at least a gluon must be exchanged. For example, the color neutrality is restored by exchange of a single gluon in Fig. A.12 (a). In this case there is a factor of $N_c$ from choosing a quark from the first baryon, a factor of $N_c$ from choosing a quark in the second baryon, and a factor of $1/N_c$ from the gluon couplings giving altogether an amplitude for this process $O(N_c)$. The situation is the same for higher order corrections such as in
Fig. A.12 (b) where two gluons are exchanged. One have a combinatoric factor of $N_c^3$ and a factor of $1/N_c^2$ from the coupling constants. So, the dominant contribution to the baryon-baryon interaction is $\mathcal{O}(N_c)$.

One natural way to interpret the physics contained in the diagrams of Figs. A.12 is by considering them at the hadronic level as a one meson exchange between baryons. Actually one can generalize this picture for two mesons exchanges and so on. A typical two-meson exchange diagram is represented in Fig. A.13.

It is straightforward to see that this kind of diagrams grow as $N_c^2$ since one has $N_c^4$ from combinatorics and four quark-gluon vertices which contribute with $1/N_c^2$. This apparent divergence of the perturbative series when we add more and more meson exchanges can be understood if we note that we are iterating the underlying $\mathcal{O}(N_c)$ potential\(^2\).

\footnote{As was pointed out by Banerjee, Cohen and Gelman \cite{142} to preserve the spin-flavor structure of the NN potential necessary cancellations between meson retardation in direct box diagrams and crossed box diagrams take place. In the TPE case the $\Delta$-isobar embodying the contracted $SU(4)$ symmetry was explicitly needed.}

**Figure A.12**: Typical $N_c$ contributions to the baryon-baryon interaction in the large-$N_c$. In a hadron-based picture this kind of diagrams are regarded as one meson exchanges.

**Figure A.13**: Typical $N_c^2$ contribution to the baryon-baryon interaction in the large-$N_c$. In a hadron-based picture this kind of diagrams are regarded as two meson exchanges.

### A.3 Contracted $SU(4)$ spin-flavor symmetry

The large $N_c$ counting rules imply non-trivial constraints on baryonic static matrix elements which can be derived by considering meson-baryon scattering at low energies. As we have seen according to Witten’s large-$N_c$ counting rules the scattering amplitude for the process meson + baryon $\rightarrow$ meson + baryon is $\mathcal{O}(1)$ but the meson-baryon vertex is $\mathcal{O}(\sqrt{N_c})$. Individual baryon-meson diagrams describing the absorption and then emission of a meson by a baryon contain two baryon-meson vertices and grow as...
\( N_c \). If this is so, the total scattering amplitude would grow as \( N_c \) which violates unitarity and Witten’s counting rules unless there are cancellations among the \( \mathcal{O}(N_c) \) meson-baryon scattering diagrams. In fact, these cancellations are described in the large-\( N_c \) by means of consistency conditions which constrain meson-baryon couplings.

The easiest way to derive these consistency conditions is by considering low-energy pions, or equivalent, baryon axial current matrix elements \([138, 139]\). At low energies the pion-nucleon vertex is given by chiral symmetry through derivative coupling,

\[
\frac{\partial \mu \pi^a}{f_\pi} \langle N| q^\mu \gamma_5 \tau^a q|N \rangle ,
\]

(A.25)

where \( a = 1, 2, 3 \) is the isospin index of the pion. This vertex is \( \mathcal{O}(\sqrt{N_c}) \) given the fact that \( g_A \sim \mathcal{O}(N_c) \) as the baryon is a coherent state of \( N_c \) quarks and the pion couples to each of these quarks, the pion-baryon axial vector coupling constant is order \( N_c \) \([139]\) and \( f_\pi \sim \mathcal{O}(\sqrt{N_c}) \) as it is a meson decay constant (see Sec. A.2.2).

In the large-\( N_c \) limit, the nucleon is infinitely heavy compared with the pion, so the time component of Eq. (A.25) vanishes and it reduces to the static-nucleon axial current that we write as,

\[
\langle N| q^\mu \gamma_5 \tau^a q|N \rangle = g_{Nc} \langle N| X^a|N \rangle ,
\]

(A.26)

where we have kept the \( N_c \) dependence explicit, being \( \langle N| X^a|N \rangle \) and \( g \sim g_A/N_c \) of \( \mathcal{O}(1) \) and \( X^a \) a spin and isospin one operator.

The leading contributions to pion-nucleon scattering amplitude come from the two diagrams of Fig. A.14. The propagator of the intermediate nucleon in Fig. A.14 (a) reduces on-shell \((v^2 = 1)\),

\[
S_a(P^\mu) = \frac{i(P + M)}{P^2 - M} \rightarrow i(1 + \not{v}) = \frac{i(1 + \not{v})}{k \cdot v} ,
\]

(A.27)

and in the nucleon rest frame \((k \cdot v = k^0 = E_\pi)\) we get \( S_a(P^\mu) = i/E_\pi \). Equivalently we get for Fig. A.14 (b) \( S_b(P^\mu) = -i/E_\pi \). Therefore, in the static limit with on-shell nucleons and in the nucleon rest frame the scattering amplitude can be written as

\[
\mathcal{M} = -i \frac{q^\mu q^\nu}{E_\pi} \frac{N^2 g^2}{f_\pi^2} \left[ X^a, X^b \right] ,
\]

(A.28)

where the product of \( X \)'s matrices sums over possible spins and isospins of the intermediate baryon states and only these intermediate baryons which are degenerate with the initial and final nucleon in the large \( N_c \) limit contribute in the sum. Since \( f_\pi \sim \mathcal{O}(\sqrt{N_c}) \) the overall amplitude is \( \mathcal{O}(N_c) \) which

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3These can be also derived for general mesons coupling to baryons but the derivation is much more laborious [326].
contradicts Witten’s counting rules. Therefore, we need other intermediate states apart of the nucleon $I = J = 1/2$ which cancel this divergence. We can generalize $X^{ia}$ to be an operator on this degenerate set of intermediate baryon states with the following constraint,

$$N_c [X^{ia}, X^{jb}] \leq O(1). \quad (A.29)$$

If we use an $1/N_c$ expansion of the $X^{ia}$ operator,

$$X^{ia} = X_0^{ia} + \frac{1}{N_c} X_1^{ia} + \frac{1}{N_c^2} X_2^{ia} + \cdots \quad (A.30)$$

then the large-$N_c$ consistency condition imply,

$$[X_0^{ia}, X_0^{jb}] = 0. \quad (A.31)$$

The physical content of Eq. (A.31) is the following. Consider the scattering process $N + \pi \rightarrow N + \pi$. If the only intermediate baryon involved in the scattering is the nucleon ($I = J = 1/2$), then Eq. (A.31) must be satisfied for the nucleon matrix elements of the operator $X_0^{ia}$ which is proportional to the nucleon matrix elements of $\sigma^i \tau^a$. But the nucleon matrix elements of $\sigma^i \tau^a$ do not commute and Eq. (A.31) cannot be satisfied. Then, there must be additional baryon states degenerate with the nucleon that participate as intermediate states in the scattering. Let us assume that it is the first nucleon excited state $\Delta$ ($I = J = 3/2$) and let us include it. The consistency condition Eq. (A.31) allows us to determine $g_{\pi \Delta N}$ in terms of $g_{\pi NN}$. Now consider the process $N + \pi \rightarrow \Delta + \pi$ with intermediate states being N and $\Delta$. Eq. (A.31) allows to determine $g_{\pi \Delta \Delta}$ in terms of $g_{\pi \Delta N}$ and $g_{\pi NN}$. But as far as the $\Delta$ is included, the process $\Delta + \pi \rightarrow \Delta + \pi$ should be also consider which cannot be satisfied unless there is an additional baryon state with $I = J = 5/2$. This process continues up to infinity, i.e., the solution of the large-$N_c$ consistency condition requires an infinite tower of degenerate baryons with $I = J = 1/2, 3/2, 5/2, \ldots$.

Since $X^{ia}$ is a tensor operator with spin one and isospin one, the commutation relations with the spin and isospin generators $J^i$ and $I^a$ are:

$$[J^i, X_0^{jb}] = i \epsilon^{ijk} X_0^{kb},$$
$$[I^a, X_0^{jb}] = i f^{abc} X_0^{jc}. \quad (A.32)$$

The generators $J^i$ and $I^a$ satisfy the usual $SU(2)$ algebra,

$$[J^i, J^j] = i \epsilon^{ijk} J^k,$$
$$[I^a, I^b] = i f^{abc} I^c,$$
$$[I^a, J^i] = 0. \quad (A.33)$$

The commutation relations Eqs. (A.31), (A.32) and (A.33) form the contracted spin-flavor $SU(2N_f)_c$ algebra in the large-$N_c$ limit with $N_f$ the number of flavors with the spin $J^i$, flavor $I^a$ and the amplitudes $X^{ia}$ the generators of the group. To see that it is in fact a contracted algebra let us compare the $SU(2N_f)_c$ spin-flavor algebra which generators are $J^i$, $I^a$ and the Gamow-Teller operator
\[ G^{i\alpha} \text{ satisfying the commutation relations,} \]
\[
\begin{align*}
[J^i, J^j] &= i\epsilon^{ijk} J^k, \\
[I^a, I^b] &= i f^{abc} I^c, \\
[I^a, J^j] &= 0,
\end{align*}
\] (A.34)

and
\[
\begin{align*}
[J^i, G^{jb}] &= i\epsilon^{ijk} G^{kb}, \\
[I^a, G^{jb}] &= i f^{abc} G^{jc}, \\
[G^{ia}, G^{jb}] &= \frac{i}{2N_f} \delta^{ab}\epsilon^{ijk} J^k + \frac{i}{4} \epsilon^{ijk} f^{abc} I^c + \frac{i}{2} \epsilon^{ijk} d^{abc} G^{kc}.
\end{align*}
\] (A.35)

We obtain the contracted \( SU(2N_f) \) algebra from the \( SU(2N_f) \) algebra by rescaling the Gamow-Teller operator \( G^{i\alpha} \) and taking the large-\( N_c \) limit,
\[
X^i_0 = \lim_{N_c \to \infty} \frac{G^{i\alpha}}{N_c}.
\] (A.36)

As we can see this contraction affects only to the commutator \([G^{i\alpha}, G^{jb}]\) which becomes \([X^i_0, X^j_0] = 0\) because the right hand side vanish as the baryon operators \( J \) and \( I \) are at most \( O(N_c) \). In [327] Dashen et al. found all possible irreducible representations of the contracted spin-flavor algebra. However, for large and finite \( N_c \) one can work with the \( SU(2N_f) \) algebra with \( N_f \) the number of flavors rather than the contracted one. It is shown that for \( N_f = 2 \) the ground state baryon representation produces a finite tower of baryon states with \( I = J \),
\[
I = J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{N_c}{2},
\] (A.37)

which reduces to the states \( N \) and \( \Delta \) for \( N_c = 3 \).

### A.4 Spin-flavor structure of the NN interaction

We have seen in Sec. A.3 that meson-baryon interactions respect a manifest \( SU(2N_f) \) spin-flavor symmetry. The Lie algebra is given in terms of fifteen generators \( I^a, J^i \) and \( G^{i\alpha} \) where the spin \( i \) and isospin indexes \( a \) take values \( i, a = 1, 2, 3 \). The six generators \( I^a, J^i \) are the usual generators for \( SU(2) \)-isospin and \( SU(2) \)-spin. This important result can provide information about the structure of the NN interaction using the \( 1/N_c \) expansion.

The general form of the NN potential for elastic and non-relativistic scattering from general invariance principles such as translations, Galilean, rotations, parity and time-reversal invariance is,
\[
V_{NN} = \sum V_{i} \text{ terms,}
\]
\[
V_{NN} = V_C + V_S \sigma_1 \cdot \sigma_2 + V_{LS} L \cdot S + V_{T} S_{12} + V_{Q} Q_{12} \\
\left[ W_C + W_S \sigma_1 \cdot \sigma_2 + W_{LS} L \cdot S + W_{T} S_{12} + W_{Q} Q_{12} \right] \tau_1 \cdot \tau_2,
\] (A.38)
where
\[
S_{12} = 3 \sigma_1 \cdot \hat{r} \sigma_2 \cdot \hat{r} - \sigma_1 \cdot \sigma_2, \quad (A.39)
\]
\[
Q_{12} = \frac{1}{2} \{\sigma_1 \cdot L, \sigma_2 \cdot L\}, \quad (A.40)
\]
and the subscripts C, S, T, LS and Q stand for central, spin-spin, tensor, spin-orbit and quadratic spin-orbit interactions. In Refs. [140, 141] the strength of the $V$'s and $W$'s functions were determined in the $1/N_c$ expansion. They consider baryons as described by a Hartree Hamiltonian in the case of two light flavors and work with the one quark operator basis of the quark-model to make group theory computations easily. In this basis one has,
\[
\hat{j}^i = q^i \frac{\sigma^i}{2} q, \quad \hat{r}^a = q^i \frac{\sigma^a}{2} q, \quad G^{ia} = q^i \frac{\sigma^i \sigma^a}{4} q, \quad (A.41)
\]
where $q = (u, d)$ and $q^i$ are annihilation and creation operators for the $u$ and $d$ quark flavors. These operators are one-body operators in the quark representation as they act on a single quark line.

In the large-$N_c$ limit baryons are color singlet states of $N_c$ quarks. In general we want to calculate baryon matrix elements of one-body QCD operators $O_{QCD}^{1\text{-body}} = \hat{q} \Gamma \hat{q}$ where $\Gamma$ is a Dirac bilinear. The baryon matrix element of $O_{QCD}^{1\text{-body}}$ is obtained by inserting the operator on any of the $N_c$ quark lines. Since there are $N_c$ insertions and each one is $O(1)$, the one-body QCD operator has a matrix element which is at most $O(N_c)$. The matrix element is not necessarily $O(N_c)$ since there may be cancellations among the $N_c$ insertions as we will see soon.

We can write the QCD operator as a quark operator expansion as follows,
\[
O_{QCD}^{1\text{-body}} = N_c \sum_n c_n \frac{1}{N_c} O_n, \quad (A.42)
\]
where the sum is over all possible $n$-body quark operators $n = 0, \ldots, N_c$ with the same spin-flavor quantum numbers as $O_{QCD}^{1\text{-body}}$ and the unknown coefficient $c_n \sim O(1)$ containing the complicated QCD dynamics. A $n$-body operator $O_n$ is typically $O(N_c^n)$ because each one-body operator is order $N_c$. That means that all of the terms in Eq. (A.42) are of the same order in the $1/N_c$ expansion as the leading term. Let us emphasize that the operators in the r.h.s. of Eqs. (A.41) are one-body quark operators. An example of QCD operator expressed as quark operator expansion is the baryon mass,
\[
M = m_0 N_c 1 + m_2 \frac{1}{N_c} j^2 + \cdots \quad (A.43)
\]

The matrix elements of the operators $O_n$ have a nontrivial dependence on $N_c$. The ground state baryon spin tower $J = I = 1/2, 3/2, \ldots, N_c/2$ contains baryons with spin $J$ and isospin $I$ of order unity at the bottom of the spin tower, as well as baryons with spin and isospin of $O(N_c)$ at the top of the spin tower but all of the baryon states in the multiplet have matrix elements of $G^{ia}$ that are $O(N_c)$. For low-spin baryons with $J \sim O(1)$ the hyperfine splitting $J^2/N_c$ is order $1/N_c$, whereas for baryons with $J \sim O(N_c)$, the hyperfine splitting is order $N_c$, of the same order as the average mass of the baryon multiplet. Therefore, near the top of the tower the $1/N_c$ correction to the baryon mass is order $N_c$ and hence is not a small perturbation. However it is small at the bottom of the tower. The $1/N_c$ expansion is then under control for baryons with fixed and finite spin and isospin when the large-$N_c$ limit is taken, but the expansion breaks down for baryons with spin and isospin order $N_c$. 

Appendix A. Large $N_c$ QCD and the Nucleon-Nucleon interaction
As a consequence, the baryon matrix elements of the operators (A.41) between ground state baryon states \( B \) restricted to have \( J = I \simeq \mathcal{O}(1) \) satisfy,

\[
\langle B|\hat{J}/N_c|B\rangle \sim \langle B|\hat{I}/N_c|B\rangle \sim 1/N_c, \quad \langle B|\hat{G}/N_c|B\rangle \sim 1, \quad (A.44)
\]

It is not difficult to see intuitively that this scaling laws constrain the structure of the NN potential Eq. (A.38) to the following hierarchy

\[
V_C \sim W_S \sim W_T \sim N_c, \quad (A.45)
\]

\[
W_C \sim V_S \sim V_T \sim V_{LS} \sim W_{LS} \sim W_Q \sim 1/N_c, \quad (A.46)
\]

\[
V_Q \sim 1/N_c^3. \quad (A.47)
\]

This scaling rules are usually referred to as Kaplan-Savage-Manohar (KSM) counting rules. Therefore, for two flavors, the leading structure of the NN potential from spin-flavor symmetry is,

\[
V_{NN} = V_C + \tau_1 \cdot \tau_2 [W_S \sigma_1 \cdot \sigma_2 + W_T S_{12}], \quad (A.48)
\]

that is, the strongest interaction is the central force terms \( V_C \) and \( W_S \) as well as the tensor force \( W_T \). The remaining contributions, with the exception of \( V_Q \), are relatively suppressed by \( \mathcal{O}(1/N_c^2) \). The isospin invariant quadratic spin-orbit force \( W_Q \) is suppressed \( \mathcal{O}(1/N_c^3) \) compared to the central potential.

It is worthy to connect the KSM counting rules and the potential structure with the meson exchange picture at the hadronic level. The meson-baryon couplings connecting baryon states are given by baryon matrix elements of \( I^a, J^i, G^{ia} \) and the identity operator \( 1 \). The non-relativistic meson-baryon couplings are shown in Table A.1 for mesons with a mass near the nucleon mass (~1 GeV).

<table>
<thead>
<tr>
<th>Meson</th>
<th>Coupling</th>
<th>Scaling</th>
<th>Order</th>
</tr>
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<tr>
<td>Scalar</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( I = 0 )</td>
<td>( \sigma )</td>
<td>( B^iB\phi )</td>
<td>( \sqrt{N_c} )</td>
</tr>
<tr>
<td>( I = 1 )</td>
<td>( a_0 )</td>
<td>( B^i I^a B\phi^a )</td>
<td>( 1/\sqrt{N_c} )</td>
</tr>
<tr>
<td>Pseudo-scalar</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( I = 0 )</td>
<td>( \eta )</td>
<td>( B^i J^i B\phi^i )</td>
<td>( 1/\sqrt{N_c} )</td>
</tr>
<tr>
<td>( I = 1 )</td>
<td>( \pi )</td>
<td>( B^{i \dagger} G^{ia} B^i \phi^a )</td>
<td>( \sqrt{N_c} )</td>
</tr>
<tr>
<td>Vector</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( I = 0 )</td>
<td>( \omega^0 )</td>
<td>( B^i BV^i )</td>
<td>( \sqrt{N_c} )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( B^i \epsilon_{ijk} J^k B \phi^i \phi^j )</td>
<td>( 1/\sqrt{N_c} )</td>
<td>NLO</td>
</tr>
<tr>
<td>( \rho^0 )</td>
<td>( B^i I^a BV^{ia} )</td>
<td>( 1/\sqrt{N_c} )</td>
<td>NLO</td>
</tr>
<tr>
<td>( \rho )</td>
<td>( B^{i \dagger} \epsilon_{ijk} G^{ka} B^j \phi^{ia} )</td>
<td>( \sqrt{N_c} )</td>
<td>LO</td>
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<tr>
<td>Axial</td>
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<tr>
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<td>( f_1 )</td>
<td>( B^i J^i BA^i )</td>
<td>( 1/\sqrt{N_c} )</td>
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<td>( I = 1 )</td>
<td>( a_1 )</td>
<td>( B^{i \dagger} G^{ia} BA^i )</td>
<td>( \sqrt{N_c} )</td>
</tr>
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</table>

Table A.1: Non-relativistic meson-baryon couplings and its corresponding \( 1/N_c \) scalings.

Matrix elements of the operators \( 1 \) and \( G^{ia} \) are of order \( N_c \). However, matrix elements containing only \( J^i \) or \( I^a \) are suppressed \( \mathcal{O}(1/N_c) \), i.e., are of order \( N_c^0 \). The \( 1/N_c \) scaling of a given meson-baryon coupling constant is obtaining by dividing the corresponding current matrix element by the meson decay constant which scale \( \mathcal{O}(\sqrt{N_c}) \). These scalings are shown in the fifth column of Table A.1. Each meson can contribute to different spin-flavor components of the NN potential, but the KSM countings are always preserved. In fact, it can be shown (see Appendix. B) that those contributions which are sub-leading
according to KSM countings are suppressed by inverse powers of the baryon mass $1/M_B \sim 1/N_c$. The order at which each meson contributes to the expansion is in the last column of the table.

Therefore, as a minimum requirement for a One-Boson-Exchange model for the NN interaction fulfilling KSM counting rules one has to include at leading order the $\sigma$, $\pi$, $\omega$-time component, $\rho$-space component and eventually the $a_1$. This is what we do in Appendix B \footnote{It is important to remark that not all the component of the NN potential generated by a given LO meson are leading in the expansion. Relativistic corrections such as spin-orbit and non-local contributions are suppressed as multiplicative $1/M_N$ factors arise.}. The rest of mesons in Table A.1 should be incorporated in a second step as next-to-leading order corrections.
Appendix B

One Boson Exchange Potential

In this appendix we shall derive the non-relativistic One Boson Exchange (OBE) potential used along the thesis. To do so, we will start by considering invariant Feynman amplitudes for the simplest tree level diagrams and eventually keep only terms with a well-behaved $1/N_c$ scaling. In Sec. B.1 we set the notation and conventions. In Sec. B.2 the non-relativistic potential and the S-matrix are related and in Sec. B.3 we list the Lagrangian densities to calculate Feynman amplitudes. We derive then the NN potential, in the Born approximation, in Sec. B.4. The exchange of one scalar, one pseudo-scalar, one vector, one tensor and one axial will be considered. Expressions in $p$-space and $r$-space are presented. In Sec. B.5 an overview of estimates of couplings from several sources is presented. Finally we also show expressions for transition potentials in Sec. B.6 and calculate the $\Delta$ decay width in Sec. B.7.

B.1 Notation and convections

B.1.1 Dirac and Rarita-Schwinger spinors

For positive energy spin-$\frac{1}{2}$ particles such as the nucleon, the bispinor satisfying the Dirac equation

$$(\not{p} - M_N)u(p, s) = 0,$$  \hspace{1cm} (B.1)

with normalization\footnote{We use Bjorken-Drell [328] normalization.}

$$\bar{u}(p, s)u(p, r) = \delta_{rs},$$  \hspace{1cm} (B.2)

is given by

$$u(p, s) = \sqrt{\frac{E_N + M_N}{2M_N}} \left( \frac{1}{\not{E} + M_N} \right) \chi_s^{(1/2)},$$  \hspace{1cm} (B.3)
with $E_N = \sqrt{p^2 + M_N^2}$ and the two-component spinors

$$\chi_{+1/2}^{(1/2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-1/2}^{(1/2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad \text{(B.4)}$$

The projector operator is

$$A_{+}^{(1/2)}(p) \equiv \sum_{r=1}^{2} u(p, r) \bar{u}(p, r) = \frac{p + M_N}{2M_N}. \quad \text{(B.5)}$$

For positive energy spin-$\frac{3}{2}$ particles such as the $\Delta$-isobar, the bispinor is described by the Rarita-Schwinger equation [329]

$$(p - M_\Delta) u^{\mu}(p, s) = 0, \quad \text{(B.6)}$$

with the constraints

$$\gamma_\mu u^{\mu}(p, s) = 0, \quad \text{(B.7)}$$

$$p_\mu u^{\mu}(p, s) = 0, \quad \text{(B.8)}$$

where $p$ is on-shell since spinors are on-shell by definition

$$(p^2 - M_\Delta^2) u^{\mu}(p, s) = 0. \quad \text{(B.9)}$$

The on-shell positive energy projection operator is

$$A_{+}^{(3/2)}(p)^{\mu\nu} \equiv \sum_{r=1}^{4} u^{\mu}(p, r) \bar{u}^{\nu}(p, r)$$

$$= \begin{pmatrix} p + M_\Delta \\ 2M_\Delta \end{pmatrix} \left( -g^{\mu\nu} + \frac{1}{3} \gamma^{\mu} \gamma^{\nu} + \frac{2p^{\mu}p^{\nu}}{3M_\Delta^2} - \frac{p^{\mu}p^{\nu} - p^{\nu}p^{\mu}}{3M_\Delta^2} \right) \bigg|_{p^2 = M_\Delta^2} \quad \text{(B.10)}$$

The field $u^{\mu}(p, s)$ is known as the Rarita-Schwinger field and it can be constructed by direct product of a spin-1 and a spin-$\frac{1}{2}$ state, which in the rest frame leads to,

$$u^{\mu}(0, s') = \sum_{r, s} \left< 1 \right| \hat{S}^{\frac{3}{2}} \hat{S}^{\frac{3}{2}} | s' > u(0, s) \epsilon^{\mu}_{s} \equiv \sum_{s=1}^{2} S^s(0, s) u^{\mu}(0, s), \quad \text{(B.11)}$$

where we have defined the transition operator $S^{\mu} = (0, S)$ as

$$S = \begin{pmatrix} -\hat{\epsilon}_{-} & 0 \\ \sqrt{2/3} \hat{\epsilon}_{0} & -\sqrt{1/3} \hat{\epsilon}_{-} \\ -\sqrt{1/3} \hat{\epsilon}_{+} & \sqrt{2/3} \hat{\epsilon}_{0} \\ 0 & -\hat{\epsilon}_{+} \end{pmatrix} \quad \text{(B.12)}$$

with $\hat{\epsilon}_{\pm} = \mp \frac{1}{\sqrt{2}} (\hat{x} \pm i\hat{y})$, $\hat{\epsilon}_{0} = \hat{z}$. We can obtain the general expression for the Rarita-Schwinger field by boosting from the rest frame to an arbitrary momentum frame

$$u^{\mu}(p, s) = L^{(3/2)}(p)^{\mu}_{\nu} u^{\nu}(0, s). \quad \text{(B.13)}$$
The boost operator can be factorized into one for a spin-1 particle of mass $M$:

$$L^{(1)}(p)_\mu = \frac{1}{M} \begin{pmatrix} E_\Delta & p_1 & p_2 & p_3 \\ p_1 & p_2 & M_\Delta \delta_{ij} + \frac{p_i p_j}{E_\Delta + M_\Delta} \end{pmatrix},$$  \hspace{1cm} (B.14)

and one spin-$\frac{1}{2}$ boost:

$$u(p, s) = L^{(1/2)}(p) \ u(0, s) = \left[2M_\Delta(E_\Delta + M_\Delta)\right]^{-1/2} (\gamma \cdot p + M_\Delta) \ u(0, s) = \left(\frac{E_\Delta + M_\Delta}{2M_\Delta}\right)^{1/2} \left(\frac{1}{\gamma \cdot p + M_\Delta}\right) \chi_s^{1/2}. \hspace{1cm} (B.15)$$

So, we can write the spin-$\frac{3}{2}$ field in a finite-momentum frame as

$$u_\mu(p, s) = \left(\frac{E_p + M_\Delta}{2M_\Delta}\right)^{1/2} L^{(1)}(p)_\mu \left(\frac{1}{\gamma \cdot p + M_\Delta}\right) (S^1)^{\nu} \chi_s^{3/2}. \hspace{1cm} (B.16)$$

These spinors are normalized according to

$$\bar{u}(p, r)u_\mu(p, s) = -\delta_{rs}, \hspace{1cm} (B.17)$$

and we also have the relation

$$\bar{u}(p, r)\gamma^\nu u_\mu(p, s) = -\frac{p^\nu}{M_\Delta}. \hspace{1cm} (B.18)$$

### B.1.2 Dirac bilinears

When calculating amplitudes we have to deal with bilinear combinations of $\Gamma$-matrices $\Gamma = 1, \gamma_5, \gamma^\mu, \gamma_5 \gamma^\mu, \sigma^{\mu\nu}$ and Dirac fields. Using our definition for Dirac fields Eq. (B.3), the Dirac tensor matrix elements are given by

$$\bar{u}(p, r)\Gamma p_s M(p', p) \chi_s = \chi_s^+ M(p', p) \chi_s, \hspace{1cm} (B.19)$$

where $p$ is the four-momentum of the in-going particle and $p'$ the one for the out-going. $\chi_s$ are the two-component Pauli spinor and the matrix elements $M(p', p)$ are displayed in Table. B.1.

Neither non-relativistic reductions nor on-shell particles have been consider here yet.

### B.2 Covariance, S-matrix and the non-relativistic potential

The relativistic interaction between two particles is described in a covariantly fashion by the Bethe-Salpeter (BS) equation which in operator notation reads,

$$\mathcal{M} = \mathcal{V} + \mathcal{G} \mathcal{M}, \hspace{1cm} (B.20)$$
with \( \mathcal{M} \) the invariant Feynman amplitude for the two particle scattering process, \( \mathcal{V} \) the sum of all two-particle irreducible diagrams, in order to avoid double counting when we iterate the equation, and \( \mathcal{G} \) the relativistic two-particle propagator. Since this four-dimensional integral equation is very hard to solve, the so-called three dimensional reduction is usually done to obtain a more readily solved three-dimensional equation. This reduction leaves the equation covariant and satisfy relativistic elastic unitarity. Although this reduction is not unique, all they basically consist on replacing Eq. (B.20) by two coupled equations

\[
\mathcal{M} = \mathcal{V} + \mathcal{W} \cdot g \cdot \mathcal{M}, \tag{B.21}
\]

\[
\mathcal{W} = \mathcal{V} + \mathcal{V} (\mathcal{G} - g) \mathcal{W}, \tag{B.22}
\]

where \( g \) is a three-dimensional propagator with the same analytic structure as \( \mathcal{G} \). In particular as suggested by Blankenbecler and Sugar (BbS) \[330\] \text{BbS} can be chosen with the same discontinuity across the right-hand cut as \( \mathcal{G} \) and in such a way that if we take the approximation \( \mathcal{W} = \mathcal{V} \) in Eq. (B.22), one can obtain from Eq. (B.21) the following equation for the scattering amplitude (see [3] for details)

\[
\mathcal{F}(q', q) = \mathcal{V}(q', q) + \int \frac{d^3k}{(2\pi)^3} \mathcal{V}(q', k) \frac{M^2}{E_k} \frac{1}{q'^2 - k^2 + i\epsilon} \mathcal{F}(k, q). \tag{B.23}
\]

If we define

\[
\hat{\mathcal{F}}(q', q) = \sqrt{\frac{M}{E_{k'}}} \mathcal{F}(q', q) \sqrt{\frac{M}{E_k}}, \tag{B.24}
\]

\[
\hat{\mathcal{V}}(q', q) = \sqrt{\frac{M}{E_{k'}}} \mathcal{V}(q', q) \sqrt{\frac{M}{E_k}}, \tag{B.25}
\]

which is know as \textit{minimal relativity}, we can rewrite Eq. (B.23) as a nonrelativistic Lippmann-Schwinger equation. Thus Eq. (B.25) defines a relativistic potential which can be consistently applied to nonrelativistic nuclear problems.

To the same conclusion we arrive if we define our potential by comparing the scattering amplitude of the non-relativistic potential theory with the one for the quantum field theory both in the Born approximation. The interaction potential is defined in the Born approximation from the \( T \)-matrix which
in turn is defined in terms of the S-matrix. The non-relativistic Hamiltonian for two interacting particles is

\[ H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(r_1 - r_2) \]

\[ = \frac{p^2}{2\mu} + \frac{P^2}{2M} + V(r), \]  \hspace{1cm} (B.26)

where in the last step we have transformed into the reduced system. The scattering amplitude is

\[ S_{fi} = \delta_{fi} - 2\pi i\delta(E_f - E_i) \langle f|V|i \rangle. \]  \hspace{1cm} (B.27)

Using plane wave states normalized in a box of volume Ω

\[ |i\rangle = \psi_i(r_1, r_2) = \frac{e^{i p_1 \cdot r_1} e^{i p_2 \cdot r_2}}{\sqrt{\Omega}}, \hspace{1cm} (B.28) \]

\[ |f\rangle = \psi_f(r_1, r_2) = \frac{e^{i p_1' \cdot r_1} e^{i p_2' \cdot r_2}}{\sqrt{\Omega}}, \hspace{1cm} (B.29) \]

the matrix element becomes

\[ \langle f|V|i \rangle = \int d^3r_1 d^3r_2 \psi_f^*(r_1, r_2)V(r_1 - r_2)\psi_i(r_1, r_2) \]

\[ = \frac{1}{\Omega} \int d^3re^{i(p_1' - p_1) \cdot r}V(r)\delta(p_1' + p_2' - p_1 - p_2), \hspace{1cm} (B.30) \]

\[ (B.31) \]

where we have used \( r_1 = R + \frac{r}{2} \) and \( r_2 = R - \frac{r}{2} \) and we have integrated the \( R \) variable getting a momentum conservation delta function. Fermi’s Golden Rule gives the transition probability per unit time as

\[ \frac{dP_{i\rightarrow f}}{dt} = \sum_{p_1', p_2'} (2\pi) |\langle f|V|i \rangle|^2 \delta(E_1' + E_2' - E_1 - E_2). \hspace{1cm} (B.32) \]

If we divide by the incident flux \( J = |v_1 - v_2|/\Omega \) being \( v_1, v_2 \) the incident particle velocities, we get the differential cross section in the infinite volume limit \( \Omega \to \infty \)

\[ d\sigma = \frac{dP_{i\rightarrow f}}{dt} \cdot \frac{1}{J} \]

\[ \rightarrow \int \frac{d^3p_1'}{(2\pi)^3 |v_1 - v_2|} |V(p_1', p_1)|^2 \delta(E_1' + E_2' - E_1 - E_2), \hspace{1cm} (B.33) \]

where energy conservation is assumed \( p_1' + p_2' = p_1 + p_2 \) and the momentum space potential has been defined

\[ V(p_1', p_1) = \int d^3re^{i(p_1' - p_1) \cdot r}V(r). \hspace{1cm} (B.34) \]

In the c.m. system and working on-shell \( (m_i = M, E_i = E_i' = E) \) we have

\[ |v_1 - v_2| = \frac{\partial E}{\partial p_1}, \hspace{1cm} (B.35) \]

\[ \delta(E_1' + E_2' - E_1 - E_2) = \frac{1}{2} \frac{1}{\partial E/\partial p_1} \delta(p_1' - p_1), \hspace{1cm} (B.36) \]
and then

\[ d\sigma = \frac{1}{8\pi^2} \int d\Omega_1 |f_B|_{\text{nonrel}}^2, \quad (B.37) \]

with

\[ |f_B|_{\text{nonrel}} = \frac{p_1}{\partial E/\partial p_1} |V(p_1', p_1)|. \quad (B.38) \]

Now, the relativistic differential cross section is (see e.g. Appendix B of [328])

\[ d\sigma = \int \frac{d^3p_1'}{(2\pi)^3} \frac{d^3p_2'}{(2\pi)^3} (2\pi)^3 \mathcal{M}^2 \frac{m_1 m_2 m_{11} m_{22}}{E_1 E_2 E'_1 E'_2} \delta(p_1' + p_2' - p_1 - p_2) \frac{1}{|v_1 - v_2|} \quad (B.39) \]

being here \( p = (E, \mathbf{p}) \). Assuming the identification \( p_{\text{nonrel}} = p_{\text{rel}} \), i.e., the same incident flux for both prescriptions, we get for the relativistic case an equation similar to Eq. (B.37) but now with

\[ |f_B|_{\text{rel}} = \frac{p_1}{\partial E/\partial p_1} M |M| \quad (B.40) \]

So, using \( |\partial E/\partial p_1|_{\text{nonrel}} = p_1/M \) and \( |\partial E/\partial p_1|_{\text{rel}} = p_1/E \) and matching the scattering amplitudes \( |f| \) we get the usual definition of a non-relativistic potential from the Feynman amplitude \( M \)

\[ V(q', q) = \int d^2r e^{ik \cdot r} V(r) = \frac{M}{E} |\mathcal{M}|, \quad (B.41) \]

or in general for off-shell nucleons \( E \neq E' \),

\[ V(q', q) = \int d^2r e^{ik \cdot r} V(r) = \sqrt{\frac{M}{E'}} |\mathcal{M}| \sqrt{\frac{M}{E}}, \quad (B.42) \]

where \( q = p_1 = -p_2, q' = p_1' = -p_2' \) and \( k = q' - q \) is the momentum transfer. Note that this multiplicative energy factor cancels the spinor normalization factors Eq. (B.3) in the low energy expansion of the potential, i.e.,

\[ \frac{M}{E} \left( \frac{E + M}{2M} \right)^2 = 1 + \mathcal{O}(p^4/M^4), \quad (B.43) \]

which means that we can forget the spinor normalization and this \( M/E \) factor if we are only interested in the low energy expansion of the potential. Therefore, we define the momentum space potential in the c.m. system for the exchange of a given \( \alpha \)-meson (see Fig. B.1) as,

\[ (-i)V_\alpha(q', q) = \bar{u}_1(q') \Gamma^{(\alpha)}_1 u_1(q) \frac{P_\alpha}{(q' - q)^2 - m_\alpha^2} \bar{u}_2(-q') \Gamma^{(\alpha)}_2 u_2(-q), \quad (B.44) \]

where we have denoted here \( \Gamma^{(\alpha)}_i \) as the \( i \)-vertex structure for the \( \alpha \)-meson and

\[ P_\alpha = \begin{cases} i, & \text{if } \alpha \text{ is scalar or pseudo-scalar} \\ i(-g_{\mu\nu} + k_\mu k_\nu/m_\alpha^2), & \text{if } \alpha \text{ is vector, tensor or axial} \end{cases} \quad (B.45) \]
### B.3 Meson-Baryon $SU(3)$ chiral lagrangians

For our OBE model of the NN interaction we consider as starting point the exchange of mesons with a mass below the nucleon mass, namely, $\pi$, $\sigma$, $\rho$ and $\omega$. The meson-nucleon system is described by the following effective Lagrangian density \[ L = L_0 + L_{\text{int}}, \] (B.46)

where $L_0$ is the free field Lagrangian

\[
L_0 = \bar{N} (i\gamma_\mu \partial^\mu - M_N) N + \frac{1}{2} (\partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2) + \frac{1}{2} (\partial_\mu \pi \partial^\mu \pi - m_\pi^2 \pi^2) + \frac{1}{2} m_\omega^2 \omega^\mu \omega^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_\rho^2 \rho^\mu \rho^\mu - \frac{1}{4} G_{\mu\nu} G^{\mu\nu},
\] (B.47)

with the following tensor fields

\[ F_{\mu\nu} = \partial_\nu \omega_\mu - \partial_\mu \omega_\nu, \quad G_{\mu\nu} = \partial_\nu \rho_\mu - \partial_\mu \rho_\nu, \] (B.48)

and where $L_{\text{int}}$ is the interacting part

\[ L_{\text{int}} = L_{\pi NN} + L_{\sigma NN} + L_{\rho NN} + L_{\omega NN} \] (B.49)

The following interactions are consider

\[
L_{\pi NN} = -g_{\pi NN} \bar{N} \sigma N, \quad L_{\sigma NN} = -g_{\sigma NN} \bar{N} i\gamma_5 \tau \cdot \pi N - \frac{f_{\pi NN} N}{m_\pi} \bar{N} \gamma_5 \gamma_\mu \tau \cdot \partial^\mu \pi N, \quad (B.50)
\]

\[
L_{\rho NN} = -g_{\rho NN} \bar{N} \gamma_5 \sigma N - \frac{f_{\rho NN} N}{2\Lambda_N} \bar{N} \sigma_\mu \partial^\mu \omega^\nu N, \quad (B.51)
\]

\[
L_{\omega NN} = -g_{\omega NN} \bar{N} \gamma_\mu \omega N - \frac{f_{\omega NN} N}{2\Lambda_N} \bar{N} \sigma_\mu \partial^\mu \omega N, \quad (B.52)
\]

Further comments should be added:

- We have defined a scaling mass $\Lambda_N = 3M_N/N_c$ which takes the nucleon mass value for $N_c = 3$. 

---

**Figure B.1:** Tree level diagram for the exchange of a $\alpha$-meson between two nucleons. Solid lines represent nucleons while dashed represent mesons.
• Near the nucleon mass it is also the $a_1$-meson ($m_{a_1} = 1275$ MeV) whose Lagrangian is given by

\[
\mathcal{L}_{a_1 NN} = -g_{a_1 NN} \bar{N} \gamma_\mu \gamma_5 \tau \cdot a_1^\mu N - i \frac{f_{a_1 NN}}{\Lambda_N} N \gamma_\mu \tau \cdot \partial_\mu a_1^\mu N. \tag{B.54}
\]

It is customary to take $f_{a_1 NN} = 0$ because with this choice this coupling is part of the $a_1$-interaction to pions and nucleons, in such a way that the $a_1$ couples to a partially conserved axial current (PCAC). From Appendix A we know that this meson should be included in principle since appears to be leading order in $N_c$. For that reason we study its effects eventually.

• Here the pseudo-scalar (PS) and pseudovector (PV) couplings have been explicitely consider but as we will show later the differences come out in a piece in the non-local contributions which vanish off-shell. As a consequence both couplings lead to the same OPE potential in the non-relativistic reduction provided one has the relation $g_{\pi NN}/2M_N = f_{\pi NN}/m_\pi$ and on-shell nucleons.

• The $\rho$($\omega$) coupling to the nucleon field has two parts: a vector part $\gamma_\mu \rho^\mu(\omega^\mu)$ and a tensor part $\sigma^{\mu\nu} \partial_\nu \rho^\mu(\omega^\mu)$. Usually the tensor couplings are given by the ratio to the vector couplings through the weak couplings, i.e., $f_{\rho NN} = \kappa_\rho g_{\rho NN}$ and $f_{\omega NN} = \kappa_\omega g_{\omega NN}$ and then we can re-write the corresponding Lagrangian in a more compact way,

\[
\mathcal{L}_{\omega NN} = -g_{\omega NN} \bar{N} \left( \gamma_\mu + \frac{\kappa_\omega}{2\Lambda_N} \sigma^{\mu\nu} \partial_\nu \right) \omega^\mu N, \tag{B.55}
\]

\[
\mathcal{L}_{\rho NN} = -g_{\rho NN} \bar{N} \left( \gamma_\mu + \frac{\kappa_\rho}{2\Lambda_N} \sigma^{\mu\nu} \partial_\nu \right) \tau \cdot \rho^\mu N. \tag{B.56}
\]

• In addition to these mesons, the exchange of the scalar-isovector $a_0$ and the pseudoscalar-isoscalar $\eta$ should be consider in a second step when we want to include sub-leading contributions in the $1/N_c$-expansion. The lagrangeans are,

\[
\mathcal{L}_{a_0 NN} = -g_{a_0 NN} \bar{N} \tau \cdot a_0 N, \tag{B.57}
\]

\[
\mathcal{L}_{\eta NN} = -g_{\eta NN} \bar{N} i\gamma_5 \eta N. \tag{B.58}
\]

Finally, if the delta resonance $\Delta$ is included, isospin conservation restrict the possible mesons to be involved and only those with $I = 1$ can be exchanged. The interaction lagrangeans for the $\pi$ and $\rho$ are in these cases,

\[
\mathcal{L}_{\pi N\Delta} = -\frac{f_{\pi N\Delta}}{m_\pi} \bar{N} T \cdot \partial^\mu \pi \Delta_\mu + \text{h.c.}, \tag{B.59}
\]

\[
\mathcal{L}_{\rho N\Delta} = i \frac{f_{\rho N\Delta}}{m_\rho} \bar{N} \gamma_5 \gamma_\mu T \cdot (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) \Delta_\nu + \text{h.c.}, \tag{B.60}
\]

where $T$ is the isospin transition operator. In the non relativistic limit the lagrangeans Eqs. (B.59) and (B.60) lead to the same expressions for the OBE potentials than the one obtained from Eqs. (B.51) and (B.53) with the appropriate replacements

\[
\sigma \to S, \quad \tau \to T, \quad f \to f_\Delta. \tag{B.61}
\]

These are the so-called transition potentials that we write for the pion case in Sec. B.6.
Appendix B. One Boson Exchange Potential

B.4 OBE potential at leading order in $N_c$

We now proceed to derive the OBE potential using the previous lagrangean densities and using the large-$N_c$ scaling rules to discriminate between leading $O(N_c)$ and sub-leading contributions $O(1/N_c)$. We will calculate the potential up to $O(1/N_c)$ keeping only on-shell terms ($E = E' = M_N$, $|q| = |q'|$) and expanding energy factors in powers of the nucleon momentum $q$, i.e., the so called adiabatic limit ($q^2 \ll M_N^2$), in order to obtain a non-relativistic potential in $r$-space. In this case it is possible to omit in our expressions the spinor normalization factors as they vanish by Eq. (B.43). We will write the potential in $q$-space in terms of $k = q' - q$ and $P = \frac{1}{2}(q' + q)$. Some useful relations used throughout this section are,

\[
\begin{align*}
q \cdot q' &= P^2 - \frac{k^2}{4}, \\
q' \times q &= k \times P, \\
q'^2 + q^2 &= 2P^2 + \frac{k^2}{2}, \\
(q \cdot q')(q \cdot q) &= P^2 - \frac{k^2}{4} + i\sigma \cdot k \times P, \\
(\sigma_1 \sigma_1 \cdot q + \sigma_1 \cdot q \sigma_1) \cdot (\sigma_2 \sigma_2 \cdot q + \sigma_2 \cdot q \sigma_2) &= \sigma_1 \cdot k \sigma_2 \cdot k - k^2 \sigma_1 \cdot \sigma_2 + 4P^2 + 4iS \cdot k \times P, \\
\sigma_1 \cdot q' \sigma_1 \cdot q \sigma_2 \cdot q' \sigma_2 \cdot q &= \left( P^2 - \frac{k^2}{4} \right)^2 + 2i(P^2 - \frac{k^2}{4})S \cdot k \times P - 2(S \cdot k \times P)^2 + (k \times P)^2, \\
\end{align*}
\]

where $S = \frac{1}{2}(\sigma_1 + \sigma_2)$. To calculate the vertex structures we refer the reader to Table. B.1.

B.4.1 Momentum space potential

B.4.1.1 Pseudoscalar meson

In general the pion couples to the nucleon via pseudo-scalar (ps) and pseudo-vector (pv) couplings, i.e.,

\[
\begin{align*}
\mathcal{L}_{\pi NN}^{(ps)} &= - g_{\pi NN} \bar{N} \gamma_5 \tau \cdot \pi N, \\
\mathcal{L}_{\pi NN}^{(pv)} &= - \frac{f_{\pi NN}}{m_{\pi}} \bar{N} \gamma_5 \gamma_{\mu} \tau \cdot \partial^\mu \pi N. \\
\end{align*}
\]

The vertex structures of these couplings are

\[
\begin{align*}
\bar{u}(q') \gamma_5 u(q) &= \frac{\sigma \cdot q}{E + M_N} - \frac{\sigma \cdot q'}{E' + M_N} \\
& \simeq - \frac{1}{2M_N} \left[ \sigma \cdot k \left( 1 - \frac{q^2 + q'^2}{2M_N^2} \right) - \sigma \cdot (q + q') \frac{q'^2 - q^2}{2M_N^2} \right], \\
\bar{u}(q') \gamma_5 \gamma_{\mu} k^\mu u(q) &= - \left[ \sigma \cdot k + \frac{\sigma \cdot q'}{E' + M_N} \sigma \cdot k \frac{\sigma \cdot q}{E + M_N} \right] \\
& \simeq - \left[ \sigma \cdot k \left( 1 - \frac{q'^2 + q^2}{2M_N^2} \right) + \sigma \cdot (q + q') \frac{q'^2 - q^2}{2M_N^2} \right],
\end{align*}
\]

where $q$ and $q'$ are the initial and final nucleon momentum in the c.m. system, $M_N$ is the nucleon mass and $k^\mu = (0, k)$ is the four-momentum transfer with $k = q' - q$. The multiplicative $k^\mu$ factor comes from $\partial^\mu \pi = -ik^\mu \pi$ and the low energy expansion for energy factors has been used in the second step.

Using these structures we can trivially obtain the OPE potential for both couplings following Eq. (B.44),

\[
V^{(\psi)}(q', q) = - \frac{g^{2}_{\pi NN}}{4M_{N}^{2}} \frac{\tau_{1} \cdot \tau_{2}}{k^{2} + m_{\pi}^{2}} \left[ \sigma_{1} \cdot k \sigma_{2} \cdot k - \sigma_{1} \cdot k \sigma_{2} \cdot k q'^{2} + q^{2} \right] - \frac{\sigma_{1} \cdot k \sigma_{2} \cdot (q + q') + \sigma_{1} \cdot (q + q') \sigma_{2} \cdot k}{8M_{N}^{2}} (q'^{2} - q^{2}) \right], \quad (B.67)
\]

\[
V^{(\mu)}(q', q) = - \frac{g^{2}_{\pi NN}}{m_{\pi}^{2}} \frac{\tau_{1} \cdot \tau_{2}}{k^{2} + m_{\pi}^{2}} \left[ \sigma_{1} \cdot k \sigma_{2} \cdot k - \sigma_{1} \cdot k \sigma_{2} \cdot k q'^{2} + q^{2} \right] + \frac{\sigma_{1} \cdot k \sigma_{2} \cdot (q + q') + \sigma_{1} \cdot (q + q') \sigma_{2} \cdot k}{8M_{N}^{2}} (q'^{2} - q^{2}) \right]. \quad (B.68)
\]

The first term in Eqs. (B.67) and (B.68) is the usual local OPE potential, and as we can see both couplings give the same interaction in the static approximation up to $O(1/M_{N})$ provided the identification $g_{\pi NN}/(2M_{N}) = f_{\pi NN}/m_{\pi}$ is satisfied. The second and third terms in Eqs. (B.67) and (B.68) are non-local contributions to the OPE potential. The differences between the pseudoscalar and pseudovector couplings arise as a sign change in the third term which actually vanish on-shell having as a consequence the same static non-local potential as well. Henceforth we will choose the pseudo-scalar coupling and write the OPE potential as follows,

\[
V^{(L)}(q', q) = - \frac{g^{2}_{\pi NN}}{4M_{N}^{2}} \frac{\sigma_{1} \cdot k \sigma_{2} \cdot k}{k^{2} + m_{\pi}^{2}} \tau_{1} \cdot \tau_{2}, \quad (B.69)
\]

\[
V^{(NL)}(q', q) = + \frac{g^{2}_{\pi NN}}{16M_{N}^{4}} \frac{\sigma_{1} \cdot k \sigma_{2} \cdot k}{k^{2} + m_{\pi}^{2}} \left( q'^{2} + q^{2} \right) \tau_{1} \cdot \tau_{2}. \quad (B.70)
\]

Note that because $g_{\pi NN}/2M_{N} \sim f_{\pi NN}/m_{\pi} \sim \sqrt{N_{c}}$ then $V^{(L)} \sim O(N_{c})$ but $V^{(NL)} \sim O(1/N_{c})$, i.e., non-local contributions are sub-leading in the $1/N_{c}$ counting. The 1π-exchange potential is then,

\[
V_{\pi}(k, P) = - \frac{g^{2}_{\pi NN}}{4M_{N}^{2}} \frac{\sigma_{1} \cdot k \sigma_{2} \cdot k}{k^{2} + m_{\pi}^{2}} \tau_{1} \cdot \tau_{2} + \frac{g^{2}_{\pi NN}}{8M_{N}^{2}} \frac{\sigma_{1} \cdot k \sigma_{2} \cdot k}{k^{2} + m_{\pi}^{2}} \left( P^{2} + \frac{k^{2}}{4} \right) \tau_{1} \cdot \tau_{2} + O(1/N_{c}^{2}). \quad (B.71)
\]

### B.4.1.2 Scalar meson

For the scalar,

\[
\mathcal{L}^{(s)}_{\sigma NN} = - g_{\sigma NN} \bar{N} N \sigma, \quad (B.72)
\]

the vertex structure is

\[
\bar{u}(q') u(q) = 1 - \frac{\sigma \cdot q' \sigma \cdot q}{(E' + M_{N})(E + M_{N})} \approx 1 - \frac{1}{4M_{N}} \sigma \cdot q' \sigma \cdot q, \quad (B.73)
\]
where we have kept only terms up to $1/M_N^2 \sim 1/N_c^2$ because $g_{\sigma NN} \sim \sqrt{N_c}$. Using again Eq. (B.44) we arrive at the usual $1\sigma$-exchange potential,

$$V_\sigma(k, P) = -\frac{g_{\sigma NN}^2}{k^2 + m_\sigma^2} \left[ 1 - \frac{P^2}{2M_N^2} + \frac{k^2}{8M_N^2} \left( \sigma_1 + \sigma_2 \right) \cdot (k \times P) \right] + O(1/N_c^2). \quad (B.74)$$

Note that again the non-local contribution and spin-orbit are sub-leading.

### B.4.1.3 Vector meson

If we forget the isospin for a moment we can consider the following general lagrangean for vectors,

$$\mathcal{L}_{vNN} = -g_v \bar{N} \gamma_\mu \varphi_\mu N - \frac{f_v}{2\Lambda_N} \bar{N} \sigma_{\mu\nu} \varphi_\mu \varphi_\nu N, \quad (B.75)$$

Now, using Gordon’s identity

$$\bar{u}(q') \gamma^\mu u(q) = \bar{u}(q') \left[ \frac{(q' + q)^\mu}{2M_N} + \frac{i\sigma^{\mu\nu} (q'_\nu - q_\nu)}{2M_N} \right] u(q), \quad (B.76)$$

we can rewrite the lagrangean as follows

$$\mathcal{L}_{vNN} = -g_v \bar{N} \gamma_\mu \varphi_\mu N - \frac{f_v}{2\Lambda_N} \bar{N} [2M_N \gamma_\mu - (q' + q)_\mu] N \varphi_\mu^\mu \quad (B.77)$$

It is useful in this case to label the external legs explicitly, i.e., the incoming particles having momentum $q = p_1 = -p_2$ and the outgoing particles $q' = p_3 = -p_4$. With this notation the potential is,

$$(-i)V_v(q', q) = (-i)^2 \bar{u}_1(p_3) \left[ \bar{g}_v \gamma_\mu - \frac{f_v}{2\Lambda_N} (p_3 + p_1)_\mu \right] u_1(p_1),$$

$$P^{\mu\nu}(k^2, m_v^2) \bar{u}_2(p_4) \left[ \bar{g}_v \gamma_\nu - \frac{f_v}{2\Lambda_N} (p_4 + p_2)_\nu \right] u_2(p_2) \quad (B.78)$$

where we have define the effective coupling $\bar{g}_v = g_v + f_v \frac{\Lambda_N}{\Lambda_N}$ in order to simplify expressions. The Proccia propagator can be simplified to,

$$P_{\mu\nu}(k^2, m_v^2) = iP(k^2, m_v^2) \left[ -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_v^2} \right] \quad (B.79)$$

$$\rightarrow -\frac{i g_{\mu\nu}}{k^2 - m_v^2}, \quad (B.80)$$
because the second term gives a null contribution on-shell, being \( P(k^2, m^2) \) the Klein-Gordon propagator. Therefore, there are three terms we have to evaluate, namely,

\[
\begin{align*}
\frac{g^2}{4N} & \times \left[ \bar{u}_1(p_3) \gamma_\mu u_1(p_1) \bar{u}_2(p_4) \gamma^\mu u_2(p_2) \right] , \\
\frac{f_v^2}{4\Lambda_N^2} & \times \left[ \bar{u}_1(p_3)(p_3 + p_1)_{\mu} u_1(p_1) \bar{u}_2(p_4)(p_2 + p_4)^{\mu} u_2(p_2) \right] , \\
- \frac{g_v f_v}{2\Lambda_N^2} & \times \left[ \bar{u}_1(p_3) \gamma_\mu u_1(p_1) \bar{u}_2(p_4)(p_2 + p_4)^{\mu} u_2(p_2) \\
& \quad + \bar{u}_1(p_3)(p_3 + p_1)_{\mu} u_1(p_1) \bar{u}_2(p_4) \gamma^\mu u_2(p_2) \right] .
\end{align*}
\] (B.81)

Note that from Table A.1 in Appendix A \( g_{\omega NN} \sim f_{\rho NN} \sim \sqrt{N_c} \) while \( g_{\rho NN} \sim f_{\omega NN} \sim 1/\sqrt{N_c} \) and \( \Lambda_N \sim N_c \). Since we have here generic couplings \( f_v \) and \( g_v \) we keep term up to \( \mathcal{O}(1/M_N^2) \) and specify to the corresponding meson at the end. Considering nucleons on-shell one obtains,

\[
\begin{align*}
\bar{u}_1(p_3) & \gamma_\mu u_1(p_1) \bar{u}_2(p_4)(p_2 + p_4)^{\mu} u_2(p_2) \simeq \\
& \left[ 1 + \frac{1}{(E + M_N)^2} (\sigma_1 \cdot q' \sigma_1 \cdot q + \sigma_2 \cdot q' \sigma_2 \cdot q) \\
& \quad + \frac{1}{(E + M_N)^4} (\sigma_1 \cdot q' \sigma_1 \cdot q \sigma_2 \cdot q' \sigma_2 \cdot q) \\
& \quad + \frac{1}{(E + M_N)^2} (\sigma_1 \sigma_1 \cdot q + \sigma_1 \cdot q \sigma_1) \cdot (\sigma_2 \sigma_2 \cdot q + \sigma_2 \cdot q \sigma_2) \right] ,
\end{align*}
\] (B.82)

\[
\begin{align*}
\bar{u}_1(p_3)(p_3 + p_1)_{\mu} u_1(p_1) \bar{u}_2(p_4)(p_2 + p_4)^{\mu} u_2(p_2) & \simeq \\
& \left( 4E^2 + (q + q')^2 \right) \left[ 1 - \frac{1}{(E + M_N)^2} (\sigma_1 \cdot q' \sigma_1 \cdot q + \sigma_2 \cdot q' \sigma_2 \cdot q) \\
& \quad + \frac{1}{(E + M_N)^4} (\sigma_1 \cdot q' \sigma_1 \cdot q \sigma_2 \cdot q' \sigma_2 \cdot q) \right] ,
\end{align*}
\] (B.83)

\[
\begin{align*}
\bar{u}_1(p_3) & \gamma_\mu u_1(p_1) \bar{u}_2(p_4)(p_2 + p_4)^{\mu} u_2(p_2) + \\
\bar{u}_1(p_3)(p_3 + p_1)_{\mu} u_1(p_1) \bar{u}_2(p_4) \gamma^\mu u_2(p_2) & \simeq \\
& 4E \frac{4E}{(E + M_N)^3} \left[ (q \cdot q')^2 + i (\sigma_1 + \sigma_2) \cdot q' \times q (q \cdot q') - \sigma_1 \cdot q' \times q \sigma_2 \cdot q' \times q \right] \\
& \quad + \frac{1}{(E + M_N)^3} \left[ 2(q + q')^2 + 2i(\sigma_1 + \sigma_2) \cdot q' \times q \right] \\
& \quad - \frac{1}{(E + M_N)^3} \left[ 2q \cdot q' (q + q')^2 + i (\sigma_1 + \sigma_2) \cdot q' \times q (q + q')^2 \right. \\
& \quad \left. \quad + 2i q \cdot q' (\sigma_1 + \sigma_2) \cdot q' \times q - 4 \sigma_1 \cdot q' \times q \sigma_2 \cdot q' \times q \right] .
\end{align*}
\] (B.84)
Now, expanding in momentum the energy factors $E + M_N$ and keeping only term up to $O(1/M_N^4)$ one arrives, with the help of Eqs. (B.62), at the following one-vector-exchange potential,

$$V_v(k, P) = \frac{1}{k^2 + m_v^2} \left\{ g_v^2 \left[ 1 - \frac{k^2}{2M_N^2} + \frac{3P^2}{2M_N^2} - \frac{k^2}{4M_N^2} \sigma_1 \cdot \sigma_2 + \frac{1}{4M_N^2} \sigma_1 \cdot k \sigma_2 \cdot k + \frac{3i}{2M_N^2} S \cdot k \times P \right] 
+ f_v \left[ -\frac{k^2}{M_N} \sigma_1 \cdot \sigma_2 + \frac{1}{M_N} \sigma_1 \cdot k \sigma_2 \cdot k + \frac{4i}{M_N} S \cdot k \times P \right] 
+ \frac{f_v^2}{4\Lambda_N^2} \left[ -\frac{k^2}{2M_N} \sigma_1 \cdot \sigma_2 + \sigma_1 \cdot k \sigma_2 \cdot k + \frac{k^4}{4M_N^2} \right. 
\left. + \left( \frac{k^4}{8M_N^2} + \frac{k^2 P^2}{2M_N^2} \right) \sigma_1 \cdot \sigma_2 
- \left( \frac{k^2}{8M_N^2} + \frac{P^2}{2M_N^2} \right) \sigma_1 \cdot k \sigma_2 \cdot k 
- \frac{3i k^2}{2M_N} S \cdot k \times P - \frac{2}{M_N} \sigma_1 \cdot k \times P \sigma_2 \cdot k \times P \right\} \right\}
+ O(1/N_c^2),
$$

(B.87)

and therefore one has the $1\omega$-exchange potential

$$V_{\omega}(k, P) = \frac{1}{k^2 + m_v^2} \left\{ \frac{1}{2} \left[ 1 - \frac{k^2}{2M_N^2} + \frac{3P^2}{2M_N^2} - \frac{k^2}{4M_N^2} \sigma_1 \cdot \sigma_2 + \frac{1}{4M_N^2} \sigma_1 \cdot k \sigma_2 \cdot k + \frac{3i}{2M_N^2} S \cdot k \times P \right] 
+ f_{\omega N N\omega N N} \left[ -\frac{k^2}{M_N} \sigma_1 \cdot \sigma_2 + \frac{1}{M_N} \sigma_1 \cdot k \sigma_2 \cdot k + \frac{4i}{M_N} S \cdot k \times P \right] 
+ \frac{f_{\omega N N}^2}{4\Lambda_N^2} \left[ -\frac{k^2}{2M_N} \sigma_1 \cdot \sigma_2 + \sigma_1 \cdot k \sigma_2 \cdot k \right] \right\} \right\}
+ O(1/N_c^2),
$$

(B.88)

and the $1\rho$-exchange potential

$$V_{\rho}(k, P) = \frac{1}{k^2 + m_v^2} \left\{ \frac{1}{2} \left[ 1 - \frac{k^2}{2M_N^2} + \frac{3P^2}{2M_N^2} - \frac{k^2}{4M_N^2} \sigma_1 \cdot \sigma_2 + \frac{1}{4M_N^2} \sigma_1 \cdot k \sigma_2 \cdot k + \frac{3i}{2M_N^2} S \cdot k \times P \right] 
+ f_{\rho N N\rho N N} \left[ -\frac{k^2}{M_N} \sigma_1 \cdot \sigma_2 + \frac{1}{M_N} \sigma_1 \cdot k \sigma_2 \cdot k + \frac{4i}{M_N} S \cdot k \times P \right] 
+ \frac{f_{\rho N N}^2}{4\Lambda_N^2} \left[ -\frac{k^2}{2M_N} \sigma_1 \cdot \sigma_2 + \sigma_1 \cdot k \sigma_2 \cdot k + \frac{k^4}{4M_N^2} \right. 
\left. + \left( \frac{k^4}{8M_N^2} + \frac{k^2 P^2}{2M_N^2} \right) \sigma_1 \cdot \sigma_2 
- \left( \frac{k^2}{8M_N^2} + \frac{P^2}{2M_N^2} \right) \sigma_1 \cdot k \sigma_2 \cdot k 
- \frac{3i k^2}{2M_N} S \cdot k \times P - \frac{2}{M_N} \sigma_1 \cdot k \times P \sigma_2 \cdot k \times P \right\} \right\} \tau_1 \cdot \tau_2 
+ O(1/N_c^2),
$$

(B.89)
### B.4.1.4 Axial meson

Finally for the axial we write,

\[ L_{a_{1}NN}^{(ps)} = -g_{a_{1}NN} \bar{\gamma}_{5} \gamma_{\mu} N \cdot a_{1} \cdot N. \]

The full Procca propagator Eq. (B.80) should be kept in this case because the term \( k_{\mu}k_{\nu}/m_{a_{1}}^{2} \) gives a finite contribution on-shell. This second term in the propagator gives potentials which are exactly of the form as those of pseudo-vector exchange because

\[ (\bar{N} \gamma_{5} \gamma_{\mu} N) k^{\mu} k^{\nu} (\bar{N} \gamma_{5} \gamma_{\mu} N) = (-i \bar{N} \gamma_{5} \gamma_{\mu} N) (\gamma_{5} \gamma_{\mu} N). \]

The vertex structure is then,

\[
\bar{u}(q') \gamma_{5} \gamma_{\mu} u(q) = \left[ \frac{\sigma \cdot q'}{E' + M_N} + \frac{\sigma \cdot q}{E + M_N} \right] \frac{\sigma + \frac{\sigma \cdot q'}{E' + M_N} \sigma - \frac{\sigma \cdot q}{E + M_N}}{2M_N},
\]

\[ \approx \left[ \frac{\sigma \cdot (q' + q)}{2M_N} - \frac{\sigma \cdot q' \sigma \cdot q}{4M_N^2} \right], \]

\[ \bar{u}(q') \gamma_{5} \gamma_{\mu} k^{\mu} u(q) = -\left[ \sigma \cdot k + \frac{\sigma \cdot q'}{E' + M_N} \sigma \cdot k \frac{\sigma \cdot q}{E + M_N} \right],
\]

\[ \approx -\left[ \sigma \cdot k \left(1 - \frac{q'^2 + q^2}{8M_N^2}\right) + \sigma \cdot (q + q') \frac{q'^2 - q^2}{8M_N^2} \right]. \]

Going on-shell and keeping terms up to \( \mathcal{O}(1/M_N) \) because \( g_{a_{1}NN} \sim \sqrt{N_c} \), one gets,

\[
V_{a_{1}}^{(ps)}(q', q) = -\frac{g_{a_{1}NN}}{k^2 + m_{a_{1}}^2} \left\{ \frac{1}{4M_N^2} \sigma_{1} \cdot (q' + q') \sigma_{2} \cdot (q' + q') + \sigma_{1} \cdot \sigma_{2}
\right.
\]

\[ + \frac{1}{2M_N^2} (\sigma_{1} \cdot q' \sigma_{2} \cdot q + \sigma_{2} \cdot q' \sigma_{1} \cdot q) - \frac{1}{4M_N^2} \sigma_{1} \cdot \sigma_{2} (\sigma_{1} \cdot q' \sigma_{2} \cdot q + \sigma_{2} \cdot q' \sigma_{1} \cdot q)
\]

\[ \left. + \frac{1}{m_{a_{1}}^2} \sigma_{1} \cdot k \sigma_{2} \cdot k \left(1 + \frac{q'^2 + q^2}{4M_N^2}\right) \right\} \tau_{1} \cdot \tau_{2} + \mathcal{O}(1/N_c^2). \]

The potentials given by Eqs. (B.71), (B.74), (B.88), (B.89) and (B.94) contain terms in \( P^2 \) which generate non-localities in coordinate space, i.e., terms in \( \nabla^2 \). They also have terms of the form \( S \cdot k \times P \) which generate the spin-orbit \( (L \cdot S) \) when we transform into coordinate space. However, all these terms scale with \( 1/M_N \sim 1/N_c \) and therefore are sub-leading in the \( 1/N_c \) expansion. In what follows we will consider the leading order (LO) in the \( 1/N_c \) expansion, i.e., we will only keep term which preserve the spin-flavor structure of the NN interaction at first order (see Sec. A.4 in Appendix A). These terms are those which
give a contribution to the potential like $V_{NN} \sim N_c$. This simplify enormously the potential having,

$$V^{(LO)}(k) = -\frac{g_{\pi NN}^2}{4M_N^2} \frac{\sigma_1 \cdot k \sigma_2 \cdot k}{k^2 + m_\pi^2} \tau_1 \cdot \tau_2$$

$$- \frac{g_{\omega NN}^2}{k^2 + m_\omega^2} + \frac{f_{\rho NN}^2}{4M_N^2} \frac{1}{k^2 + m_\rho^2} \left[ -k^2 \sigma_1 \cdot \sigma_2 + \sigma_1 \cdot k \sigma_2 \cdot k \right] \tau_1 \cdot \tau_2$$

$$- \frac{g_{a_1 NN}^2}{k^2 + m_{a_1}^2} \left[ \sigma_1 \cdot \sigma_2 + \frac{1}{m_{a_1}^2} \sigma_1 \cdot k \sigma_2 \cdot k \right] + \mathcal{O}(1/N_c).$$

Note that we have included the axial ($a_1$) meson for consistency but its mass is higher than the nucleon mass giving an influence only at very short distances. Usually, OBE models only include mesons with a mass below the nucleon mass.

### B.4.2 Coordinate space potential

Because of the two independent variables, $q$ and $q'$, the direct Fourier transform of an off-shell potential result in a highly non-local coordinate expression which cannot be given in any analytical form. As it has been already pointed out, in order to overcome this difficulty we have to go to the adiabatic limit ($q^2 \ll M_N^2$). In this situation the potentials Eqs. (B.71), (B.74), (B.88), (B.89) and (B.94) can be written as functions of $q$ and $k$ since on-shell one has $P^2 = q^2 - \frac{k^2}{4}$. The potentials in coordinate space are given by the Fourier transform of $V(k, q)$,

$$V(r, \tilde{q}^2) = \int \frac{d^3k}{(2\pi)^3} V(k, q)e^{ik \cdot r},$$

where $\tilde{q}^2$ means a differential operator in coordinate space $\tilde{q}^2 = (\tilde{q}^2 + \tilde{\nabla}^2)/2$ and $\tilde{q} = -i\tilde{\nabla}$. Note that distributional contributions proportional to $\delta(x)$ and derivatives may appear. We discard them by just assuming that we are in a region $r > r_c$ where $r_c$ is a short distance radial cut-off. The Fourier transforms are listed in Table. B.2 where we have used the following set of operator

<table>
<thead>
<tr>
<th>Operator</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$\sigma_1 \cdot \sigma_2$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>$\sigma_1 \cdot k \sigma_2 \cdot k$</td>
</tr>
<tr>
<td>$P_4$</td>
<td>$iS \cdot k \times P$</td>
</tr>
<tr>
<td>$P_5$</td>
<td>$\sigma_1 \cdot k \times P \sigma_2 \cdot k \times P$</td>
</tr>
</tbody>
</table>

$^3$These terms are in fact inessential under renormalization of the corresponding Schrödinger equation via the coordinate boundary condition method.
and we have defined

\[ Y(x) = \frac{e^{-x}}{x}, \quad (B.102) \]
\[ T(x) = \left(1 + \frac{3}{x} + \frac{3}{x^2}\right)Y(x), \quad (B.103) \]
\[ Z(x) = \left(\frac{1}{x} + \frac{1}{x^2}\right)Y(x). \quad (B.104) \]

The angular momentum, tensor and quadratic spin-orbit operators are defined by

\[
\mathbf{L} = \mathbf{r} \times \mathbf{P}, \quad (B.105) \\
S_{12}(\hat{r}) = 3 \hspace{1pt} \mathbf{\sigma}_1 \cdot \hat{r} \hspace{1pt} \mathbf{\sigma}_2 \cdot \hat{r} - \mathbf{\sigma}_1 \cdot \mathbf{\sigma}_2, \quad (B.106) \\
Q_{12} = \frac{1}{2} \{ \mathbf{\sigma}_1 \cdot \mathbf{L}, \mathbf{\sigma}_2 \cdot \mathbf{L} \} \\
= 2(\mathbf{L} \cdot \mathbf{S})^2 + \mathbf{L} \cdot \mathbf{S} - L^2. \quad (B.107)
\]

In our case the Fourier transforms are straightforward for the leading \(N\) OBE potential, which, according

<table>
<thead>
<tr>
<th>(q)-space term</th>
<th>(r)-space term</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{k^2 + m^2})</td>
<td></td>
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<tr>
<td>(\frac{1}{k^2 + m^2})</td>
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<tr>
<td>(\frac{1}{k^2 + m^2})</td>
<td></td>
</tr>
</tbody>
</table>

Table B.2: Fourier transforms of \(q\)-space potentials to \(r\)-space where \(m\) is the meson mass. In the last one \(\{,\}\) stands for anti-commutator.

To their increasing mass, reads

\[
V_\tau(r) = \frac{1}{12} \frac{g^2_{\pi NN}}{4\pi} \frac{m^3_\pi}{A_N^2} \left[ Y(m_\pi r) \hspace{1pt} \mathbf{\sigma}_1 \cdot \mathbf{\sigma}_2 + T(m_\pi r) \hspace{1pt} S_{12}(\hat{r}) \right] \mathbf{\tau}_1 \cdot \mathbf{\tau}_2, \quad (B.108) \\
V_\sigma(r) = \frac{g^2_{\sigma NN}}{4\pi} m_\sigma Y(m_\sigma r), \quad (B.109) \\
V_\rho(r) = \frac{1}{12} \frac{g^2_{\rho NN}}{4\pi} \frac{m^3_\rho}{A_N^2} \left[ 2 \hspace{1pt} Y(m_\rho r) \hspace{1pt} \mathbf{\sigma}_1 \cdot \mathbf{\sigma}_2 - T(m_\rho r) \hspace{1pt} S_{12}(\hat{r}) \right] \mathbf{\tau}_1 \cdot \mathbf{\tau}_2, \quad (B.110) \\
V_\omega(r) = \frac{g^2_{\omega NN}}{4\pi} m_\omega Y(m_\omega r), \quad (B.111) \\
V_{a_1}(r) = \frac{g^2_{a_1 NN}}{4\pi} m_{a_1} \left[ -2 \hspace{1pt} Y(m_{a_1} r) \hspace{1pt} \mathbf{\sigma}_1 \cdot \mathbf{\sigma}_2 + T(m_{a_1} r) \hspace{1pt} S_{12}(\hat{r}) \right] \mathbf{\tau}_1 \cdot \mathbf{\tau}_2. \quad (B.112)
\]
B.5 Review of coupling constants

A crucial point in the present framework corresponds to the choice of coupling constants, \( g_{\pi NN}, g_{\sigma NN}, f_{\rho NN}, g_{\omega NN} \) and \( g_{\rho NN} \) and masses, \( m_\pi, m_\sigma, m_\rho \) and \( m_\omega \), entering the calculation. We review here reasonable ranges on the basis of several sources that we refer to as natural values of the coupling constants.

- **Pseudo-scalar coupling \( g_{\pi NN} \)**

  According to the Goldberger-Treiman relation the pion nucleon coupling constant is given by

  \[
  g_{\pi NN} = g_A M_N / f_\pi \quad \text{where} \quad f_\pi = 92.4 \text{MeV} \]

  from \( \beta \)-decay. This yields a different value of \( g_{\pi NN} \) = 12.8. However, a phase shift analysis of NN scattering [264] yields \( g_{\pi NN} = 13.1083 \) which is compatible with a value of \( g_A = 1.29 \). This later value is the one we adopt in our calculations. Nevertheless, the latest determinations from the Goldberger-Miyazawa-Oehme (GMO) sum rule [333] yields the value \( g_{\pi NN} = 13.3158 \), this variation at the 5% level dominates the uncertainties in the \( 1\pi \) exchange calculations.

- **Scalar coupling \( g_{\sigma NN} \)**

  The Goldberger-Treiman relation for scalar mesons in the linear sigma model yields a scalar coupling constant \( g_{\sigma NN} = M_N / f_\pi = 10.1 \). However, if we consider contributions from excited scalar mesons we may expect a somewhat different number. Actually, QCD sum rules yield [334] \( g_{\sigma NN} = 14.4 \pm 3.7 \) for the Ioffe current nucleon interpolator and a smaller value \( g_{\sigma NN} = 7 \pm 3 \) for more general interpolators [335]. A recent quark model calculation yields \( g_{\sigma NN} = 14.5 \pm 2 \) [163].

- **Vector coupling \( g_{\rho NN} \):**

  The vector \( g_{\rho NN} \) coupling constant is after Sakurai’s universality \( g_{\rho NN} = g_{\rho \pi \pi} / 2 \) while the current-algebra KSFR relation\(^4\) provides \( g_{\rho \pi \pi} = m_\rho / (\sqrt{2} f_\pi) \), yielding \( g_{\rho NN} = 2.9 \). The \( \rho \pi N \) Vertex in Vector Dominance Models was also determined in the old analysis [336] yielding \( g_\rho = 2.9(1) \), a value confirmed in Ref. [337].

- **Tensor coupling \( f_{\rho NN} \)**

  The tensor \( f_{\rho NN} \) coupling is usually given by the ratio to the vector coupling \( f_{\rho NN} = \kappa_\rho g_{\rho NN} \). In single vector meson dominance models \( \kappa_\rho = \mu_p - \mu_n - 1 \) with \( \mu_p = 2.79 \) and \( \mu_n = -1.91 \) the magnetic moments (in nuclear magneton units \( e / (2 M_p) \)) of proton and neutron respectively, yielding \( \kappa_\rho = 3.7 \) and hence \( f_{\rho NN} = 10.7(4) \) for \( g_{\rho NN} = 2.9(1) \).

- **Vector coupling \( g_{\omega NN} \)**

  The relation \( g_{\omega NN} = 3 g_{\rho NN} (= 8.7(3) \text{ for } g_{\rho NN} = 2.9(1)) \) is the \( SU(3) \) prediction for the ideal \( \omega - \phi \) mixing case corresponding to the OZI rule where \( g_{\phi NN} = 0 \) as well, i.e., where the \( \phi \)-meson is a pure \( ss \) state that does not couple to the nucleon [59]. A better estimate is obtained from the e.m. decay widths \( \omega \rightarrow e^+e^- \) and \( \rho^0 \rightarrow e^+e^- \) which suggest \( g_{\omega NN} / g_{\rho NN} = [(m_\omega \Gamma_{\rho \rightarrow e^+e^-}) / (m_\rho \Gamma_{\omega \rightarrow e^+e^-})]^{1/2} = 3.4 \pm 0.1 \) accounting for a \( SU(3) \) breaking of \( g_{\omega NN} = 3.5 g_{\rho NN} = 10.2(4) \text{ for } g_{\rho NN} = 2.9(1) \).

\(^4\)The unfamiliar reader can find a short review on vector meson dominance in Appendix C.
\begin{itemize}
\item **Tensor coupling $f_{\omega NN}$**

The tensor $f_{\omega NN}$ coupling is also given by its the ratio to the vector coupling $f_{\omega NN} = \kappa_\omega g_{\omega NN}$.

In single vector meson dominance models $\kappa_\omega = \mu_p + \mu_n - 1$ yielding $\kappa_\omega = -0.12$ and hence $f_{\omega NN} = -0.3(1)$ for $g_{\omega NN} = 3 - 3.5$.

\item **Axial coupling $g_{a_1NN}$**

In $SU(3)$ the axial-vector coupling constant is closely related to the pseudovector coupling constant due to the mixing between axial-vector and pseudoscalar fields. With the choice $Z_\pi = 1/2$ for the pseudoscalar renormalization constant one gets [59] $g_{a_1NN} = (m_{a_1}/m_\pi)f_{\pi NN} = 8.4$ which is the familiar old results by Weinberg [338], Schwinger [268] and Wess-Zumino [332]. Axial-vector dominance establish [269] $m_{a_1} = \sqrt{2}m_\rho$ although this value is not fully supported by experiment (e.g. in PDG $m_{a_1} = 1230$MeV).

Nucleon Electromagnetic Form factors with high energy QCD constraints also provide information on vector meson couplings. Ref. [339] yields $g_{\omega NN} = 20.86(25)$, $\kappa_\omega = -0.16(1)$, $f_{\omega NN} = -3.41(24)$ and $\kappa_\rho = 6.1(2)$, and more recently [340] it was found $g_{\omega NN} = 20(3)$ and $f_{\omega NN} = 3(7)$. On the other hand, QCD sum rules yield for the $\rho NN$ coupling a spread of values $g_{\rho NN} = 2.4 \pm 0.6$ and $f_{\rho NN} = 7.7 \pm 1.9$ [341] and $g_{\rho NN} = 3.2 \pm 0.9$ and $f_{\rho NN} = 36.8 \pm 13.0$ [342] and for the $\omega NN$ coupling the values [341] $g_{\omega NN} = 7.2 \pm 1.8$ and $f_{\omega NN} = -2.2 \pm 0.6$.

Phase-shift analyzes of NN scattering below 160 MeV based on the $\epsilon_1$ mixing angle were argued to be an indication for a strong tensor force [343], an issue further qualified in Ref. [266]. The strong tensor coupling is $\kappa_\rho = f_{\rho NN}/g_{\rho NN} = 6.1(6)$ and the weak is $\kappa_\rho = \mu_p - 1 - \mu_n = 3.7$ corresponding to vector meson dominance saturated with a single state. Note that the value $f_{\rho NN} = g_{\tau NN} = 13.1$ for which the tensor force $1/r^3$ singularity disappears corresponds to $\kappa_\rho = 4.5(2)$ a value in between weak and strong.

As we see there are a big amount of different values for the coupling constants coming from different sources. Probably this is one of the reasons why must of the OBE models fit directly their coupling constants to NN data. These fits should be in agreement with model independent determinations, a requirement that not always is fulfilled [3, 53].

\section{B.6 Transition potentials}

Finally, we would like to write the explicit expressions for the transition potentials in the case of the pseudoscalar $\pi$ meson (for the rest of isovectors we proceed in a similar fashion). In general, the OPE-transition potentials for the process $AB \to CD$ can be written as,

$$V^\tau_{A,B,CD}(r) = \left(\tau_A \cdot \tau_{CD}\right)\{\sigma_{AB} \cdot \sigma_{CD}[W_{\pi,\tau}(r)]_{AB,CD} + [S_{12}]_{AB,CD}[W_{\pi,\tau}(r)]_{AB,CD}\}, \quad (B.113)$$

where the tensor term and radial functions are defined by

$$[S_{12}]_{AB,CD} = \frac{3}{4\pi} \frac{f_{\pi AC}f_{\pi BD}}{m_\pi} Y_{0,2}(m_\pi r), \quad (B.114)$$

with $Y_0(x) = e^{-x}/x$ and $Y_2(x) = Y_0(x)(1 + 3/x + 3/x^2)$. Applying the rules Eqs. (B.61) to Eq. (B.113)
we get for the processes of Fig. B.2 the following transition potentials,

\[
V_{NN,N\Delta}^\pi (r) = \{\sigma_1 \cdot S_2 [W_{S}^\pi (r)]_{NN,NN} + [S_{12}]_{NN,N\Delta} [W_{T}^\pi (r)]_{NN,N\Delta}\} T_1 \cdot \tau_2,
\]

(B.116)

\[
V_{NN,\Delta\Delta}^\pi (r) = \{S_1 \cdot S_2 [W_{S}^\pi (r)]_{NN,NN} + [S_{12}]_{NN,\Delta\Delta} [W_{T}^\pi (r)]_{NN,\Delta\Delta}\} T_1 \cdot T_2,
\]

(B.117)

with the tensor operators,

\[
[S_{12}]_{NN,N\Delta} = 3(\sigma_1 \cdot \hat{r})(S_2 \cdot \hat{r}) - \sigma_1 \cdot S_2,
\]

(B.118)

\[
[S_{12}]_{NN,\Delta\Delta} = 3(S_1 \cdot \hat{r})(S_2 \cdot \hat{r}) - S_1 \cdot S_2,
\]

(B.119)

and the radial functions

\[
[W_{S,T}^\pi (r)]_{NN,NN} = \frac{m_\pi f_{\pi NN} f_{\pi N\Delta}}{3} Y_{0,2}(m_\pi r),
\]

(B.120)

\[
[W_{S,T}^\pi (r)]_{NN,\Delta\Delta} = \frac{m_\pi f_{\pi NN} f_{\pi N\Delta}}{3} Y_{0,2}(m_\pi r),
\]

(B.121)

where \(S\) and \(T\) are the appropriate spin and isospin transition operators between nucleon and \(\Delta\). The coupling constant \(f_{\pi N\Delta}\) can be determined from quarks models or from the \(\Delta\) decay width.

### B.7 \(\Delta \to \pi N\) decay

Consider the decay process \(\Delta \to \pi N\). In the \(\Delta\)-isobar c.m. system we have,

\[
E_1 = M_\Delta,
\]

(B.122)

\[
E_2 = \sqrt{p_N^2 + M_\Delta^2} = \sqrt{q_{cm}^2 + M_N^2},
\]

(B.123)

\[
E_3 = \sqrt{p_\pi^2 + m_\pi^2} = \sqrt{q_{cm}^2 + m_\pi^2}.
\]

(B.124)

From Bjorken-Drell \[328\] the decay rate for this process is given by,

\[
d\Gamma = \frac{1}{2} \frac{d^3p_N}{E_N (2\pi)^3} \frac{d^3p_\pi}{E_\pi (2\pi)^3} \frac{1}{2E_\pi} |\mathcal{M}|^2 (2\pi)^4 \delta^3(-p_N - p_\pi)\delta(E_\Delta - E_N - E_\pi),
\]

(B.125)
and in the case of an angular independent Feynman amplitude we obtain,

\[ \Gamma_{\Delta \to \pi N} = \frac{M_N}{2\pi} \frac{|q_{c.m.}|}{M_\Delta} |\mathcal{M}|^2. \]  

(B.126)

As we will see the Feynman amplitude is indeed angular independent. To calculate it we consider the lagrangean density Eq. (B.59) at the \( \pi N \Delta \) vertex. We have then,

\[ |\mathcal{M}|^2 = \frac{f_{\pi N\Delta}^2}{m_\pi^2} \bar{u}(p', s')q_{0}u^\mu(p, s)\bar{u}^\nu(p, s)q_{\nu}u(p', s'), \]  

(B.127)

where \( u(p, s) \) is a Dirac spinor Eq. (B.3), \( u^\mu(p, s) \) is a Rarita-Schwinger spinor Eq. (B.16) and the momentum factors come from the derivative coupling of the pion. We now have to sum over final spin states and average over initial spin states. The sum over final spin states gives a spin-\( \frac{1}{2} \) projector operator according to Eq. (B.5) and then we only have to calculate the quantity,

\[ q_\mu \left( \sum_{s=-3/2}^{+3/2} u^\mu(p, s)u^\nu(p, s) \right) q_\nu = q_\mu A^{(3/2)}_+(p)^{\mu\nu} q_\mu. \]  

(B.128)

Using Eq. (B.10) and

\[ q^2 = m_\pi - |q_{c.m.}|^2, \]  

(B.129)

\[ p \cdot q = M_\Delta m_\pi, \]  

(B.130)

we obtain

\[ q_\mu A^{(3/2)}_+(p)^{\mu\nu} q_\mu = \frac{2}{3} \frac{p + M_\Delta}{2M_\Delta} |q_{c.m.}|^2. \]  

(B.131)

Therefore the sum over final spin states and average over initial spin states give,

\[ \frac{1}{4} \sum_{s, s'} |\mathcal{M}|^2 = \frac{1}{4} \frac{f_{\pi N\Delta}^2}{3 m_\pi^2} |q_{c.m.}|^2 Tr \left[ \frac{p' + M_N p + M_\Delta}{2M_N} \right] \times \frac{2M_N}{2M_\Delta} \left[ \frac{p' + M_N p + M_\Delta}{2M_N} \right] \]  

(B.132)

By pure kinematics we have the relations,

\[ |q_{c.m.}|^2 = \frac{M_N^4 - 2(M_N^2 + m_\pi^2)M_\Delta^2 + (M_\Delta^2 - m_\pi^2)^2}{4M_\Delta^4}, \]  

(B.133)

\[ E_N + M_N = \frac{1}{2M_\Delta} [(M_N + M_\Delta)^2 - m_\pi^2], \]  

(B.134)

and finally the decay width,

\[ \Gamma_{\Delta \to \pi N} = \frac{f_{\pi N\Delta}^2}{24\pi m_\pi^2} \frac{(M_N + M_\Delta)^2 - m_\pi^2}{M_\Delta^2} |q_{c.m.}|^3. \]  

(B.135)

For the experimental value \( \Gamma_{\Delta \to \pi N}^{(exp)} = 115 \text{ MeV} \) we get \( f_{\pi N\Delta}^2/(4\pi) = 0.36. \)
Appendix C

Short review of Vector Meson Dominance (VMD)

In this appendix we shall review the main results of Sakurai’s VMD model. This model will be useful to estimate natural values for vector meson coupling constants to the nucleon.

C.1 Electromagnetic structure and vector mesons

In 1957 Nambu [344] postulated the existence of a neutral vector meson in order to explain the electromagnetic structure of nucleons. Until then the nucleon structure was thought due mainly to the pion cloud. However this assumptions lead to inconsistencies between theory and experiment because in a theory with pions only, a photon couples to the nucleon through a pion pair such as in Fig. C.1. Because $\pi\pi$ is a system with $J^P = 1^-$, $T = 1$ and $T_3 = 0$ this couples to the proton and neutron with opposite signs (for $T = 1$, $T_3 = 0$ the NN system is $\bar p p - \bar n n$). This gives for example $\langle r^2 \rangle_p = -\langle r^2 \rangle_n$ in total disagreement with the experiment $\langle r^2 \rangle_p^{1/2} = 0.86$fm and $\langle r^2 \rangle_n^{1/2} = 0$fm. We need a contribution which has the same sign for both $p$ and $n$ in order to cancel the negative charge cloud of the neutron. A system with $J^P = 1^-$, $T = 0$ satisfy this requirement. We can think in a picture where the photon can convert directly into a vector meson ($\rho$ or $\omega$) which couples to the nucleon (see Fig. C.2). This is the so-called \textit{vector meson dominance model}. 

\vspace{0.2cm}

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figC1}
\caption{A photon coupled to a nucleon by two-pions $\gamma \rightarrow \pi \pi \rightarrow \bar N N$.}
\end{figure}
Appendix C. Short review of Vector Meson Dominance (VMD)

Figure C.2: A photon coupled to a nucleon by transforming into a vector meson. Here (a) $\gamma \to \omega \to \bar{N}N$ and (b) $\gamma \to \rho \to \bar{N}N$.

C.2 Universality

As a starting point let us concentrate in the isovector $\rho$-meson. To couple this vector to a conserved hadronic current we follow the simplest minimal principle used for electromagnetic interactions. We know that the e.m. field $A_\mu$ couples to a conserved e.m. current $j_{\mu}^{em}$ with an universal charge $e$, and this coupling is archived by the replacement,

$$\partial_\mu \psi \to D_\mu \psi = (\partial_\mu - ieA_\mu)\psi \quad (C.1)$$

In the same spirit we couple the isovector field $\rho_\mu$ to the conserved isospin current $j_{\mu}^{ih}$ with an universal charge $f_\rho$,

$$D_\mu \psi = (\partial_\mu - if_\rho T \cdot \rho_\mu)\psi \quad (C.2)$$

where $T = \tau/2$ for nucleons. The isovector hadronic current $j_{\mu}^{ih}$ is,

$$j_{\mu}^{ih} = \bar{N}\gamma^{\mu}TN + \pi \times \partial^{\mu}\pi + \rho_\nu \times \partial^{\mu}\rho_\nu, \quad (C.3)$$

therefore the interaction lagrangean is,

$$L_{int}^{(1)} = f_\rho \rho_\mu \cdot j_{\mu}^{ih} = \rho_\mu \cdot \left[ f_{\rho NN} \bar{N}\gamma^{\mu}TN + f_{\rho\pi\pi} \pi \times \partial^{\mu}\pi + f_{\rho\rho\rho} \rho_\nu \times \partial^{\mu}\rho_\nu \right], \quad (C.4)$$

and universality requires $f_{\rho NN} = f_{\rho\pi\pi} = \ldots = f_\rho$, i.e., the vertices representing the $\rho NN$ or $\rho\pi\pi$ interaction are characterized by one and the same charge $f_\rho$.

C.3 Vector meson dominance and universality

The two pictures we have mentioned, one based on an attempt to understand the electromagnetic structure of nucleons and the other on universality and conserved currents are in fact intimately related.

VMD asserts that a virtual photon converts to a neutral vector meson first and then couples to a hadron, and universality claims that this coupling does not depend on the hadron in particular (nucleon, $\pi$, $\ldots$). To accommodate such a hypothesis within a theoretical framework we need an effective lagrangean in addition to Eq. (C.4) for the interaction vertex in which a photon annihilates into a neutral vector meson.
Appendix C. Short review of Vector Meson Dominance (VMD)

and vice-versa. We write this lagrangean as,
\[ L^{(2)}_{\text{int}} = \gamma_\rho \rho^3 A^\mu, \]  \hspace{1cm} (C.5)

where the third component of the vector meson field account for the neutral \( \rho^0 \)-meson. An interaction like this would arise from gauge coupling if the third component of the isovector electromagnetic current \( j_3^\mu \) is itself proportional to the \( \rho^0 \)-field,
\[ e j_3^\mu = \gamma_\rho \rho^3_\mu, \]  \hspace{1cm} (C.6)

because we have
\[ e j^{e.m.}_\mu = e \left( j_3^\mu + \frac{1}{2} j_Y^\mu \right), \]  \hspace{1cm} (C.7)

where \( j_3^\mu \) is the same than the third component of \( j^\mu_h \) and \( j_Y^\mu \) is the isoscalar current. The interaction Eq. (C.5) arise from the gauge coupling \( j^{e.m.}_\mu A^\mu \). The generalization of Eq. (C.6) is called current-field identity (CFI),
\[ e j_\alpha^\mu = \gamma_\rho \rho_\alpha^\mu, \quad \alpha = 1, 2, 3. \]  \hspace{1cm} (C.8)

To appreciate the physical content of VMD it is illustrative to study e.m. form factors of hadrons. Consider for example the process \( e^- \pi^+ \rightarrow e^- \pi^+ \) draw in Fig. C.3 (a) where all the electromagnetic structure of the pion, represented by a bubble, is parameterized with a form factor \( F_\pi(q^2) \) at the \( \gamma \pi \pi \)-vertex. In VMD this e.m. structure arises solely from the \( \rho^0 \)-meson, as shown in Fig. C.3 (b). We can then calculate the pion form factor by equating both Feynman amplitudes. Naively we have,
\[ e \frac{1}{q^2} e F_\pi(q^2) = e \frac{1}{q^2} \gamma_\rho \frac{1}{m_\rho^2 - q^2} \gamma_\rho \pi \pi \Rightarrow e F_\pi(q^2) = \frac{\gamma_\rho \gamma_\rho \pi \pi}{m_\rho^2 - q^2}. \]  \hspace{1cm} (C.9)

Figure C.3: Elastic \( e^- \pi^+ \) scattering. The pion form factor \( F_\pi(q^2) \) is represented by a bubble in (a). In VMD the e.m. structure arises from the \( \rho^0 \)-meson (b).
Appendix C. Short review of Vector Meson Dominance (VMD)

From $F_\pi(q^2 = 0) = 1$ and universality we obtain,

\[ \gamma_\rho = \frac{e m^2_\rho}{f_\rho}, \quad \text{(C.10)} \]
\[ j_\mu^\alpha = \frac{m^2_\rho}{f_\rho} \rho^\alpha_\mu. \quad \text{(C.11)} \]

In a more realistic approach one has to consider the width of the $\rho^0$-meson, which for $q^2 = m^2_\rho$ is $\Gamma_\rho = 152 \text{MeV}$. The form factor then becomes,

\[ F_\pi(q^2) = \frac{m^2_\rho}{m^2_\rho - q^2 - i m_\rho \Gamma_\rho}. \quad \text{(C.12)} \]

As was shown by Brown et al. [345] it turns out that the form given by Eq. (C.12) fit quite well the experimental data. The charge radius in this model is

\[ \left\langle r^2 \right\rangle^{1/2} = \left(6 \frac{d F_\pi}{dq^2} \bigg|_{q^2=0} \right)^{1/2} = \frac{\sqrt{6}}{m_\rho} = 0.62 \text{fm}, \quad \text{(C.13)} \]

in comparison to the experimental value $\left\langle r^2 \right\rangle^{1/2} = (0.66 \pm 0.01) \text{fm}$.

To estimate how good is the universal relation $f_{\rho NN} = f_{\rho\pi\pi} = \ldots = f_\rho$ consider the leptonic decay $\rho^0 \to e^+ e^-$ where the $\rho^0$-meson converts into a photon that annihilates into a $e^- e^+$ pair with coupling $\gamma_\rho$ at the meson-photon vertex (see Fig C.4 (a)). The decay width for this process is,

\[ \Gamma(\rho^0 \to e^+ e^-) = \frac{1}{3} \alpha^2 m_\rho \left( \frac{4\pi}{f_\rho^2} \right) \left[ 1 + 2 \left( \frac{m_e}{m_\rho} \right)^2 \right] \left[ 1 - 4 \left( \frac{m_e}{m_\rho} \right)^2 \right]^{1/2}. \quad \text{(C.14)} \]

Equating this to the experimental value of 6.8 KeV one gets $f_\rho = 5.1$. In the decay process $\rho^0 \to \pi^+ \pi^-$ of Fig C.4 (b) the $\rho^0$-meson couples directly to the pionic current in the form,

\[ L_{int}^{\rho\pi\pi} = f_{\rho\pi\pi} \mathbf{p}_\pi \cdot \mathbf{\pi} \times \partial^\mu \mathbf{\pi}, \quad \text{(C.15)} \]

which gives the following decay width,

\[ \Gamma(\rho^0 \to \pi^+ \pi^-) = \frac{2}{3} \left( \frac{f_{\rho\pi\pi}}{4\pi} \right) \left| \mathbf{p}_\pi \right|^3 \left| m_\rho \right|, \quad \text{(C.16)} \]

where $\mathbf{p}_\pi$ is the 3-momentum of the pion that is fixed by kinematic $m_\rho = 2\sqrt{m^2_\rho + |\mathbf{p}_\pi|^2}$. To fit the experimental value $\Gamma(\rho^0 \to \pi^+ \pi^-) = 153 \text{MeV}$ one needs $f_{\rho\pi\pi} = 5.9$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{Figure_C.4.png}
\caption{(a) Decay of a $\rho^0$-meson to leptons $\rho^0 \to e^+ e^-$. (b) Decay of a $\rho^0$-meson to hadrons $\rho^0 \to \pi^+ \pi^-$.}
\end{figure}
Appendix C. Short review of Vector Meson Dominance (VMD)

In the baryon sector we can estimate the product $f_{\rho\pi\pi} f_{\rho NN}$ by considering low-energy S-wave $\pi N$ scattering which is described by the chiral lagrangean,

$$\mathcal{L}^{(s-wave)}_{\pi N} = -\frac{1}{2f_\pi^2} N \gamma^\mu T \cdot (\pi \times \partial^\mu \pi) N,$$

(C.17)

with $T = \tau/2$. This lagrangean gives contact s-wave scattering with coupling constant $1/2f_\pi^2$ (see Fig. C.5 (a)) but in the VMD model (see Fig. C.5 (b)) the effective coupling transforms to,

$$f_{\rho\pi\pi} \frac{1}{m_\rho^2 - q^2} f_{\rho NN},$$

(C.18)

and at $q^2 = 0$ we obtain the so-called KSFR relation,

$$\frac{1}{2f_\pi^2} = \frac{f_{\rho\pi\pi} f_{\rho NN}}{m_\rho^2},$$

(C.19)

which relates the decay constant $f_\pi$ with the hadronic charge of the $\rho^0$-meson. Taking $f_\pi = 92.4$ MeV and $m_\rho = 770$ MeV one gets $f_{\rho NN} = 5.88$.

C.4 Virtual vector meson exchange in NN scattering

We have estimate the value of $f_{\rho NN}$ from VMD. In general the $\rho$-exchange process in the NN interaction give rise to a term proportional to $(\tau_1 \cdot \tau_2) (\sigma_1 \cdot \sigma_2)$, which is attractive in even-L waves and repulsive in odd-L waves. It also produces a tensor interaction of opposite sign to the $\pi$-exchange.

This can be deduced from the $\rho$ meson-nucleon interaction lagrangean\(^1\) given by,

$$\mathcal{L}_{\rho NN} = g_{\rho NN} \left[ N \gamma^\mu T \cdot \rho^\mu N + \frac{\kappa_\rho}{2M_N} N \sigma_{\mu\nu} \partial^\mu \rho^\nu N \right],$$

(C.20)

In the context of VMD and the effective lagrangean Eq. (C.4), the proper coupling constant is $f_{\rho NN} = 2g_{\rho NN}$ which refers to using the isospin $T = \tau/2$ instead of $\tau$ in Eq. (C.20). Using previous results one obtain a value of $g_{\rho NN} \simeq 2.9$.

An important remark: note that in Appendix B we have called $f_{\rho NN} = g_{\rho NN} \kappa_\rho$ but this coupling constant is not the same than the one used in this appendix. The reader should not confuse the vector meson charge $f_\rho = f_{\rho NN} = 2g_{\rho NN} \simeq 5.1$ used here with the tensor coupling constant to the nucleon $f_{\rho NN} = g_{\rho NN} \kappa_\rho \simeq 10.7$ of Appendix B.

\(^1\)See Sec. B.4.1.3 in Appendix B for example.
Appendix D

Squared transition potential products

In this appendix the explicit calculations for the squared transition potential products of Chapter 2 are shown. To carry out these products we make use of the following properties

\[
\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k, \quad \tau_i \tau_j = \delta_{ij} + i \epsilon_{ijk} \tau_k
\]

\[
S_i S_j^+ = \delta_{ji} - \frac{1}{3} \sigma_i \sigma_j, \quad T_i T_j^+ = \delta_{ji} - \frac{1}{3} \tau_i \tau_j
\]

together with

\[
\epsilon_{ijk} \epsilon^{ijl} = 2 \delta_{kl}, \quad \epsilon_{ijk} \epsilon^{ilm} = \delta_{jm} \delta_{kl} - \delta_{jl} \delta_{km}
\]

\[
\epsilon_{ijk} \epsilon^{mn} = \delta_{im} (\delta_{jn} \delta_{kl} - \delta_{jl} \delta_{km}) - \delta_{in} (\delta_{jm} \delta_{kl} - \delta_{jl} \delta_{km}) + \delta_{il} (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km})
\]

where \(\sigma_i(\tau_i)\) are the spin(isospin) Pauli matrices, \(\epsilon_{ijk}\) is the Levi-Civita tensor and \(\delta_{ij}\) is the Kronecker delta. We also define the following tensor operators in each channel

\[
S^{I}_{12} = 3(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r}) - \vec{\sigma}_1 \cdot \vec{\sigma}_2
\]

\[
S^{II}_{12} = 3(\vec{\sigma}_1 \cdot \vec{r})(\vec{S}_2 \cdot \vec{r}) - \vec{\sigma}_1 \cdot \vec{S}_2
\]

\[
S^{III}_{12} = 3(\vec{S}_1 \cdot \vec{r})(\vec{S}_2 \cdot \vec{r}) - \vec{S}_1 \cdot \vec{S}_2
\]
Appendix D. Squared transition potential products

- Isospin factors

\[
\left( \tilde{T}_1 \cdot \tilde{r}_2 \right) \left( \tilde{T}_2 \cdot \tilde{T}_1^\dagger \right) = \tau_2^j \tau_2^j T_1^j T_1^{j*} \\
= \tau_2^j \tau_2^j \left( \delta_{ij} - \frac{1}{3} \tau_1^j \tau_1^i \right) \\
= \frac{3}{2} \left( \delta_{ij} + i \epsilon_{ijk} \tau_1^k \right) \left( \delta_{ij} + i \epsilon_{ijl} \tau_1^l \right) \\
= 3 \left( \delta_{ij} \tau_1^j \tau_1^i \right) \\
= 2 + \frac{2}{3} \left( \tilde{T}_1 \cdot \tilde{r}_2 \right)
\]

\[
\left( \tilde{T}_1 \cdot \tilde{T}_2 \right) \left( \tilde{T}_2^* \cdot \tilde{T}_1^\dagger \right) = T_1^j T_2^* T_2^j T_1^{j*} \\
= \left( \delta_{ij} - \frac{1}{3} \tau_1^j \tau_1^i \right) \left( \delta_{ij} - \frac{1}{3} \tau_2^j \tau_2^i \right) \\
= 3 \left( \delta_{ij} \tau_2^j \tau_2^i \right) \\
= 2 - \frac{2}{3} \left( \tilde{T}_1 \cdot \tilde{r}_2 \right)
\]

- Products in channel $NN \rightarrow N\Delta$

\[
\left( \tilde{\sigma}_1 \cdot \tilde{S}_2 \right) \left( \tilde{\sigma}_1 \cdot \tilde{S}_2^\dagger \right) = \sigma_1^j \sigma_1^j S_2^j S_2^{j*} \\
= \sigma_1^j \sigma_1^j \left( \delta_{ij} - \frac{1}{3} \sigma_1^j \sigma_1^i \right) \\
= \frac{3}{2} \left( \delta_{ij} + i \epsilon_{ijk} \sigma_1^k \right) \left( \delta_{ij} + i \epsilon_{ijl} \sigma_1^l \right) \\
= 2 + \frac{2}{3} \left( \tilde{\sigma}_1 \cdot \tilde{S}_2 \right)
\]

\[
S_{12}^t \left( \tilde{\sigma}_1 \cdot \tilde{S}_2 \right) + \left( \tilde{\sigma}_1 \cdot \tilde{S}_2 \right) S_{12}^t = \\
= 3 \sigma_1^j S_1 S_2 x^t x^j - \sigma_1^j S_1 S_2^t x^j S_1^x \sigma_1^k \\
+ \sigma_1^k S_2^t \left( 3 S_1 S_2 x^t x^j - S_2^t x^j S_1^x \sigma_1^k \right) \\
= 3 \sigma_1^j \sigma_1^k S_2^t S_1^x x^j S_1^t \sigma_1^k \\
+ 3 \sigma_1^j \sigma_1^k S_2^t S_1^x x^j S_2^t \sigma_1^k \\
= 3 \sigma_1^j \sigma_1^k \left( \delta_{ij} - \frac{1}{3} \sigma_2^j \sigma_2^i \right) x^t x^j - \sigma_1^j \sigma_1^k \left( \delta_{ik} - \frac{1}{3} \sigma_2^k \sigma_2^i \right) x^t x^j \\
+ 3 \sigma_1^j \sigma_1^k \left( \delta_{ij} - \frac{1}{3} \sigma_2^j \sigma_2^i \right) x^t x^j - \sigma_1^j \sigma_1^k \left( \delta_{ik} - \frac{1}{3} \sigma_2^k \sigma_2^i \right) x^t x^j \\
= 3 \left( \delta_{ij} + i \epsilon_{ijk} \sigma_1^k \right) x^t x^j - \left( \delta_{ik} + i \epsilon_{ikm} \sigma_1^m \right) \left( \delta_{jk} + i \epsilon_{jkl} \sigma_1^l \right) x^t x^j \\
+ \frac{1}{3} \left( \delta_{ik} + i \epsilon_{ikm} \sigma_1^m \right) \left( \delta_{ki} + i \epsilon_{kij} \sigma_1^j \right) x^t x^j \\
= - \frac{2}{3} S_{12}^t
\]
Appendix D. Squared transition potential products

\[ S_{12}^{II}S_{12}^{II+} = (3\sigma_1^J S_2^I x^J - \sigma_1^I S_2^J)(3S_2^{+m}\sigma_1^m x^m x^n - S_2^{-m}\sigma_1^m) \]

\[ = 9\sigma_1^I S_2^I S_2^{+m}\sigma_1^m x^J x^m x^n + \sigma_1^I S_2^I S_2^{-m}\sigma_1^m \]

\[ - 3\sigma_1^I S_2^I S_2^{+m}\sigma_1^m x^J - 3\sigma_1^I S_2^I S_2^{-m}\sigma_1^m x^m x^n \]

\[ = 9\sigma_1^I (\delta_{mj} - 1/3\sigma_2^m) x^J x^m x^n + \sigma_1^I (\delta_{mi} - 1/3\sigma_2^m) \]

\[ - 3\sigma_1^m (\delta_{ mj} - 1/3\sigma_2^m) x^J - 3\sigma_1^m (\delta_{ mi} - 1/3\sigma_2^m) x^m x^n \]

\[ = 3 + 3\sigma_1^I \sigma_2^m x^J - 3\sigma_1^I \sigma_2^m x^J x^m x^n \]

\[ + \sigma_1^m \sigma_2^m x^J + \sigma_1^m \sigma_2^m x^J x^m x^n - \frac{1}{3} \sigma_1^m \sigma_2^m \]

\[ = 3 + 3(\delta_{ij} + i\epsilon_{ijk}\sigma_1^j) x^J \]

\[ - 3(\delta_{in} + i\epsilon_{ink}\sigma_1^k)(\delta_{jm} + i\epsilon_{jml}\sigma_2^l)x^J x^m x^n \]

\[ + (\delta_{in} + i\epsilon_{ink}\sigma_1^k)(\delta_{jm} + i\epsilon_{jml}\sigma_2^l)x^J \]

\[ - \frac{1}{3}(\delta_{in} + i\epsilon_{ink}\sigma_1^k)(\delta_{jm} + i\epsilon_{jml}\sigma_2^l) \]

\[ = 4 + \frac{2}{3}\tau_{12} - \frac{2}{3}(\bar{\sigma}_1 \cdot \bar{\sigma}_2) \]

- Products in channel \( NN \rightarrow \Delta \Delta \)

\[ (\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{S}_2)^+ = S_1^I S_2^I S_2^{+I} S_1^{+I} \]

\[ = (\delta_{ij} - \frac{1}{3}\sigma_1^i \sigma_1^j)(\delta_{ij} - \frac{1}{3}\sigma_2^i \sigma_2^j) \]

\[ = 3 - \frac{1}{3}\bar{\sigma}_1^2 - \frac{1}{3}\bar{\sigma}_2^2 + \frac{1}{9}\sigma_1^I \sigma_2^I \]

\[ = 1 + \frac{1}{9}(\delta_{ij} + i\epsilon_{ijk}\sigma_1^k)(\delta_{ij} + i\epsilon_{ijk}\sigma_2^k) \]

\[ = \frac{4}{3} - \frac{2}{9}(\bar{\sigma}_1 \cdot \bar{\sigma}_2) \]
\[ S_{12}^{II}\left( \vec{S}_1 \cdot \vec{S}_2 \right)^+ + \left( \vec{S}_1 \cdot \vec{S}_2 \right) S_{12}^{III+} \]
\[ = (3S_i^1 S_{2i}^1 x^j - S_i^2 S_i^1) S_{2i}^{+k} S_k^1 \]
\[ + S_i^1 S_{2i}^1 (3S_2^{+i} S_i^{1j} x^j - S_{2i}^{+1} S_{1i}^{+}) \]
\[ = 3(S_i^1 S_{2i}^1 S_{2i}^{+k} S_k^1 + S_i^1 S_{2i}^1 S_{2i}^{+1} S_{1i}^{+}) x^j \]
\[ - 2(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{S}_2)^+ \]
\[ = 3 \left[ (\delta_{ki} - \frac{1}{3} \sigma_1^k \sigma_2^i)(\delta_{kj} - \frac{1}{3} \sigma_2^k \sigma_2^j) \right] \]
\[ + (\delta_{ij} - \frac{1}{3} \sigma_1^i \sigma_1^j)(\delta_{ik} - \frac{1}{3} \sigma_2^k \sigma_2^j) x^j \]
\[ - 2(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{S}_2)^+ \]
\[ = 6 \left[ \delta_{ij} - \frac{1}{3} \sigma_1^i \sigma_1^j \right] \]
\[ + (\delta_{ik} - \frac{1}{3} \sigma_2^k \sigma_2^j) x^j \]
\[ - 2(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{S}_2)^+ \]
\[ = \frac{2}{9} S_{12}^I \]

\[ S_{12}^{II} S_{12}^{III+} = (3S_i^1 S_{2i}^{+1} x^j - S_i^2 S_i^1)(3S_{2i}^{+m} S_{1i}^{m} x^m x^n - S_{2i}^{+m} S_{1i}^{m}) \]
\[ = 9S_i^1 S_{2i}^1 S_{2i}^{+m} S_{1i}^{m} x^j x^n x^m \]
\[ - 3S_i^1 S_{2i}^{+m} S_{2i}^{+1} S_{1i}^{m} x^j x^n x^m \]
\[ + S_i^1 S_{2i}^1 S_{2i}^{+m} S_{1i}^{+n} \]
\[ = 9(\delta_{im} - \frac{1}{3} \sigma_1^i \sigma_1^m)(\delta_{jn} - \frac{1}{3} \sigma_2^j \sigma_2^n) x^j x^n x^m \]
\[ - 3(\delta_{in} - \frac{1}{3} \sigma_2^i \sigma_2^m) x^j \]
\[ - 3(\delta_{im} - \frac{1}{3} \sigma_1^i \sigma_1^m)(\delta_{jn} - \frac{1}{3} \sigma_2^j \sigma_2^n) x^j x^n x^m \]
\[ + (\delta_{in} - \frac{1}{3} \sigma_2^i \sigma_2^m)(\delta_{in} - \frac{1}{3} \sigma_2^i \sigma_2^m) \]
\[ = 9 \left( \frac{4}{9} - \frac{1}{9} \epsilon_{ink} \epsilon_{jnl} T_{k}^{j} \right) x^j x^n x^m \]
\[ - 6 \left( \frac{4}{9} - \frac{1}{9} \epsilon_{ink} \epsilon_{jnl} T_{k}^{j} \right) \]
\[ + \frac{4}{3} \left( \frac{1}{9} \epsilon_{ink} \epsilon_{jnl} T_{k}^{j} \right) \]
\[ = \frac{8}{3} \left( \frac{2}{9} S_{12}^I + \frac{2}{9}(\vec{S}_1 \cdot \vec{S}_2) \right) \]
Appendix E

NN interaction and renormalization with boundary condition

In this appendix we review rather general concepts about the NN interaction. In Sec. E.1 we review the partial wave decomposition. In Sec. E.2 we review the S-matrix and the scattering phase shifts and in Sec. E.3 we describe how can be obtained the partial-wave phase shifts for the case of a regular potential. We also review the concept of renormalization with boundary conditions in Sec. E.4.

E.1 Partial wave decomposition of the NN interaction

E.1.1 General form of the NN interaction

The general form of the NN interaction can be derived from invariance principles such as translations, Galilean transformations, rotations, parity and time reversal. In the non relativistic case the interaction reads

\[ V_{NN}(r) = V_C(r) + \tau W_C(r) \]

\[ + [V_S(r) + \tau_1 \cdot \tau_2 W_S(r)] \sigma_1 \cdot \sigma_2 \]

\[ + [V_T(r) + \tau_1 \cdot \tau_2 W_T(r)] S_{12} \]

\[ + [V_{LS}(r) + \tau_1 \cdot \tau_2 W_{LS}(r)] \mathbf{L} \cdot \mathbf{S} \]

\[ + [V_{LL}(r) + \tau_1 \cdot \tau_2 W_{LL}(r)] (\mathbf{L} \cdot \mathbf{S})^2 \]  

(E.1)

where the operators \( \tau, \sigma, S_{12} \) and \( \mathbf{L} \cdot \mathbf{S} \) are given by

\[ \tau = \tau_1 \cdot \tau_2 = 2(t+1) - 3, \]  

(E.2)

\[ \sigma = \sigma_1 \cdot \sigma_2 = 2(s+1) - 3, \]  

(E.3)

\[ S_{12} = 3 (\sigma_1 \cdot \hat{r})(\sigma_2 \cdot \hat{r}) - \sigma_1 \cdot \sigma_2 , \]  

(E.4)

\[ \mathbf{L} \cdot \mathbf{S} = \frac{1}{2} [j(j+1) - l(l+1) - s(s+1)] \]  

(E.5)
E.1.2 Central potential

In the case of a central potential the Schrödinger equation in the c.m. system can be written as

\[ \left[ -\frac{\nabla^2}{2\mu} + \frac{L^2}{2\mu r^2} + V(r) \right] \psi(r) = E\psi(r) \]  \hspace{1cm} (E.6)

where \( \mu \) is the reduced mass of the system, \( L \) is the angular momentum operator, \( E \) is the c.m. energy and \( V(r) \) is the central potential. Since we have a central potential the Hamiltonian commutes with \( L^2 \), i.e., \( [H, L^2] = 0 \), and we can write the solution of the Schrödinger equation as a combination of eigenfunctions of \( L^2 \)

\[ \psi(r) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \psi_{lm}(r) \]  \hspace{1cm} (E.7)

where \( \psi_{lm}(r) \) are given by

\[ \psi_{lm}(r) = \frac{u_l(r)}{r} Y_{lm}(\hat{r}) \]  \hspace{1cm} (E.8)

with \( Y_{lm}(\hat{r}) \) the spherical harmonic which contains the angular dependence, and \( u_l(r) \) the reduced wave function, which contains the radial dependence and satisfy the reduced Schrödinger equation

\[ -u''_l(r) + \left[ U(r) + \frac{l(l+1)}{r^2} \right] u_l(r) = k^2 u_l(r) \]  \hspace{1cm} (E.9)

with \( U(r) = 2\mu V(r) \) the reduced potential and \( k = \sqrt{2\mu E} \).

E.1.3 Non-central potential

If we add a tensor force to the potential the Schrödinger equation in the c.m. system can be written as

\[ \left[ -\frac{\nabla^2}{2\mu} + \frac{L^2}{2\mu r^2} + V(r) \right] \Psi(r) = E\Psi(r) \]  \hspace{1cm} (E.10)

where we now take the potential to consist of a non-tensorial part \( V_{NT}(r) \) (which is the sum of central, spin-spin, spin-orbit and quadratic spin-orbit terms) and a tensor part \( V_T(r) \),

\[ V(r) = V_{NT}(r) + S_{12}V_T(r) \]  \hspace{1cm} (E.11)

The tensor operator \( S_{12} \), as defined in Eq. (E.4), satisfy the commutation relations,

\[ [S_{12}, L^2] \neq 0, \quad [S_{12}, J^2] = 0, \quad [S_{12}, S^2] = 0 \]  \hspace{1cm} (E.12)

\(^1\)We work in natural units where \( \hbar = c = 1 \).
This means that we can construct the total wave function $\Psi(r)$ as a sum of components with good total angular momentum $\Psi_{jm}(r)$ and also that we can differentiate between singlet ($s=0$) and triplet ($s=1$) spin channels. In singlet channels $l = j$ while in triplet $l = j - 1, j, j + 1$. Thus the tensor force is operative in spin triplet channels only mixing states with different angular momentum $l = j - 1, j, j + 1$. For a given value of the total angular momentum $j$ we can write

$$\Psi_{jm}(r) = \sum_{l} \frac{u_{jl}(r)}{r} \mathcal{Y}_{jm}^{ls}(\hat{r})$$

where the sum has only one term for singlet states and three for triplet. The generalized spherical harmonics $\mathcal{Y}_{jm}^{ls}(\hat{r})$ are simultaneous eigenvectors of $L^2$, $S^2$, $J^2$ and $J_3$ and contain all the angular dependence of the wave function. They are defined by

$$\mathcal{Y}_{jm}^{ls}(\hat{r}) = \sum_{m_l, m_s} (lsm_l m_s jm) Y_{lm_l}(\hat{r}) \chi_{sm_s}$$

with $(lsm_l m_s jm)$ a Clebsch-Gordan coefficient and $\chi_{sm_s}$ the two components spinor. Note that taken into account $m = m_s + m_l$ the sum over $m_s$ can be omitted. In the singlet channel, $s = 0$, and therefore $\mathcal{Y}_{jm}^{00}(\hat{r}) = Y_{lm_l}(\hat{r})$.

Introducing Eq. (E.13) in Eq. (E.10) we arrive to the radial Schrödinger equation for a given $j$

$$-u''_{jl}(r) + \left[ U_{l,NT}^{s,j}(r) + \frac{l(l+1)}{r^2} \right] u_{jl}(r) + U_T(r) \sum_{l'} S_{jl'} u_{jl'}(r) = k^2 \ u_{jl}(r)$$

being $U_{l,NT}^{s,j}$ the reduced potential having all the non-tensorial components but depending on the spin $s$, orbital $l$ and total $j$ angular momentum of the system. The matrix element $S_{jl'l'}$ is defined by

$$S_{jl'l'} = \langle jml | S_{ll'} | jml' \rangle = \int d^2r \mathcal{Y}_{jm}^{ls}(\hat{r}) S_{ll'} \mathcal{Y}_{jm}^{l's}(\hat{r})$$

and its explicit form can be evaluated to give,

$$S_{jl'l'} = \begin{cases} 0 & \text{singlet channel ($s=0$)}, \\ \frac{2j(j+1)}{2j+1} & \text{triplet channel ($s=1$)} \end{cases}$$

where the matrix indices represent the orbital angular momentum $l, l' = j - 1, j, j + 1$. As we can see from Eq. (E.18) triplet states with $l = j$ are un-coupled. This can be seen clearly by writing

$$U_{ll'}^{s,j} = \delta_{ll'} U_{l,NT}^{s,j} + S_{jl'l'} U_T$$

and defining $u \equiv u_{jj}$, $v \equiv u_{jj-1}$, $w \equiv u_{jj+1}$. Then we have for the NN interaction three different cases:

- Singlet channel ($s=0$, $l = j$)

$$-u''(r) + \left[ U_{jj}^{ij}(r) + \frac{j(j+1)}{r^2} \right] u(r) = k^2 \ u(r)$$

(E.20)
• Triplet un-coupled channel \( (s = 1, l = j) \)

\[
-u''(r) + \left[ U_{jj}^{ij}(r) + \frac{j(j+1)}{r^2} \right] u(r) = k^2 u(r) \tag{E.21}
\]

• Triplet coupled channel \( (s = 1, l = j-1, j+1) \)

\[
-v''(r) + \left[ U_{j-1,j-1}^{ij}(r) + \frac{(j-1)j}{r^2} \right] v(r) + U_{j-1,j+1}^{ij}(r) w(r) = k^2 v(r)
\]
\[
-u''(r) + \left[ U_{j+1,j+1}^{ij}(r) + \frac{(j+1)(j+2)}{r^2} \right] w(r) + U_{j+1,j-1}^{ij}(r) v(r) = k^2 w(r) \tag{E.22}
\]

Taking into account the potential decomposition Eq. (E.1) but dropping the quadratic spin-orbit term, one obtains

\[
U_{jj}^{0j}(r) = M \left[ (V_C - 3V_S) + \tau(W_C - 3W_S) \right] \tag{E.23}
\]
\[
U_{jj}^{1j}(r) = M \left[ (V_C + V_S - V_{LS}) + \tau(W_C + W_S - W_{LS}) + 2(V_T + \tau W_T) \right] \tag{E.24}
\]
\[
U_{j-1,j-1}^{ij}(r) = M \left[ (V_C + \tau W_C + V_S + \tau W_S) + (j-1)(V_{LS} + \tau W_{LS}) - \frac{2(j-1)}{2j+1}(V_T + \tau W_T) \right] \tag{E.25}
\]
\[
U_{j+1,j+1}^{ij}(r) = \frac{6\sqrt{j(j+1)}}{2j+1} M (V_T + \tau W_T) \tag{E.26}
\]

E.2 S-matrix and phase shifts

We must now turn to the analysis of scattering properties of the NN interaction and look for the S-matrix and phase shifts. We shall initially restrict ourselves to singlet states \( (s = 0) \), so that the complications of spin and the action of the tensor force do not enter. For a un-coupled channel with orbital angular momentum \( l \) described by Eq. (E.9) in the case of a finite range potential, the asymptotic behaviour of the radial wave function can be expressed as follow

\[
u_l(r) \to \hat{h}_l^{-}(kr) - \hat{h}_l^{+}(kr)S_l(k) \tag{E.27}
\]

where \( \hat{h}_l^{\pm}(x) \) are the reduced spherical Hankel functions of order \( l \) defined by \( \hat{h}_l^{\pm}(x) = x\hat{h}_l^{\pm}(x) \) with

\[
\hat{h}_l^{\pm}(x) = x^{l} \left( -\frac{1}{x} \frac{d}{dx} \right)^{l} \frac{e^{\pm ix}}{x} \tag{E.28}
\]

that simplify to \( \hat{h}_0^{\pm}(x) = e^{\pm ix} \) for s-waves. \( S_l(k) \) is the partial wave S-matrix element (on-shell). From the unitarity of the S-matrix, \( S_l S_l^* = 1 \), it follows that \( S_l \) does correspond to the exponential of a phase,
and this phase is defined as the phase shifts $\delta_l(k)$, i.e.,

$$S_l(k) = e^{2i\delta_l(k)} \quad (E.29)$$

Using the phase shifts we can write for the real part of the radial wave function

$$u_l(r) \to \hat{j}_l(kr) - \cot \delta_l(k) \hat{y}_l(kr) \quad (E.30)$$

where $\hat{j}_l(x) = x j_l(x)$ and $\hat{y}_l(x) = xy_l(x)$ are the reduced spherical Bessel functions of order $l$ with

$$j_l(x) = x (\frac{d}{dx})^l \sin x \quad \text{and} \quad y_l(x) = -x (\frac{d}{dx})^l \cos x \quad (E.31)$$

that simplify to just $\hat{j}_0(x) = \sin x$ and $\hat{y}_0(x) = -\cos x$ for s-waves. We can also define the partial wave M-matrix element $M_l(k)$ through

$$M_l(k) = i \frac{S_l(k) + 1}{S_l(k) - 1} = \cot \delta_l(k) \quad (E.32)$$

which can be expanded as an effective range expansion at low momenta, i.e.,

$$\hat{M}_l(k) \equiv k^{2l+1} \cot \delta_l(k) = -\frac{1}{\alpha_l} + \frac{1}{2} r_l k^2 + v_{2,l} k^4 + v_{3,l} k^6 + \cdots \quad (E.33)$$

where $\alpha_l$ is the scattering length, $r_l$ the effective range and $v_{i,l}$ the curvature parameters.

When we include a tensor force the admixture of states with different orbital angular momentum occurs and hence $l$ is not a good quantum number. However, if we want our Hamiltonian to be invariant under rotations we can decompose the S-matrix in blocks with good total angular momentum $j$, i.e., rotational invariance imply $S$ to be diagonal in $j,m$ and independent of $m$,

$$\langle l's'j'm'|S(k)|lsjm\rangle = \delta_{jj'} \delta_{mm'} \langle l's'|S^j(k)|ls\rangle \quad (E.34)$$

If in addition we impose certain symmetries the form of the matrix elements simplify notably. If parity is conserved then transitions like $(-1)^j \leftrightarrow (-1)^{j\pm 1}$ are forbidden and we say that the state $l = j$ is un-coupled to the rest,

$$\langle l's'|S^j(k)|ls\rangle = 0, \quad l, l' = j, j \pm 1 \quad (E.35)$$

Time reversal symmetry imply $S$ to be symmetric, i.e.,

$$\langle l's'|S^j(k)|ls\rangle = \langle ls|S^j(k)|l's'\rangle \quad (E.36)$$

Now, since $S$ is symmetric and unitary, can be diagonalized by a real orthogonal matrix,

$$\langle l's'|S^j(k)|ls\rangle = \sum_{\alpha} \langle l's'|U^j|\alpha\rangle e^{2i\delta^j_\alpha} \langle \alpha|U^j|ls\rangle \quad (E.37)$$

where the eigen values $\delta^j_\alpha$ are real quantities.
In the NN system we have for the singlet channel \((s = 0), l = l' = j\) and
\[
\langle l'|S^j(k)|l\rangle \equiv S_{jj}^0 = e^{2i\delta_j^0(k)} \quad \text{(E.38)}
\]
while for the triplet \((s = 1), l, l' = j, j \pm 1, \text{ and } S\) is a \(3 \times 3\) matrix
\[
\langle l'|S^j(k)|l\rangle \equiv S_{jj'}^{1j} = \begin{pmatrix}
S_{j-1,j-1}^j & 0 & S_{j-1,j+1}^j \\
0 & S_{j,j}^j & 0 \\
S_{j+1,j-1}^j & 0 & S_{j+1,j+1}^j
\end{pmatrix} \quad \text{(E.39)}
\]
where again unitarity imply \(S_{jj}^j = e^{2i\delta_j^j(k)}\) for triplet un-coupled channels. For triplet coupled channels, the remanent \(2 \times 2\) matrix \(S_{jj'}^{1j}\) with \(l, l' = j \pm 1\) can be diagonalized by a real orthogonal matrix in the form [346]
\[
S_{jj'}^{1j} = \begin{pmatrix}
\cos \epsilon_j & -\sin \epsilon_j \\
\sin \epsilon_j & \cos \epsilon_j
\end{pmatrix}
\begin{pmatrix}
e^{2i\delta_j^1} & 0 \\
0 & e^{2i\delta_{j+1}^1}
\end{pmatrix}
\begin{pmatrix}
\cos \epsilon_j & \sin \epsilon_j \\
-\sin \epsilon_j & \cos \epsilon_j
\end{pmatrix} \quad \text{(E.40)}
\]
This is known as the Blatt-Biedenharn (BB) parameterization and \(\delta_j^1\) and \(\epsilon_j\) as the eigen phase shifts. For low momenta the eigen phase shifts behave like
\[
\delta_j^{1j} \sim k^{2j-1}, \quad \delta_j^{1j} \sim k^{2j+3} \quad \text{and} \quad \epsilon_j \sim k^2 \quad \text{(E.41)}
\]
For \(j > 1\) and small values of \(k\) the mixing parameter \(\epsilon_j\) can have already a sizeable value when the phases are both still practically zero. For that reason there are another parameterization where the behaviour of the mixing parameter is nicer. This is the Stapp-Ypsilantis-Metropolis (SYM) [347] in which \(S\) is written in terms of the nuclear-bar phase shifts \(\bar{\delta}_j^{1j}\) and \(\bar{\epsilon}_j\) as
\[
S_{jj'}^{1j} = \begin{pmatrix}
\cos (2\bar{\epsilon}_j)e^{2i\bar{\delta}_j^{1j}} & i\sin (2\bar{\epsilon}_j)e^{i(\bar{\delta}_j^{1j} + \bar{\delta}_{j+1}^{j+1})} \\
\sin (2\bar{\epsilon}_j)e^{i(\bar{\delta}_j^{1j} + \bar{\delta}_{j+1}^{j+1})} & \cos (2\bar{\epsilon}_j)e^{2i\bar{\delta}_j^{j+1}}
\end{pmatrix} \quad \text{(E.42)}
\]
For low momenta the nuclear bar phase shifts behave like
\[
\bar{\delta}_j^{1j} \sim k^{2j-1}, \quad \bar{\delta}_j^{1j} \sim k^{2j+3} \quad \text{and} \quad \bar{\epsilon}_j \sim k^{2j+1} \quad \text{(E.43)}
\]
Both parameterizations are related by the following equations
\[
\begin{align*}
\delta_j^{1j} + \bar{\delta}_j^{1j} & = \bar{\delta}_j^{1j} + \bar{\delta}_{j+1}^{j+1} \\
\sin(\bar{\delta}_j^{1j} - \bar{\delta}_{j}^{1j}) & = \tan 2\bar{\epsilon}_j \\
\sin(\bar{\delta}_{j+1}^{j+1} - \bar{\delta}_{j+1}^{j+1}) & = \sin 2\bar{\epsilon}_j
\end{align*} \quad \text{(E.44-46)}
\]
Then, in the case of a finite range potential, for triplet un-coupled channels with total angular momentum \(j\) described by Eq. (E.21), the asymptotic behaviour of the radial wave function is as Eq. (E.30) with the corresponding triplet phase shift \(\delta_j = \delta_j^{1j}\)
\[
u_j(r) \rightarrow j_j(kr) - \cot \delta_j(k)\tilde{\delta}_j(kr) \quad \text{(E.47)}
\]
For triplet coupled channels with total angular momentum \( j \) described by Eq. (E.22), the asymptotic behaviour of the radial wave functions \( v_j \) and \( w_j \) can be expressed as follow

\[
v_j(r) \rightarrow a_{j-1}(k)\hat{h}_{j-1}(kr) - b_{j-1}(k)\hat{h}_{j-1}^+(kr) \tag{E.48}
\]

\[
w_j(r) \rightarrow a_{j+1}(k)\hat{h}_{j+1}(kr) - b_{j+1}(k)\hat{h}_{j+1}^+(kr) \tag{E.49}
\]

where the coefficients \( a_{j\pm 1} \) and \( b_{j\pm 1} \) are related through the S-matrix

\[
\begin{pmatrix}
  b_{j-1}(k) \\
  b_{j+1}(k)
\end{pmatrix} = \begin{pmatrix}
  S^j_{j-1,j-1} & S^j_{j-1,j+1} \\
  S^j_{j+1,j-1} & S^j_{j+1,j+1}
\end{pmatrix} \begin{pmatrix}
  a_{j-1}(k) \\
  a_{j+1}(k)
\end{pmatrix} \tag{E.50}
\]

In general, the Schrödinger equation for this case has two linearly independent solutions that we label as \( \alpha \) and \( \beta \) solutions, which asymptotic behavior can be expressed in terms of the eigen phase shifts as,

\[
v_{k,j,\alpha}(r) \rightarrow \hat{j}_{j-1}(kr)\cot\frac{\delta_{1j}^j}{2} - \hat{y}_{j-1}(kr) \tag{E.51}
\]

\[
w_{k,j,\alpha}(r) \rightarrow \tan\epsilon\left[\hat{j}_{j+1}(kr)\cot\frac{\delta_{1j}^j}{2} - \hat{y}_{j+1}(kr)\right] \tag{E.52}
\]

\[
v_{k,j,\beta}(r) \rightarrow \tan\epsilon\left[\hat{j}_{j-1}(kr)\cot\frac{\delta_{2j}^j}{2} - \hat{y}_{j-1}(kr)\right] \tag{E.53}
\]

\[
w_{k,j,\beta}(r) \rightarrow \hat{j}_{j+1}(kr)\cot\delta_{2j}^j - \hat{y}_{j+1}(kr) \tag{E.54}
\]

### E.3 Triplet-channel phase shifts for regular potentials

The extracting of eigen-phase shifts in the case of a regular potential is not trivial. A practical way of doing so is by taking two linearly independent solutions \((u_1, w_1)\) and \((u_2, w_2)\) behaving at the origin, \( r \rightarrow 0 \), as,

\[
u_1(r) \rightarrow r^j, \quad w_1(r) \rightarrow 0, \tag{E.55}
\]

\[
u_2(r) \rightarrow 0, \quad w_2(r) \rightarrow r^{j+2}, \tag{E.56}
\]

One knows that asymptotically, \( r \rightarrow \infty \), these solutions must be linear combinations of Bessel functions that can be written as,

\[
u_1(r) \rightarrow A_1 \hat{j}_{j-1} + B_1 \hat{y}_{j-1}, \quad w_1(r) \rightarrow C_1 \hat{j}_{j+1} + D_1 \hat{y}_{j+1},
\]

\[
u_2(r) \rightarrow A_2 \hat{j}_{j-1} + B_2 \hat{y}_{j-1}, \quad w_2(r) \rightarrow C_2 \hat{j}_{j+1} + D_2 \hat{y}_{j+1}. \tag{E.57}
\]

Then, integrating upward the Schrödinger equation from \( r = 0 \) to infinity starting with Eqs. (E.55), (E.56) and matching the numerical solution to the asymptotic solutions Eqs. (E.57) one knows the coefficients \( A_{1,2}, B_{1,2}, C_{1,2}, D_{1,2} \).
Appendix E. NN interaction and renormalization with boundary condition

Once we know the value of this coefficients, we form the usual $\alpha$ and $\beta$ solutions,

\[
\begin{align*}
  u^{(\alpha)}(r) &= \alpha_1 u_1(r) + \alpha_2 u_2(r), \\
  w^{(\alpha)}(r) &= \alpha_1 w_1(r) + \alpha_2 w_2(r), \\
  u^{(\beta)}(r) &= \beta_1 u_1(r) + \beta_2 u_2(r), \\
  w^{(\beta)}(r) &= \beta_1 w_1(r) + \beta_2 w_2(r),
\end{align*}
\]  

\hspace{1cm} (E.58)

with asymptotic behaviour,

\[
\begin{align*}
  u^{(\alpha)}(r) &\rightarrow \cot \delta_{j-1} - \hat{y}_{j-1}, \\
  w^{(\alpha)}(r) &\rightarrow \tan \epsilon \left( \cot \delta_{j+1} - \hat{y}_{j+1} \right), \\
  u^{(\beta)}(r) &\rightarrow \tan \epsilon \left( \hat{j}_{j-1} \cot \delta_2 - \hat{y}_{j-1} \right), \\
  w^{(\beta)}(r) &\rightarrow \hat{j}_{j+1} \cot \delta_2 - \hat{y}_{j+1}.
\end{align*}
\]  

\hspace{1cm} (E.59)

Therefore, introducing Eqs. (E.57) into Eqs. (E.58) and equating to its asymptotic behaviour Eqs. (E.59) we obtain the following relations, e.g. using the $\alpha$ solution,

\[
\begin{align*}
  \frac{A_1 + \alpha A_2}{B_1 + \alpha B_2} &= \frac{C_1 + \alpha C_2}{D_1 + \alpha D_2} = -\cot \delta_1, \\
  \frac{D_1 + \alpha D_2}{B_1 + \alpha B_2} &= \tan \epsilon
\end{align*}
\]  

\hspace{1cm} (E.60, E.61)

we have defined the coefficient $\alpha \equiv \alpha_2/\alpha_1$. The first equation, in terms of $\alpha$, have two solutions, one corresponds to the $\alpha$-state eigen-phase shift and the other to the $\beta$ one. The second equation can be used to obtain the mixing angle unambiguously. If we were used the $\beta$ state instead we were found the same result.

### E.4 Renormalization with boundary conditions

Now we will derive constraints on the short distance boundary condition. Working with energy independent potentials \(^2\) we will show what this requirement implies for the renormalization program. Consider a potential which can be singular at short distances and with which we want to solve the Schrödinger equation,

\[
-\frac{1}{M} \nabla^2 \Psi_k(x) + V(x)\Psi_k(x) = E_k \Psi_k(x),
\]  

\hspace{1cm} (E.62)

where $\Psi(x)$ is a spin-isospin vector with $4 \times 4 = 16$ components, which usually satisfies the out-going wave boundary condition,

\[
\Psi_k(x) \rightarrow \left[ e^{ik \cdot x} + f(\hat{k}' \cdot \hat{k}) \frac{e^{ikr}}{r} \right]_l^{s,m} \chi_{l,m}^{s,m},
\]  

\hspace{1cm} (E.63)

\(^2\)We expect genuine energy dependence to show up as sub-threshold inelastic (e.g. pion production) effects.
with \( f(\hat{k}, \hat{k}) \) the quantum mechanical scattering matrix amplitude and \( \chi_{l,m}^p \) a 4x4 total spin-isospin state. We introduce a renormalization scale in the problem by considering a radial cut-off \( r_c \) such that the local potential \( V(x) \) is valid for the long distance region \( r > r_c \). The precise form of the interaction for the short distance region \( r < r_c \) is not necessary as the limit \( r_c \to 0 \) will be taken at the end. Any distributional terms \( \sim \delta^{(3)}(x) \) arising from the long distance potential \( V(x) \) are necessarily included in the inner region, \( r < r_c \). The inner wave function \( \Phi_k(x) \) satisfies

\[
- \frac{1}{M} \nabla^2 \Phi_k(x) + V(x) \Phi_k(x) = E_k \Phi_k(x),
\]

and will be assumed to be regular at the origin \(^3\). Using the second Green identity we get for the inner and outer regions

\[
(E_k - E_p) \int_{r < r_c} d^3x \Phi_k^\dagger(x) \Phi_p(x) = - \frac{1}{M} \int dS \cdot \left[ \nabla \Phi_k^\dagger(x) \Phi_p(x) - \Phi_k^\dagger(x) \nabla \Phi_p(x) \right]_{r = r_c},
\]

and

\[
(E_k - E_p) \int_{r > r_c} d^3x \Phi_k^\dagger(x) \Psi_p(x) = + \frac{1}{M} \int dS \cdot \left[ \nabla \Psi_k^\dagger(x) \Psi_p(x) - \Psi_k^\dagger(x) \nabla \Psi_p(x) \right]_{r = r_c},
\]

respectively where the difference in sign from the inner to the outer integration comes from opposite orientations in the integration surface. If we use a radial cut-off and integrate in a sphere \( r_c \) we get, e.g., for the outer region \( (\hat{x} = x/r) \),

\[
(E_k - E_p) \int d^3x \Phi_k^\dagger(x) \Psi_p(x) = \frac{1}{M} \int d\hat{x} \left[ \partial_r \Psi_k^\dagger(\hat{x} r_c) \Psi_p(\hat{x} r_c) - \Psi_k^\dagger(\hat{x} r_c) \partial_r \Psi_p(\hat{x} r_c) \right].
\]

Clearly, orthogonality of states in the whole space for different energies,

\[
\int_{r < r_c} d^3x \Phi_k^\dagger(x) \Phi_p(x) + \int_{r > r_c} d^3x \Phi_k^\dagger(x) \Psi_p(x) = 0,
\]

can be achieved by setting the general and common boundary condition,

\[
\partial_r \Phi_p(\hat{x} r_c) = L_p(\hat{x} r_c) \Phi_p(\hat{x} r_c),
\]
\[
\partial_r \Psi_p(\hat{x} r_c) = L_p(\hat{x} r_c) \Psi_p(\hat{x} r_c).
\]

with in fact \( L_p(\hat{x} r_c) = L_k(\hat{x} r_c) \). Here, \( L_p(\hat{x} r_c) \) is a self-adjoint matrix which may depend on energy, and may be chosen to commute with the symmetries of the potential \( V(x) \) \(^4\). This important result deserves further comments. Deriving with respect to the energy the inner boundary condition, Eq. (E.65), i.e.

\(^3\)Obviously in the sense that the radial wave function is regular at \( r = 0 \).

\(^4\)In practice for the large-\( N \) NN potential this would mean taking

\[
L(\hat{x} r_c) = L_C(r_c) + \tau_1 \cdot \tau_2 [L_S(r_c) \sigma_1 \cdot \sigma_2 + L_T(r_c) S_{12}]
\]

which suggests at most only three counterterms for all partial waves.
taking \( E_k = E_p + \Delta E \) and \( \Phi_k(x) = \Phi_p(x) + \Delta E \phi_p(x)/\partial E \), we get
\[
\int d\hat{x} \Phi_p^\dagger(\hat{x}_r)c \frac{\partial L_p(\hat{x}_r)c}{\partial E} \Phi_p(\hat{x}_r)c = -\frac{1}{M} \int_{r<r_c} d^3x \ \Phi_p^\dagger(\hat{x}) \Phi_p(\hat{x}) ,
\] (E.70)

The important issue here is that regardless on the representation at short distances, the boundary condition must become energy independent when \( r_c \to 0 \), namely
\[
\lim_{r_c \to 0} \frac{\partial L_p(\hat{x}_r)c}{\partial E} = 0 ,
\] (E.71)

provided one has
\[
\lim_{r_c \to 0} \int_{r<r_c} d^3x \ \Phi_p^\dagger(\hat{x}) \Phi_p(\hat{x}) = 0 ,
\] (E.72)

which is guaranteed for a finite wave function at the origin. Thus we may take a fixed energy, e.g. zero energy, as a reference state.
\[
\lim_{r_c \to 0} L_p(\hat{x}_r)c = \lim_{r_c \to 0} L_0(\hat{x}_r)c ,
\] (E.73)

The condition of Eq. (E.72) is the quite natural quantum mechanical requirement that the contribution to the total probability in the (generally unknown) short distance region is small. This is the physical basis of the renormalization program which corresponds to the mathematical implementation of short distance insensitivity. It should be noted that this requirement depends on the potential. The condition of Eq. (E.72) implies that in the limit \( r_c \to 0 \) one must always choose a normalizable outer solution \( \Psi_k(x) \) at the origin and the boundary condition must be chosen independent on energy. Note that energy dependence would be allowed if the cut-off was kept finite. Although the discussion has been done here for a local potential for simplicity it does not change at all if a non-local one is considered instead. The requirement of orthogonality in the whole space can be fulfilled for an interaction characterized by a non-local and energy independent potential in the inner region. This simultaneous disregard of both non-local and energy dependent effects was advocated long ago by Partovi and Lomon \[36\] on physical grounds, and as we see, it is a natural consequence within the renormalization approach.

The renormalization procedure is then conceptually simple since any finite energy state with given quantum numbers can be chosen as a reference state to determine the rest of the bound state spectrum and scattering states. For instance, using a bound state (the deuteron) \( \Psi_d(x) \), at long distances,
\[
\Psi_d(x) \to \frac{A_S}{\sqrt{4\pi r}} e^{-\gamma r} \left[ 1 + \frac{\eta}{\sqrt{8}} S_{12} \right] \chi_{pnm} ,
\] (E.74)

we can integrate in the deuteron equation
\[
-\frac{1}{M} \nabla^2 \Psi_d(x) + V(x) \Psi_d(x) = -\frac{\gamma^2}{M} \Psi_d(x) ,
\] (E.75)

and determine the short distance boundary condition matrix \( L(\hat{x}_r)c \) from
\[
\partial_r \Psi_d(\hat{x}_r)c = L(\hat{x}_r)c \Psi_d(\hat{x}_r)c ,
\] (E.76)
Then, using the same boundary condition matrix \( L(\hat{x}r_c) \) for the finite energy state,

\[
\partial_r \Psi_k(\hat{x}r_c) = L(\hat{x}r_c) \Psi_k(\hat{x}r_c),
\]

we integrate out the finite energy equation (E.62) whence the scattering amplitude may be obtained. In this manner the deuteron binding energy defines both the bound state spectrum and the scattering states. Renormalization is achieved by taking the limit \( r_c \to 0 \) at the end of the calculation. The conditions under which such a procedure is meaningful has been discussed at length in Refs. [126–128] in the context of chiral potentials but basically one have to check whether the observables do depend on the renormalization scale or not [129].
Appendix F

Splitting formulas for phase-shifts

We want to derive the splitting formula for phase shifts, i.e., the change in the phase shifts respect to the center of the multiplet by using distorted waves perturbation theory.

The coupled channel Schrödinger equation for the relative motion reads

\[- u''(r) + \left[ U(r) + \frac{L^2}{r^2} \right] u(r) = k^2 u(r), \tag{F.1} \]

where \( U^{S_J}_{L,L'}(r) = 2 \mu_{np} V^{S_J}_{L,L'}(r) \) is the coupled channel matrix potential which for the total angular momentum \( J > 0 \) can be written as,

\[
U^{0J}(r) = U^{0J}_{jj}
\]

\[
U^{1J}(r) = \begin{pmatrix}
U^{1J}_{-1,-1}(r) & 0 & U^{1J}_{-1,+1}(r) \\
0 & U^{1J}_{00}(r) & 0 \\
U^{1J}_{-1,+1}(r) & 0 & U^{1J}_{+1,+1}(r)
\end{pmatrix}
\]

In Eq. (F.1) \( L^2 = \text{diag}(L_1(L_1+1), \ldots, L_N(L_N+1)) \) is the angular momentum, \( u(r) \) is the reduced matrix wave function and \( k \) the c.m. momentum. In the case at hand \( N = 1 \) for the spin singlet channel with \( L = J \) and \( N = 3 \) for the spin triplet channel with \( L_1 = J - 1, L_2 = J \) and \( L_3 = J + 1 \). For ease of notation we will keep the compact matrix notation of Eq. (F.1). At long distances, we assume the asymptotic normalization condition

\[ u(r) \to \hat{\Phi}^{(-)}(r) - \hat{\Phi}^{(+)}(r)S, \tag{F.3} \]

with \( S \) the standard coupled channel unitary S-matrix. For the spin singlet state, \( S = 0 \), one has \( L = J \) and hence the state is un-coupled

\[ S^{0J}_{JJ} = e^{2i\delta^0_J}, \tag{F.4} \]

whereas for the spin triplet state \( S = 1 \), one has the un-coupled \( L = J \) state

\[ S^{1J}_{JJ} = e^{2i\delta^1_J}, \tag{F.5} \]
and the two channel coupled states \( L, L' = j \pm 1 \) which written in terms of the eigenphases are

\[
S^{1J} = \begin{pmatrix}
  \cos \epsilon_J & -\sin \epsilon_J \\
  \sin \epsilon_J & \cos \epsilon_J
\end{pmatrix}
\begin{pmatrix}
  e^{2i\delta_J^{1J} - 1} & 0 \\
  0 & e^{2i\delta_J^{1J} + 1}
\end{pmatrix}
\begin{pmatrix}
  \cos \epsilon_J & \sin \epsilon_J \\
  -\sin \epsilon_J & \cos \epsilon_J
\end{pmatrix}.
\] (F.6)

The corresponding out-going and in-going free spherical waves are given by

\[
\hat{h}^{(\pm)}(r) = \text{diag}(\hat{h}^{L,(kr)}_{L_1}(kr), \ldots, \hat{h}^{L,(kr)}_{L_N}(kr)),
\] (F.7)
with \( \hat{h}^{\pm}_l(x) \) the reduced Hankel functions of order \( l \), \( \hat{h}^{L,(kr)}_{L,(kr)} = x H^\pm_{L+1/2}(x) ( \hat{h}^0_l = e^{\pm ix} ) \), and satisfy the free Schrödinger’s equation for a free particle.

In order to determine the infinitesimal change of the \( S \) matrix, \( S \rightarrow S + \Delta S \), under a general deformation of the potential \( U(r) \rightarrow U(r) + \Delta U(r) \) we use Schrödinger’s equation (F.1) and the standard Lagrange’s identity adapted to this particular case, we get

\[
[u(r)^\dagger \Delta u'(r) - u'(r)^\dagger \Delta u(r)]' = u(r)^\dagger \Delta U(r) u(r).
\] (F.8)

The unitarity of the \( S \)-matrix,

\[
S^\dagger S = 1,
\]

yields the condition \( \Delta S^\dagger S + S^\dagger \Delta S = 0 \). We assume a mixed boundary condition at short distances, \( r = r_c \), for the unperturbed coupled channel potential, \( U(r) \),

\[
u'(r_c) + Lu(r_c) = 0,
\] (F.9)
with \( L \) a self-adjoint matrix. After integration from the cut-off radius \( r_c \) to infinity and using the asymptotic form of the matrix wave function, Eq. (F.3), as well as the condition at the origin, Eq. (F.9) yields

\[
2i\kappa S^\dagger \Delta S = \int_{r_c}^\infty dr u(r)^\dagger \Delta U(r) u(r).
\] (F.10)

If we take the Wigner symmetric states as the unperturbed problem, then \( S \), \( U(r) \) and \( u(r) \) become a diagonal matrices, so that

\[
\Delta \delta^{ST}_{JL} = -\frac{1}{2p} \int_{r_c}^\infty dr u^{ST}_L(r)^\dagger \Delta U(r) u^{ST}_L(r),
\] (F.11)

so that the perturbed eigenphases become

\[
\delta^{ST}_{JL} = \delta^{ST}_{L,L} + \Delta \delta^{ST}_{JL}
\] (F.12)

Identifying further \( \Delta U \) with the spin-orbit and the tensor potential, in the LS-coupling the result may be written as

\[
\delta^{ST}_{J,LL} = \delta^{ST}_{L,L} + \delta_{S,1}^{ST} \delta^{ST}_{T}(S^1_{J})_{L,L}
+ \delta^{ST}_{L,S} \frac{1}{2} [J(J + 1) - L(L + 1) - S(S + 1)],
\] (F.13)
where

\[
(S_{12}^J)_{J-1,J-1} = -\frac{2(J-1)}{2J+1}, \quad (F.14)
\]

\[
(S_{12}^J)_{J,J} = 2, \quad (F.15)
\]

\[
(S_{12}^J)_{J+1,J+1} = -\frac{2(J+2)}{2J+1}, \quad (F.16)
\]

and \(\delta_{S,1}\) is the Kronecker delta. The phases \(\delta_{LST}^{SST}, \delta_{LST}^{ST}\) and \(\delta_{LST}^{LS}\) represent the center of the multiplet and the splitting due to the spin-orbit and the tensor force. At this order the mixing phases vanish, \(\Delta\epsilon_J = 0\), since the S matrix is diagonal. As a consequence, this formula does not distinguish between eigen phase shift or nuclear bar parameterizations as they are second order effects.

The above equations would yield a Lande-like interval rule between spin-triplet energy levels for the spin-orbit or the tensor potentials separately. For instance, for the P-wave one has

\[
\delta_{3}^{1P} = \delta_{1P},
\]

\[
\delta_{3}^{3P} = \delta_{3P} - 4\delta_{SP} - 4\delta_{PLS},
\]

\[
\delta_{3}^{2P} = \delta_{3P} + 2\delta_{TP} - 2\delta_{PLS},
\]

\[
\delta_{3}^{1P} = -\frac{2}{5}\delta_{TP} + 2\delta_{PLS}. \quad (F.17)
\]

From the last three equations we can extract either \(\delta_{3P_C}, \delta_{3P_T}\) or \(\delta_{3P_L}\) as a function of \(\delta_{3P_0}, \delta_{3P_1}\) and \(\delta_{3P_2}\). The result is,

\[
\delta_{3P_C} = \frac{1}{9}(\delta_{3P_0} + 3\delta_{3P_1} + 5\delta_{3P_2}), \quad (F.18)
\]

\[
\delta_{3P_T} = -\frac{5}{72}(2\delta_{3P_0} - 3\delta_{3P_1} + \delta_{3P_2}), \quad (F.19)
\]

\[
\delta_{3P_L} = -\frac{1}{12}(2\delta_{3P_0} + 3\delta_{3P_1} - 5\delta_{3P_2}). \quad (F.20)
\]

Similarly for the other waves we obtain,

\[
\delta_{3D_C} = \frac{1}{15}(3\delta_{3D_1} + 5\delta_{3D_2} + 7\delta_{3D_3}), \quad (F.21)
\]

\[
\delta_{3D_T} = -\frac{7}{120}(3\delta_{3D_1} - 5\delta_{3D_2} + 2\delta_{3D_3}), \quad (F.22)
\]

\[
\delta_{3D_L} = -\frac{1}{60}(9\delta_{3D_1} + 5\delta_{3D_2} - 14\delta_{3D_3}). \quad (F.23)
\]

for triplet D-waves

\[
\delta_{3F_C} = \frac{1}{21}(5\delta_{3F_2} + 7\delta_{3F_3} + 9\delta_{3F_4}), \quad (F.24)
\]

\[
\delta_{3F_T} = -\frac{5}{112}(4\delta_{3F_2} - 7\delta_{3F_3} + 3\delta_{3F_4}), \quad (F.25)
\]

\[
\delta_{3F_L} = -\frac{1}{168}(20\delta_{3F_2} + 7\delta_{3F_3} - 27\delta_{3F_4}). \quad (F.26)
\]
for triplet F-waves

\[
\begin{align*}
\delta_{3G_C} & = \frac{1}{27} (7\delta_{3G_3} + 9\delta_{3G_4} + 11\delta_{3G_5}), \\
\delta_{3G_T} & = -\frac{77}{2160} (5\delta_{3G_3} - 9\delta_{3G_4} + 4\delta_{3G_5}), \\
\delta_{3G_{LS}} & = \frac{1}{360} (-35\delta_{3G_3} - 9\delta_{3G_4} + 44\delta_{3G_5}),
\end{align*}
\] (F.27) (F.28) (F.29)

for triplet G-waves and so on.
Appendix G

Wide resonance exchange between nucleons

Consider a wide meson, which may decay into two pions. This meson, when dealing with the NN interaction, may be either the $\sigma$ or the $\rho$. This picture of a wide exchange meson is intimately related with the two-pion-exchange nature of the NN interaction and this fact restricts the possible quantum numbers. For two pions in a relative $s$ or $p$ wave, which is typical for moderate kinetic energies, the symmetry of the $\pi\pi$ wave function restricts the possible quantum numbers to $(J^\pi = 0^+; I = 0, 2)$ and $(J^\pi = 1^-; I = 1)$. Since the NN interaction has a maximum isospin exchange of $I = 1$, only the following types of TPE processes can then exist:

1. Scalar-isoscalar $(J^\pi, I) = (0^+, 0)$ exchange, which corresponds to the quantum numbers of the $\sigma$-meson.
2. Vector-isovector $(J^\pi, I) = (1^-, 1)$ exchange, which corresponds to the quantum numbers of the $\rho$-meson.

G.1 Toy model for $\pi\pi$ scattering

To express the exchange of this wide meson of mass $m$ and width $\gamma$ between two nucleons\footnote{We call the mass and width of the Breit-Wigner resonance $m$ and $\gamma$ respectively and the one in the pole position $M$ and $\Gamma$.}, we try the following modification of the propagator for the “zero-width” pole term [227],

$$
\frac{1}{s - m^2} \rightarrow \frac{1}{s - m^2 - im\gamma f(s)},
$$

where $f(s) = \sqrt{\frac{s - 4m^2}{m^2 - 4m^2}}$, and $s = q^2$ with $q$ the four-momentum of the exchanged meson, i.e., the four-momentum transfer. This propagator can be written in the Källén-Lehmann representation,

$$
D(s) = \frac{1}{s - m^2 - im\gamma f(s)} = \int_4\! d\mu^2 \frac{\rho(\mu^2)}{s - \mu^2},
$$

where $f(s) = \sqrt{\frac{s - 4m^2}{m^2 - 4m^2}}$, and $s = q^2$ with $q$ the four-momentum of the exchanged meson, i.e., the four-momentum transfer. This propagator can be written in the Källén-Lehmann representation,
where $\rho(\mu^2)$ is the spectral function which satisfies the normalization condition

$$\int_{4m^2_\pi}^{\infty} d\mu^2 \rho(\mu^2) = 1. \quad (G.3)$$

The propagator is an analytic function in the complex $s$-plane with a $2\pi$ right cut along the $(4m^2_\pi, \infty)$ line and a left cut running from $(-\infty, 0)$. This allows us to write the propagator as a dispersion relation. If we look at the singularities confined along the real axis from $q^2 = 4m^2_\pi$ to $q^2 = \infty$ we have (see Fig. G.1)

$$\int_C \frac{D(s')}{s' - t} ds' = 2\pi i \left[ D(s) + \sum_n \frac{\gamma_n}{s_n - s} \right] = \int_{4m^2_\pi}^{\infty} \frac{ds}{s' - s} \left[ D(s' + i0^+) - D(s' - i0^+) \right]. \quad (G.4)$$

Discarding bound state pole contributions we get,

$$D(s) = \frac{1}{2\pi i} \int_{4m^2_\pi}^{\infty} ds' \frac{\text{Disc} D(s')}{s' - s} = \frac{1}{\pi} \int_{4m^2_\pi}^{\infty} ds \frac{\text{Im} D(s' + i0^+)}{s' - s}, \quad (G.5)$$

where $\text{Disc} D(s) = 2\text{Im} D(s + i0^+)$. From the Källén-Lehmann representation we obtain,

$$\rho(\mu^2) = \frac{1}{\pi} \text{Im} D(\mu^2) = \frac{1}{2\pi i} \text{Disc} D(\mu^2). \quad (G.6)$$

We then have the following spectral function,

$$\rho(\mu^2) = \frac{1}{\pi} \frac{m^2 \gamma f(\mu^2)}{\left(\mu^2 - m^2\right)^2 + m^2 \gamma^2 f^2(\mu^2)}. \quad (G.7)$$

The propagator $D(s)$ given by Eq. (G.2) is defined in the first Riemann sheet, the second Riemann sheet is determined from the usual continuity equation $D^{II}(s + i0^+) = D^I(s - i0^+)$. This can be equally defined for the S-matrix. We have,

$$S^{II}(s + i0^+) = S^I(s - i0^+) = \frac{1}{\left(S^I(s - i0^+)\right)^*}. \quad (G.8)$$
where in the second step we have used $SS^\dagger = 1$. Therefore the phase shift for the partial wave $J$ in the isospin channel $I$ reads,

$$e^{2i\delta_{IJ}(s)} = \frac{S^{II}(s + i0^+)}{S^I(s - i0^+)} = \frac{D^{II}(s + i0^+)}{D^I(s - i0^+)},$$

where in the last step we have assumed that the scattering amplitude, i.e., the S-matrix is proportional to the propagator. Thus,

$$e^{2i\delta_{IJ}(s)} = \frac{s - m^2 + im\gamma f(s)}{s - m^2 - im\gamma f(s)},$$

from which we can extract the phase shift,

$$\tan \delta_{IJ}(s) = \frac{m\gamma}{s - m^2} \left( \frac{s - 4m^2}{m^2 - 4m^2} \right),$$

We see that,

$$\delta_{IJ}(s = 4m^2) = 0 \quad (G.9)$$
$$\delta_{IJ}(s = m^2) = \frac{\pi}{2}, \quad (G.10)$$

as corresponds to a Breit-Wigner resonance. Using the definition for the scattering amplitude we can write,

$$t_{IJ}(s) = \frac{1}{2i\rho_{\pi\pi}(s)} \left( e^{2i\delta_{IJ}(s)} - 1 \right), \quad (G.11)$$

with $\rho_{\pi\pi}(s) = \sqrt{1 - 4m^2_\pi/s} = 2p/\sqrt{s}$ where we have taken into account that in the c.m. frame $s = 4(p^2 + m^2_\pi)$. As a function of $s$ and $p$ the scattering amplitude is,

$$t_{IJ}(s) = \sqrt{\frac{s - 4m^2}{2p}} e^{i\delta_{IJ}(s)} \sin \delta_{IJ}(s). \quad (G.12)$$

Let us concentrate now in the $\sigma$-meson, which manifests itself as a pole in the scattering amplitude in the second Riemann sheet,

$$t_{00}^{II}(s) \rightarrow \frac{r_{\sigma}}{s - s_\sigma} \quad (G.13)$$

with $s_\sigma$ the pole position $\sqrt{s_\sigma} = M_\sigma - i\Gamma_\sigma/2$ and $r_{\sigma}$ the residue. If one assumes $D_\sigma(s) = t_{00}(s)/r_{\sigma}$ in the whole complex plane, in particular,

$$D_\sigma^{II}(s) = \frac{t_{00}^{II}(s)}{r_{\sigma}} = \frac{1}{s - (M_\sigma - i\Gamma_\sigma/2)^2} \quad (G.14)$$

According to Ref. [227] the finite width of the scalar meson can be modeled by the propagator

$$D_\Sigma(s) = \frac{1}{s - m^2_\sigma - im_\sigma\gamma \sqrt{s - 4m^2_\sigma}}, \quad (G.15)$$
for $t \geq 4m^2_{\sigma}$. Below the elastic scattering threshold we use the standard definition $\sqrt{t-4m^2_{\sigma}} = -i\sqrt{|t-4m^2_{\sigma}|}e^{i\theta}$ where $0 \leq \theta = \text{Arg}(t-4m^2_{\sigma}) < 2\pi$. This defines the propagator in the first Riemann sheet, the second Riemann sheet is determined as we have seen from the usual continuity equation $D^{\text{II}}_t(s+i0^+) = D^{\text{I}}_t(s-i0^+)$. The pole position are solutions of the equation

$$s - m^2_{\sigma} - im_{\sigma}\gamma_{\sigma}\sqrt{\frac{s-4m^2_{\sigma}}{m^2_{\sigma}-4m^2_{\sigma}}} = 0,$$

with $\text{Im}(s) < 0$, that is,

$$s_{\sigma} = \left(M_{\sigma} - i\Gamma_{\sigma}/2\right)^2 = m^2_{\sigma} - \frac{\gamma^2_{\sigma}m^2_{\sigma} + 4m^2_{\sigma}}{2m^2_{\sigma} - 8m^2_{\sigma}} - i\frac{\gamma_{\sigma}m_{\sigma}\sqrt{4(m^2_{\sigma} - 4m^2_{\sigma})^2 - \gamma^2_{\sigma}m^2_{\sigma}}}{2m^2_{\sigma} - 8m^2_{\sigma}},$$

(G.17)

In the small width limit the position of the pole and width are

$$M_{\sigma} = m_{\sigma} - \frac{\gamma_{\sigma}^2m^2_{\sigma} + 4m^2_{\sigma}}{8m_{\sigma}(m^2_{\sigma} - 4m^2_{\sigma})} + \mathcal{O}(\gamma_{\sigma}^4),$$

(G.18)

$$\Gamma_{\sigma} = \gamma_{\sigma} + \mathcal{O}(\gamma_{\sigma}^3).$$

(G.19)

which relates the pole position with the Breit-Wigner parameters. For the $\sigma$-meson,

$$\tan \delta_{00}(s) = \frac{m_{\sigma}\gamma_{\sigma}}{s - m^2_{\sigma}} \left(\frac{s - 4m^2_{\sigma}}{m^2_{\sigma} - 4m^2_{\sigma}}\right)^2,$$

(G.20)

The parameterization is such that the standard Breit-Wigner definition of the resonance is fulfilled for the bare parameters,

$$\delta_{00}(m^2_{\sigma}) = \frac{\pi}{2}, \quad \gamma_{\sigma} = \frac{1}{m_{\sigma}\delta'_{00}(m^2_{\sigma})},$$

(G.21)

Of course, in the limit of narrow resonances both definitions are indistinguishable and we have $M_{\sigma} \rightarrow m_{\sigma}$ and $\Gamma_{\sigma} \rightarrow \gamma_{\sigma}$. If we use the pole position in the second Riemann sheet of the S-matrix or equivalently the zero in the first Riemann sheet from [43] yielding the value $M_{\sigma} = i\Gamma_{\sigma}/2 = 441^{+16}_{-8} - i272^{+0}_{-12}\text{MeV}$ we get

$$m_{\sigma} = 567(10)\text{MeV} \quad \gamma_{\sigma}/2 = 276(10)\text{MeV}.$$

(G.22)

From the small width expansion, Eq. (G.19), one gets $m_{\sigma} = 554(10)\text{MeV}$, despite the apparent large width. From Ref. [348] one has the magnitude of the residue $|R_{\sigma}| = 0.218^{+0.021}_{-0.012}\text{GeV}^2$ whereas we get $|R_{\sigma}| = 0.430\text{GeV}^2$. Note the $120(20)\text{MeV}$ shift between the Breit-Wigner and the pole position. With the above parameters the scattering length is $a_{00}m_{\sigma} = 0.36$ which is clearly off the value $a_{00}m_{\sigma} = 0.220(2)$ deduced from ChPT.

Authors of Ref. [227] claim that, in principle, one could model the wide meson mass using a two pole approximation,

$$\left[s - m^2_{\sigma} - im_{\sigma}\gamma_{\sigma}\sqrt{\frac{s-4m^2_{\sigma}}{m^2_{\sigma}-4m^2_{\sigma}}}\right]^{-1} \approx \frac{A_1}{s - m^2_1} + \frac{A_2}{s - m^2_2},$$

(G.23)
which in coordinate space is equivalent to the identification,

\[ D_S(r) = \int \frac{d^3q}{(2\pi)^3} e^{iq \cdot r} D_S(-q^2) = -\frac{1}{4\pi r} \int_{4m_r^2}^{\infty} d\mu^2 \rho(\mu^2) e^{-\mu r} \]

\[ \simeq c_1 \frac{e^{-m_1 r}}{r} + c_2 \frac{e^{-m_2 r}}{r} \]  

(G.24)

With the the spectral function given by

\[ \rho(\mu^2) = \frac{\gamma \mu^2}{\pi} \frac{m_\sigma \sqrt{m_\sigma^2 - 4m_r^2}}{\mu^2 - m_\sigma^2 + \frac{\gamma^2 m_r^2 (\mu^2 - 4m_r^2)}{m_\sigma^2 - 4m_r^2}} \]

(G.25)

we roughly obtain by fitting \( c_1, c_2, m_1, m_2 \) to match Eq. (G.24),

\[ c_1 = -6.3, \quad m_1 \simeq 1500 \text{MeV} \]

(G.26)

\[ c_2 = -7.5, \quad m_2 \simeq 464 \text{MeV} \]

(G.27)

The value of \( m_1 \) is too high as to be see in NN, however, the result for \( m_2 \) may justify the value of \( m_\sigma \simeq 500 \text{MeV} \) that usually is taken for the NN interaction \(^2\).

### G.2 Separation of the pole and background contributions

We are interested in separating the pole contribution from the two pion exchange background, and we are going to do it here for our toy model explicitly. We proceed by considering the analytic function \( \rho(z) e^{-\sqrt{z} r} \) for \( z > 0 \) and the complex plane without the cuts \((-\infty, 0) \) and \((4m_r^2, \infty)\). Actually we have two poles at the complex conjugate points \( z = (M_\sigma \pm i\Gamma_\sigma/2)^2 \) but the \( \sigma \)-pole \( z_\sigma = (M_\sigma - i\Gamma_\sigma/2)^2 \) is in the fourth quadrant of the complex plane. Before carrying out the contour integration it is worthy to remind some analytic properties. Note that from Eq. (G.11),

\[ t^{-1}(s) \equiv \rho(s) \cot \delta(s) - i\rho(s), \quad s \geq 4m_r^2 \]

(G.28)

with \( \rho(s) = \sqrt{1 - 4m^2/s} = p/\sqrt{s} \) and also that \( t^{-1}(s < 0) \) is real. Obviously we also have \( \cot \delta(s) = \frac{\text{Re} t^{-1}(s)}{\rho(s)} \). We can define the function \( F(z) \equiv t^{-1}(z) \) with,

\[ t^{-1}(s) = F(z) \bigg|_{z = s + i0^+} = F(s + i0^+) = F^*(s - i0^+) \]

(G.29)

(G.30)

where in the last step we have used the Schwartz reflection principle. In the branch cut we have the continuity conditions,

\[ F_{\Pi}(s \mp i0^+) = F_I(s \mp i0^+), \quad s \geq 4m_\sigma^2 \]

(G.31)

This is reflected in the scattering amplitude in the following fashion,

\[ t^{-1}_I(s + i0^+) = \text{Re} t^{-1}_I(s - i0^+) \]

(G.32)

\[ t^{-1}_{\Pi}(s + i0^+) = \text{Re} t^{-1}_{\Pi}(s - i0^+) \]

(G.33)

\(^2\)Actually, based on the Bethe-Salpeter approach to lowest order \([220]\) its mass is estimated to be \( m_\sigma \sim 500 \text{MeV} \) \([188]\).
Figure G.2: Integration contour used for separating the pole contribution from the $2\pi$ background. The Cauchy integral is calculated for the closed contour for which the dashed line gives a null contribution.

Now,

$$t_1^{-1}(s + i0^+) = t_1^{-1}(s - i0^+) = \left[t_1^{-1}(s - i0^+)\right]^* = \text{Re} \ t_1^{-1}(s) + i\rho(s - i0^+) \quad \text{(G.34)}$$

so we obtain $\rho(s + i0^+) = -\rho(s - i0^+)$. 

The resonance is a pole in the second Riemann sheet. To separate it from the $2\pi$ background we proceed by integrating the function $\rho(z)e^{-\sqrt{z}r}$ in the complex $z$-plane using the integration contour plotted in Fig. G.2.

$$2\pi i \ \text{Res} \bigg[ \rho(z)e^{-\sqrt{z}r}, \ z_\sigma = (M_\sigma - i\Gamma_\sigma/2)^2 \bigg]$$

$$= \int_{4m^2_\pi}^{\infty} d\mu^2 \rho_\pi(\mu^2 - i0^+)e^{-\mu r}$$

$$+ \int_{-\infty}^{-4m^2_\pi} \ i\rho_\pi(4m^2_\pi + iy)e^{-\sqrt{4m^2_\pi + iy}r} \quad \text{(G.35)}$$

which leads, after using $\rho(s + i0^+) = -\rho(s - i0^+)$ in the first integral and the change $y \to -y$ in the second, to the decomposition $D_\pi(r) = D_\sigma(r) + D_{2\pi}(r)$ with,

$$D_\sigma(r) = -\frac{i}{2r} \ \text{Res} \bigg[ \rho(z)e^{-(M_\sigma - i\Gamma_\sigma/2)r} \bigg] \quad \text{(G.36)}$$

$$D_{2\pi}(r) = -\frac{i}{4\pi r} \ \int_{-\infty}^{\infty} dy \rho_\pi(4m^2_\pi + iy)e^{-r\sqrt{4m^2_\pi + iy}} \quad \text{(G.37)}$$

In principle $D_\sigma(r)$ and $D_{2\pi}(r)$ are complex functions but given the fact that $D_\pi(r)$ is real, the imaginary parts must cancel, i.e., we have $\text{Im}D_\sigma(r) = -\text{Im}D_{2\pi}(r)$. In Fig. G.3 we checks the pole-background decomposition and shows that the total contribution, although describable by a Yukawa shape does not correspond to the pole piece. In addition the cancellation of imaginary parts, $\text{Im}D_\sigma(r) = -\text{Im}D_{2\pi}(r)$, is explicitly verified. Using the inverse relations of Eq. (G.19), in the narrow width approximation the pole contribution becomes

$$\text{Re}D_\sigma(r) = \frac{e^{-M_\sigma r}}{4\pi r} \times \left[ 1 + \frac{\Gamma_\sigma^2}{8} \left( \frac{2M_\sigma}{M^2_\sigma - 4m^2_\pi} - r \right) + \ldots \right] \quad \text{(G.38)}$$
Appendix G. *Wide resonance exchange between nucleons*

On the other hand the $2\pi$ contribution at long distances becomes

$$D_{2\pi}(r) = e^{-2m_\pi r} \rho \frac{\sigma m_\pi^2}{m_\pi^2 - 4m_\pi^2} \frac{\gamma_\sigma m_\pi^2}{m_\pi^2 - 4m_\pi^2} + \ldots \tag{G.39}$$

Finally, the $\rho$ meson propagator and the associated $(I,J) = (1,1)$ phase shift can be dealt with *mutatis mutandis* by using

$$[D_V(s)]^{-1} = s - m_\rho^2 - im_\rho \gamma_\rho \left[ s - 4m_\pi^2 \frac{m_\rho^2 - 4m_\pi^2}{m_\rho^2 - 4m_\pi^2} \right]^{\frac{3}{2}} \tag{G.40}$$

where the $p-$wave character of the $\rho \to 2\pi$ decay can be recognized in the phase space factor.

### G.3 Meson resonances at large $N_c$

Finally it is worthy to review some important results of Ref. [232]. We start by noting that analytical continuation through the unitarity right cut is implemented by,

$$t_{II}^{-1}(z) - t_1^{-1}(z) = 2i\rho(z), \quad z \in \mathbb{C}, \tag{G.41}$$

Let be $s_R = m_R^2 - im_R \Gamma_R$, the position of the pole associated to the resonance $R$. By definition $s_R$, it is solution of the equation $t_{II}^{-1}(s_R) = 0$, which can be expressed from Eq. (G.34) as,

$$\text{Re } t_1^{-1}(s_R) = -i\rho(s_R^*), \tag{G.42}$$

where we used that $\rho(s_R) = -\rho(s_R^*)$. The last equation is by definition the one which satisfy the resonance pole position.
If we consider an expansion in \( N_c \) for the pole position and \( \text{Re} \ t^{-1} \) of the form\(^3\),

\[
\begin{align*}
    s_R &= x_R + \frac{y_R}{N_c} + \mathcal{O}(N_c^{-2}), \\
    \text{Re} \ t^{-1} &= (\text{Re} \ t^{-1})_{(1)} + (\text{Re} \ t^{-1})_{(0)} + \mathcal{O}(N_c^{-1}),
\end{align*}
\]

where we have used that \( \text{Re} \ t^{-1} \) scales as \( \mathcal{O}(N_c) \) and an obvious notation in the \( N_c \) expansion of \( \text{Re} \ t^{-1} \), where \( (\text{Re} \ t^{-1})_{(a)} \) scales as \( \mathcal{O}(N_c^a) \). The large \( N_c \) expansion of Eq. (G.42) reads

\[
(\text{Re} \ t^{-1})_{(1)}(x_R) + \frac{y_R}{N_c} \left[ (\text{Re} \ t^{-1})_{(1)}' \right] (x_R) + (\text{Re} \ t^{-1})_{(0)}(x_R) + \mathcal{O}(N_c^{-1}) \] \( \mathcal{O}(N_c^0) \) (G.45)

\[
= -i\rho(x_R) + \mathcal{O}(N_c^{-1}), \quad \mathcal{O}(N_c^0) \) (G.46)

since \( \rho(s) \) scale as \( \mathcal{O}(N_c^0) \). This allows us to write the following relations order-by-order,

\[
(\text{Re} \ t^{-1})_{(1)}(x_R) = 0. \] \( \mathcal{O}(N_c^{-1}) \) (G.47)

at Leading Order (LO), and

\[
- \frac{\text{Im} \ y_R}{N_c} = \rho(x_R) \left[ (\text{Re} \ t^{-1})_{(1)}' \right] (x_R) \] \( \mathcal{O}(N_c^0) \) (G.48)

\[
\text{Re} \frac{y_R}{N_c} \left[ (\text{Re} \ t^{-1})_{(1)}' \right] (x_R) = - (\text{Re} \ t^{-1})_{(0)}(x_R). \] \( \mathcal{O}(N_c^0) \) (G.49)

at Next to Leading Order (NLO). Eq. G.47 forces \( x_R \) to be real and guarantees that \( m_R \) scales as \( \mathcal{O}(N_c^0) \) in the \( N_C \gg 3 \) limit. Eq. G.48 ensures that the resonance width, \( \Gamma_R \), scales as \( \mathcal{O}(N_c^{-1}) \), for very large values of \( N_c \).

Now, we can rewrite Eq. (G.42), with accuracy \( \mathcal{O}(N_c^{-2}) \), as

\[
\begin{align*}
    \text{Re} \ t^{-1}(s_R) &= \text{Re} \ t^{-1}(m_R^2) - i m_R \Gamma_R \left[ \text{Re} \ t^{-1} \right]' \left( m_R^2 \right) \\
    &= \frac{m_R^2 \Gamma_R^2}{2} \left[ \text{Re} \ t^{-1} \right]'' \left( m_R^2 \right) + \mathcal{O}(N_c^{-2}) \] \( \mathcal{O}(N_c^0) \) (G.50)

\[
= -i\rho(m_R^2) + m_R \Gamma_R \rho' (m_R^2) + \mathcal{O}(N_c^{-2}) = -i \rho(s_R^*). \] \( \mathcal{O}(N_c^0) \)

Thus, we find equating real and imaginary parts,

\[
\begin{align*}
    \text{Re} \ t^{-1}(m_R^2) &= m_R \Gamma_R \left\{ \rho' + \frac{m_R \Gamma_R}{2} \left[ \text{Re} \ t^{-1} \right]' \right\} \bigg|_{s=m_R^2} + \mathcal{O}(N_c^{-3}) \] \( \mathcal{O}(N_c^{-1}) \) (G.51)

\[
\frac{m_R \Gamma_R}{\mathcal{O}(N_c^{-1})} \bigg|_{s=m_R^2} + \mathcal{O}(N_c^{-3}) \) \( \mathcal{O}(N_c^{-1}) \) (G.52)

\(^3\text{For simplicity we will drop the sub-index I referring to the first Riemann sheet.}\)
Appendix G. Wide resonance exchange between nucleons

We see that at the resonance pole mass $\text{Re } t^{-1}$ scales as $O(N_c^{-1})$ instead of $O(N_c)$. Moreover, using $\cot x = -\tan(x - \frac{\pi}{2})$ we can write for the phase shift,

$$\tan \left( \delta(s) - \frac{\pi}{2} \right) = -\frac{\text{Re } t^{-1}(s)}{\rho(s)} \quad (G.53)$$

Now, using that $\tan x = x + O(x^3)$, we have,

$$\tan \left( \delta(s_{BW}) - \frac{\pi}{2} \right) = -\frac{\text{Re } t^{-1}(s_{BW})}{\rho(s_{BW})} \quad (G.54)$$

where we have used Eqs. (G.51) and (G.52) and

$$\delta'(m_R^2) = -\left[ \frac{\text{Re } t^{-1}}{\rho} \right]' \frac{1}{1 + \left( \frac{\text{Re } t^{-1}}{\rho} \right)^2} \quad (G.55)$$

The important result here is that we can find a value of $s = s_{BW}$ such that,

$$\delta(s_{BW}) = \frac{\pi}{2} + O(N_c^{-3}) \quad (G.56)$$

with

$$s_{BW} = m_R^2 \frac{\delta(m_R^2) - \pi/2}{\delta'(m_R^2)} = m_R^2 - \left[ \frac{\rho^2 \left[ \text{Re } t^{-1} \right]'}{2 \left( \left[ \text{Re } t^{-1} \right]' \right)^3} \right]_{s=m_R^2} + O(N_c^{-4}) \quad (G.57)$$

which means that the existence of a pole in the SRS guarantees the existence of a value of $s_{BW}$, which can naturally be identified with the BW position, where the phase shift is $\pi/2$, up to $O(N_c^{-3})$ corrections.

The Breit-Wigner determination of the resonance mass differs from the pole position by $O(N_c^{-2})$ terms provided the pole position is $O(N_c^0)$. In other words, assuming similar series expansions for both, the complex pole position $s_R$,

$$s_R = s_R^{(0)} + \lambda s_R^{(1)} + \lambda^2 s_R^{(2)} + \cdots \quad (G.58)$$

and the BW position of the resonance ($\delta(s_{BW}) = \pi/2$),

$$s_{BW} = s_{BW}^{(0)} + \lambda s_{BW}^{(1)} + \lambda^2 s_{BW}^{(2)} + \cdots \quad (G.59)$$

with $\lambda = 3/N_c$, we have

$$s_{BW} - \text{Re}(s_R) = O(N_c^{-2}) \quad (G.60)$$
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