Relatively weakly open subsets of the unit ball in functions spaces

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Abstract

For an infinite Hausdorff compact set $K$ and for any Banach space $X$ we show that every nonempty weak open subset relative to the unit ball of the space of $X$-valued functions that are continuous when $X$ is equipped with the weak (respectively norm, weak-$*$) topology has diameter 2. As consequence, we improve known results about nonexistence of denting points in these spaces. Also we characterize when every nonempty weak open subset relative to the unit ball has diameter 2, for the spaces of Bochner integrable and essentially bounded measurable $X$-valued functions.

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1. Introduction

The nonexistence of denting points in the unit ball of some functions spaces has been the subject of several recent researching [8,14]. A point $x_0$ in the sphere of a Banach space $X$, $S_X$, is a denting point of the unit ball in $X$, $B_X$, if there are slices, that is, subsets in the way

$$S(x^*, \alpha) = \{x \in B_X: x^*(x) > \|x^*\| - \alpha, \ x^* \in X^*, \ \alpha \in \mathbb{R}\},$$

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containing \(x_0\), with diameter arbitrarily small. From [11], \(x_0\) is a denting point of the unit ball of \(X\) if and only if \(x_0\) is an extreme point in \(B_X\) and \(x_0\) is a point of weak-norm continuity, that is, a point of continuity for the identity map from \((B_X, w)\) onto \((B_X, n)\), where \(w\) and \(n\) denote the weak and the norm topology, respectively. In particular, the existence of denting points in the unit ball of a Banach space \(X\) implies the existence of nonempty weak open subsets relative to the unit ball in \(X\) with diameter arbitrarily small.

Then the extremely opposite property to the existence of denting points in the unit ball of a Banach space is that every nonempty weak open subset relative to the unit ball has diameter 2. This is the case, for example, for \(C^*\)-algebras [2], and uniform algebras [13].

The aim of this note is to show the existence of functions and operator spaces where every nonempty weak open subset relative to the unit ball has diameter 2, by improving the results about nonexistence of denting points in [8] and [14].

Given a Banach space \(X\) and a Hausdorff compact topological space \(K\), we note \(C(K, X)\) the Banach space of all continuous functions on \(K\) into \(X\) with the norm topology, \(WC(K, X)\) the Banach space of all continuous functions on \(K\) into \(X\) with the weak topology and \(W^*C(K, X)\) the Banach space of all continuous functions on \(K\) into \(X\) with the weak-* topology, when \(X\) is a dual space. All these spaces are equipped with the supremum norm.

The first goal of this note is to prove that every nonempty weak open subset relative to the unit ball of \(C(K, X)\), \(WC(K, X)\) and \(W^*C(K, X^*)\) has diameter 2 whenever \(K\) be infinite (Corollary 2.4). This result follows from the study of the space \(C(K, (X, \tau))\) of continuous functions on a Hausdorff compact topological space \(K\) into a Banach space \(X\) endowed with a Hausdorff locally convex topology \(\tau\) (Theorem 2.3). We will obtain some consequences from this result, by using well-known identifications for this space. For example, every nonempty weak open subset relative to the unit ball in \(L(X, C(K))\) (the linear and continuous operators space) has diameter 2, whenever \(K\) be infinite (Corollary 2.6). Turning to injective tensor products, we show that for any infinite-dimensional space \(Y\) such that \(Y^*\) is isometric to an \(L_1(\mu)\)-space (the so-called \(L_1\) predual spaces [9]) and for any Banach space \(X\), every nonempty weak open subset relative to the unit ball of \(X \otimes \epsilon Y\) has diameter 2 (Corollary 2.9). As a consequence, we obtain that the space of bilinear forms on \(X \times Y\) is extremely rough, whenever \(Y\) be an infinite-dimensional \(L_1\) predual.

Also, we characterize when every nonempty weak open subset relative to the unit ball has diameter 2 for the operators space \(L(L_1(\mu), X)\), Theorem 2.11, and for the spaces \(L_1(\mu, X)\), of Bochner integrable \(X\)-valued functions and \(L_\infty(\mu, X)\), of essentially bounded and measurable \(X\)-valued functions, where \(X\) is a Banach space and \(\mu\) is a positive measure, Theorem 2.13.

2. The main result

We begin with two elementary lemmas which will be essential for the main results.

Lemma 2.1. Let \(X, Y\) Banach spaces and \(Z = X \oplus_1 Y\).
(i) Assume that $X$ has a nonempty weak open subset relative to $B_X$ with diameter less than 2. Then $Z$ has it too.

(ii) Assume that every nonempty weak open subset relative to $B_X$ or $B_Y$ has diameter 2. Then every nonempty weak open subset relative to $B_Z$ has also diameter 2.

**Proof.** (i) Let $U$ be a nonempty weak open subset relative to $B_X$ with diameter less than 2. Then $U$ must contain a nonempty finite intersection of slices $W = \{ x \in B_X : x_i^*(x) > 1 - \alpha, 1 \leq i \leq p \}$, where $0 < \alpha < 1$, $p$ is a natural number, and $x_i^* \in S_{X^*}$ for all $1 \leq i \leq p$. Consider now $V = \{ (x, y) \in B_Z : x_i^*(x) > 1 - \alpha, 1 \leq i \leq p \}$. Hence $V$ is a nonempty weak open subset relative to $B_Z$. Pick $(x, y), (x', y') \in V$. Taking into account that the diameter of $U$ is less than 2 and $\| y - y' \| < 2\alpha$, we deduce that

$$\| (x, y) - (x', y') \| = \| x - x' \| + \| y - y' \| \leq \text{diam}(U) + 2\alpha.$$

Finally, it is enough chose $\alpha$ small to conclude that the diameter of $V$ is less than 2.

(ii) Let $U$ be a nonempty weak open subset relative to $B_Z$. Then, by [7, Lemma II.1], there is a convex combination of slices $\sum_{i=1}^{n} \lambda_i S_i$ contained into $U$, where $n$ is a natural number, $\left\{ \lambda_i \right\}_{1 \leq i \leq n}$ are positive real numbers with $\sum_{i=1}^{n} \lambda_i = 1$, and for every $1 \leq i \leq n$, we put

$$S_i = \{ (x, y) \in B_Z : x_i^*(x) + y_i^*(y) > 1 - \alpha_i \},$$

where $(x_i^*, y_i^*) \in S_{Z^*}$ and $0 < \alpha_i < 1$.

Now, we split the set $\{1, \ldots, n\}$ in two disjoint subsets $I, J$, such that $\| x_i^* \| = 1$ for every $i \in I$ and $\| y_j^* \| = 1$ for every $j \in J$.

For every $i \in I$ we put $T_i = \{ x \in B_X : x_i^*(x) > 1 - \alpha_i \}$ and for every $j \in J$, we put $R_j = \{ y \in B_Y : y_j^*(y) > 1 - \alpha_j \}$. Then $\{T_i\}$ is a family of slices in $B_X$ and $\{R_j\}$ is a family of slices in $B_Y$, satisfying that $(T_i, 0) \subset S_i$ and $(0, R_j) \subset S_j$ for every $i \in I$ and $j \in J$, respectively. Thus, it follows that $\left( \sum_{i \in I} \frac{\lambda_i}{\lambda} T_i, 0 \right) \subset \sum_{i \in I} \frac{\lambda_i}{\lambda} S_i$ and $\left( \sum_{j \in J} \frac{\lambda_j}{\lambda} R_j, 0 \right) \subset \sum_{j \in J} \frac{\lambda_j}{\lambda} S_j$, where $\lambda_I = \sum_{i \in I} \lambda_i$ and $\lambda_J = \sum_{j \in J} \lambda_j$.

The sets $\sum_{i \in I} \frac{\lambda_i}{\lambda} T_i$ and $\sum_{j \in J} \frac{\lambda_j}{\lambda} R_j$ have diameter 2, since they are convex combination of slices in $B_X$ and $B_Y$, respectively, and, by hypothesis, every nonempty weak open subset relative to $B_X$, respectively $B_Y$, has diameter 2. Hence, given $\varepsilon > 0$, there are $x, x' \in \sum_{i \in I} \frac{\lambda_i}{\lambda} T_i$ and $y, y' \in \sum_{j \in J} \frac{\lambda_j}{\lambda} R_j$ verifying $\| x - x' \|, \| y - y' \| > 2 - \varepsilon$.

Note that $(\lambda_I x, \lambda_J y), (\lambda_I x', \lambda_J y') \in \sum_{i=1}^{n} \lambda_i S_i$ and so

$$\text{diam}(U) \geq \text{diam} \left( \sum_{i=1}^{n} \lambda_i S_i \right) \geq \lambda_I \| x - x' \| + \lambda_J \| y - y' \| > 2 - \varepsilon. \quad \Box$$

**Lemma 2.2.** Let $X$ be a Banach space satisfying that every nonempty weak open subset relative to $B_X$ has diameter 2. Then every nonempty weak open subset relative to $B_{X \oplus \infty Y}$ has diameter 2, where $Y$ is an arbitrary Banach space.

**Proof.** We call $Z = X \oplus Y$ and let $P : Z \to X$ be the projection from $Z$ onto $X$, which is weak open. It is clear that $B_Z = B_X \times B_Y$ and $\| P \| = 1$. Then if $W$ is a weak open in $Z$
such that $W \cap B_Z \neq \emptyset$, one has that $V = P(W \cap B_Z)$ is a nonempty weak open relative to $B_X$ and, so diam$(V) = 2$. Hence diam$(W \cap B_Z) = 2$. \hfill \Box

In order to show our main result we begin with some notation. For a Banach space $X$, a Hausdorff compact topological space $K$, and a Hausdorff locally convex topology $\tau$ on $X$, $C(K,(X,\tau))$ stands for the vector space of all continuous functions from $K$ into $(X,\tau)$. The main problem in order to study the diameter of weak open subsets relative to the unit ball of $C(K,(X,\tau))$ is to have the possibility of consider the sup norm. In general, this is not possible. A natural condition on $\tau$ for this is assume that $\tau$ is compatible for the dual pair $(X,X^*)$, since Mackey’s Theorem states that the $\tau$-bounded subsets of $X$ are also bounded for the norm topology on $X$. Recall that $\tau$ is compatible for the dual pair $(X,X^*)$ if $(X,\tau)^* = X^*$ and that every compatible topology on $X$ for the dual pair $(X,X^*)$ is coarser than the norm topology on $X$, by Mackey–Arens Theorem (for duality theory results we reference to [12,15]). So, we have that $C(K,(X,\tau))$ endowed with the sup norm is a Banach space whenever $\tau$ is a compatible topology for the dual pair $(X,X^*)$ and moreover every $\tau$-neighborhood on $X$ is also a neighborhood on $X$ for the norm topology. Now, with this facts in mind, we present our main result.

**Theorem 2.3.** Let $X$ be a Banach space and let $K$ be an infinite Hausdorff compact topological space. Let $Z = C(K,(X,\tau))$ the space of continuous functions from $K$ to $(X,\tau)$, where $\tau$ is a Hausdorff locally convex topology on $X$. Assume that $\tau$ is compatible for the dual pair $(X,X^*)$. Then every nonempty weakly open set relative to the unit ball of $Z$ has diameter $2$. Moreover, if $X$ is a dual space with predual $X_*$, then every nonempty weakly open set relative to the unit ball of $Z$ has diameter $2$, whenever $\tau$ be compatible for the dual pair $(X_*,X)$.

**Proof.** Pick $W$ a weak neighborhood of $Z$ and separate the proof in three steps.

1. Assume that $K$ have many infinite isolated points. Take $\{t_n\}$ a sequence of different isolated points of $K$ and $t_0$ an accumulation point of $\{t_n\}$. Now we have that there is $a \in S_Z \cap W$ such that $a(t_0) \neq 0$, since $S_Z \cap W$ has nonempty interior relative to the sphere $S_Z$, for the norm topology. Chose a $\tau$-neighborhood $U$ of $0$ in $X$ and a $\tau$-neighborhood $V$ of $a(t_0)$ in $X$, such that $U \cap V = \emptyset$.

   By the continuity of $a$, we know that there is $I$ an infinite set of positive integers numbers such that $\{a(t_n): n \in I\} \subset V$, and so $\{a(t_n): n \in I\} \cap U = \emptyset$. Therefore, there is $\delta > 0$ such that $\|a(t_n)\| > \delta \quad \forall n \in I$, since every $\tau$-neighborhood is also neighborhood for the norm topology in $X$. In the sequel, we assume without loss of generality that $\{t_n\}$ is a sequence of different isolated points of $K$ satisfying $\|a(t_n)\| > \delta \quad \forall n$, for suitable $\delta > 0$.

   For each $n$ we consider $x_n \in C(K)$ given by

   \[
   x_n(t) = \frac{1}{\|a(t_n)\|}, \quad x_n(t_{n+1}) = -\frac{1}{\|a(t_{n+1})\|}, \quad x_n(t) = 1, \quad \forall t \in K \setminus \{t_n, t_{n+1}\}.
   \]

   Now, $\{x_n\}$ is a bounded sequence in $C(K)$ satisfying $\lim_n x_n(t) = 1$ for each $t \in K$. Applying [4, Theorem VII.1], we obtain that $\{x_n\}$ converges to $1$ in the weak topology.
in $C(K)$, where $1$ is the constant function in $C(K)$ equal $1$. Then the sequence $\{x_n a\}$ converges to $a$ in the weak topology of $Z$, since $a \in S_Z$. Hence there is $p$ such that $x_p a, x_{p+1} a \in W \cap B_Z$, since $x_n a \in B_Z \ \forall n$.

Finally,
\[
\text{diam}(W \cap B_Z) \geq \|x_{n+1} a - x_n a\| \geq \|x_{n+1}(t_{n+1}) - x_n(t_{n+1})\| \|a(t_{n+1})\| = 2.
\]

(2) Assume that there is $a \in W \cap S_Z$ such that for all $\delta > 0$ there is $t_0 \in K'$ verifying $\|a(t_0)\| > 1 - \delta$. ($K'$ denotes the set of accumulation points of $K$.)

For each $\delta > 0$ we consider $W_0 = \{t \in K: f(a(t)) > 1 - \delta\}$, where $f \in S_{X^*}$ satisfies $f(a(t_0)) > 1 - \delta$. The existence of $f$ is guaranteed because $\tau$ is compatible for the dual pair $(X, X^*)$ (note that in the case $X$ be a dual space with predual $X_*$, the same argument works considering the dual pair $(X_*, X)$).

Now, $W_0$ is an open subset of $K$ containing $t_0$. Hence there is $\{W_n\}$ a sequence of nonempty open subset pairwise disjoint of $W_0$. For each $n$ take $t_n \in W_n$ and $x_n \in C(K)$ such that $x_n(t_n) = -1$ and $x_n(t) = 1$ whenever $t \in K \setminus W_n$. From [4, Theorem VII.1] $x_n$ converges weakly to $1$ in the weak topology of $C(K)$. Then $x_n a$ converges weakly to $a$ in the weak topology of $Z$. Thus there is $n$ such that $x_n a \in B_Z \cap W$ and
\[
\|a - x_n a\| \geq \|a(t_n)\| \geq 2 f(a(t_n)) \geq 2(1 - \delta).
\]

Finally diam$(W \cap B_Z) = 2$, since $\delta$ was arbitrary.

(3) Assume that the set of isolated points of $K$ is finite. Then we can write:
\[
Z = \ell_\infty^n(X) \oplus_\infty C(\tilde{K}, (X, \tau)),
\]
where $\tilde{K}$ is a perfect compact Hausdorff topological space and $n$ is the number of isolated points in $K$. By (2) and Lemma 2.2, the proof is complete. \qed

As the weak and the norm topologies on a Banach space $X$ are compatible for the dual pair $(X, X^*)$ and the weak-* topology on $X^*$ is compatible for the dual pair $(X^*, X)$, we obtain the following consequence.

**Corollary 2.4.** Let $X$ be a Banach space and let $K$ be an infinite Hausdorff compact topological space. Then every nonempty weakly open relative to the unit ball of $C(K, X)$, $WC(K, X)$ or $W^*C(K, X^*)$ has diameter $2$.

We want remark that the above result can be proven, in a similar way, for the Banach space $c_0(L, X)$ of continuous functions on a Hausdorff locally compact topological space $L$ which vanish at infinite with values in a Banach space $X$ equipped with the norm topology, where the space $c_0(L, X)$ is endowed with the sup norm.

**Corollary 2.5.** Let $X$ be a Banach space and let $K$ be a Hausdorff compact topological space. Let $Z$ be one of the Banach spaces $Z_1 = C(K, X)$, $Z_2 = WC(K, X)$ and, if $X$ is a dual space, $Z_3 = W^*C(K, X)$. Then
(i) Every nonempty weakly open relative to the unit ball of \( Z \) has diameter 2 if and only if \( K \) is infinite or \( K \) is finite and every nonempty weakly open relative to the unit ball of \( X \) has diameter 2.

(ii) \( Z \) has denting points if and only if \( K \) is finite and \( X \) has denting points.

(iii) \( Z \) has points of weak-norm continuity if and only if \( K \) is finite and \( X \) has points of weak continuity.

**Proof.** (i) By Corollary 2.4, we know that every nonempty weakly open relative to the unit ball of \( Z \) has diameter 2, whenever \( K \) be infinite. Assume that \( K \) is finite. Then \( Z = \ell^n_\infty(X) \), where \( n \) is the cardinality of \( K \). By Lemma 2.2, only it remains to decide the case \( K \) finite and \( X \) has a nonempty weak open subset \( V \) relative to the unit ball with diameter less than 2. Now \( \ell^n_\infty(V) \) is a nonempty weak open subset relative to the unit ball of \( Z \) with diameter less than 2 and the proof is complete.

(ii) and (iii) By Corollary 2.4, we know that \( Z \) has no denting (respectively weak-norm continuity) points whenever \( K \) be infinite. Assume that \( K \) is finite. Then \( Z = \ell^n_\infty(X) \), where \( n \) is the cardinality of \( K \). In this case is straightforward to see that a point \( z \) in \( B_Z \) is a denting (respectively weak-norm continuity) point if and only if every coordinate of \( z \) is a denting (respectively weak-norm continuity) point in \( B_X \).

In order to obtain consequences of the above result, we recall (see [5]) that the spaces \( C(K, X^*) \), \( WC(K, X^*) \) and \( W^*C(K, X^*) \) can be isometrically identified with \( K(X, C(K)) \) (the compact operators space), \( \mathcal{F}(X, C(K)) \) (the weakly compact operators space) and \( L(X, C(K)) \) (the space of all linear and continuous operators), respectively.

**Corollary 2.6.** Let \( X \) be a Banach space and let \( K \) be an Hausdorff compact topological space. Let \( Z \) be one of the Banach spaces \( K(X, C(K)) \), \( \mathcal{F}(X, C(K)) \) and \( L(X, C(K)) \). Then

(i) Every nonempty weakly open relative to the unit ball of \( Z \) has diameter 2 if and only if \( K \) is infinite or \( K \) is finite and every nonempty weakly open relative to the unit ball of \( X \) has diameter 2.

(ii) \( Z \) has denting points if and only if \( K \) is finite and \( X \) has denting points.

(iii) \( Z \) has points of weak-norm continuity if and only if \( K \) is finite and \( X \) has points of weak continuity.

Call \( \beta\mathbb{N} \) the Stone–Cech compactification of \( \mathbb{N} \). Then it is straightforward that \( L(\ell_1) \) and \( L(\ell_\infty) \) can be isometrically identified with \( L(c_0, C(\beta\mathbb{N})) \) and \( L(\ell_\infty, C(\beta\mathbb{N})) \), respectively. So we obtain the following

**Corollary 2.7.** Every nonempty weakly open subset relative to the unit ball of \( L(\ell_1) \) or \( L(\ell_\infty) \) has diameter 2.

The arguments used in the proof of Corollary 2.4 also work for the closed subspaces of \( L(X, C(K)) \) containing \( C(K, X^*) \) such that are \( C(K) \)-modules. This is the case for the spaces \( L(c_0) \) and \( L(L_1(\mu)) \) and as a consequence we obtain the following
Corollary 2.8. Every nonempty weakly open subset relative to the unit ball of \( L(c_0) \) or \( L(L_1(\mu)) \) has diameter 2.

In the following consequence, we apply our main result in deciding about weak open relative to the unit ball of injective tensor products spaces.

Corollary 2.9. Let \( X \) be a Banach space and let \( Y \) be an infinite-dimensional \( L_1 \) predual. Then every nonempty weakly open subset relative to the unit ball of \( X \otimes \varepsilon Y \) has diameter 2.

Proof. From the hypothesis, it is known that \( Y^{**} \) must be isometrically isomorphic to \( C(K) \) for suitable infinite Hausdorff compact topological space. Pick now \( W \) a nonempty open subset of \( X \otimes \varepsilon Y \). Then, by the weak-star lower semicontinuity of the dual norm,

\[
\text{diam}(W \cap B_{X \otimes \varepsilon Y}) = \text{diam}(\overline{W}^{**} \cap B(\overline{X^{**} \otimes \varepsilon Y})).
\]

It is known [6] that \( X \otimes \varepsilon Y^{**} \) is a subspace of \( (X \otimes \varepsilon Y)^{**} \) containing \( X \otimes \varepsilon Y \), from one deduce that \( \text{diam}(W \cap B_{X \otimes \varepsilon Y}) = \text{diam}(\overline{W}^{**} \cap B_{X \otimes \varepsilon Y}^{**}) \). Having into account that \( X \otimes \varepsilon Y^{**} \) is isometrically isomorphic to \( C(K, X) \) and Corollary 2.4 the proof is complete. \( \blacksquare \)

We note that the fact that every nonempty weak open subset relative to the unit ball in a Banach space has diameter 2 implies consequences in the dual and predual, when this exists.

Let \( X \) be a Banach space. We recall that for \( u \) in \( S_X \), one define the roughness of \( X \) at \( u \), \( \eta(X, u) \), by the equality

\[
\eta(X, u) := \lim_{\|h\| \to 0} \frac{\|u + h\| + \|u - h\| - 2}{\|h\|}.
\]

We remark that the absence of roughness of \( X \) at \( u \) (i.e., \( \eta(X, u) = 0 \)) is nothing but the Fréchet differentiability of the norm of \( X \) at \( u \) [3, Lemma I.1.3]. Given \( \epsilon > 0 \), the Banach space \( X \) is said to be \( \epsilon \)-rough if, for every \( u \) in \( S_X \), we have \( \eta(X, u) \geq \epsilon \). We say that \( X \) is rough whenever it is \( \epsilon \)-rough for some \( \epsilon > 0 \), and extremely rough whenever it is 2-rough.

Assume that \( X \) is a Banach space satisfying that every nonempty weak open subset relative to the unit ball in \( X \) has diameter 2. Then, by [3, Proposition I.1.11], the dual of \( X, X^* \) (respectively the predual of \( X, X_\ast \), if this exists) is extremely rough. Then, from Corollary 2.9, we deduce the following

Corollary 2.10. \( BL(X, Y) \), the space of bilinear forms on \( X \times Y \), is extremely rough, whenever \( Y \) be an infinite-dimensional \( L_1 \) predual.

Now, we characterize the existence of nonempty weak open subset relative to the unit ball, with diameter less than 2, in a new class of operator spaces.

Theorem 2.11. Let \( (\Omega, \Sigma, \mu) \) be a positive measure space and let \( X \) be a Banach space. Then the space \( L(L_1(\mu), X) \) has a nonempty weak open subset relative to the unit ball with diameter less than 2 if and only if \( L_1(\mu) \) is finite-dimensional and \( X \) has a nonempty weak open subset relative to the unit ball with diameter less than 2.
Proof. Assume that $L_1(\mu)$ is infinite-dimensional. Consider $\{A_n\}$ a sequence of pairwise disjoint measurable sets with $0 < \mu(A_n) < +\infty$. For each $n$ we define $P_n : L_1(\mu) \rightarrow L_1(\mu)$ given by $P_n(f) = f\chi_{A_n}$ for every $f \in L_1(\mu)$, where $\chi_{A_n}$ is the characteristic function of $A_n$. Then $P_n$ is a L-projection, that is, a linear and continuous projection satisfying $\|f\| = \|P_n(f)\| + \|f - P_n(f)\|$ for every $f \in L_1(\mu)$. Then, following [10], the map $Q_n : L(L_1(\mu), X) \rightarrow L(L_1(\mu), X)$ given by $Q_n(T) = T \circ P_n$ is a M-projection, that is, a linear and continuous projection satisfying $\|T\| = \text{Max}\{\|Q_n(T)\|, \|T - Q_n(T)\|\}$ for every $T \in L(L_1(\mu), X)$. Now, calling $\mu_n = \mu|_{A_n}$, we have that

$$L(L_1(\mu), X) = \bigoplus_{\infty} L(L_1(\mu_n), X) \oplus Y,$$

where $Y$ is a closed subspace of $L(L_1(\mu), X)$.

Put $X_n = L(L_1(\mu_n), X)$ and $Z = \bigoplus_{\infty} X_n$. Then $L(L_1(\mu), X) = Z \oplus Y$. In order to prove that every nonempty weak open subset relative to the unit ball of $L(L_1(\mu), X)$ has diameter 2, it is enough, by Lemma 2.2, to see that every nonempty weak open subset relative to the unit ball of $Z$ has diameter 2. For this, fix $z \in Z$ with $\|z\| = 1$. For each $i \in \mathbb{N}$, we define $w_i(n) = z_i(n) = z(n)$ if $i \neq n$ and $-w_i(n) = z_i(n) = x_0$ if $i = n$, where $x_n$ is a fixed norm one element of $X_n$ for every $n$. It is clear that $\{z_i\}, \{w_i\}$ are sequences of norm one elements of $Z$ weakly converging to $z$ with $\|z_i - w_i\| = 2$ for every $i$ so, every nonempty weak open subset relative to the unit ball of $Z$ has diameter 2.

In the case $L_1(\mu)$ is finite-dimensional then $L(L_1(\mu), X) = \ell_\infty^n(X)$, that is, the $\ell_\infty$-sum of $n$ copies of $X$, and it is enough apply Lemma 2.2. \qed

The same above argument work for closed subspaces of $L(L_1(\mu), X)$ which are closed under composition by operators in $L(L_1(\mu))$. As a consequence we obtain the following

**Corollary 2.12.** Let $(\Omega, \Sigma, \mu)$ be a positive measure space and let $X$ be a Banach space. Then the space $K(L_1(\mu), X)$ of compact operators (respectively $F(L_1(\mu), X)$ of weakly compact operators) has a nonempty weak open subset relative to the unit ball with diameter less than 2 if and only if $L_1(\mu)$ is finite-dimensional and $X$ has it too.

Now, we pass to study the same topic for the spaces $L_1(\mu, X)$, of Bochner integrable $X$-valued functions, and $L_\infty(\mu, X)$, of essentially bounded measurable $X$-valued functions, where $X$ is a Banach space and $\mu$ is a positive measure.

**Theorem 2.13.** Let $(\Omega, \Sigma, \mu)$ a positive finite measure space.

(i) The space $L_\infty(\mu, X)$ has some nonempty weak open subset relative to the unit ball with diameter less than 2 if and only if $L_\infty(\mu)$ is finite-dimensional and $X$ has it too.

(ii) The space $L_1(\mu, X)$ has some nonempty weak open subset relative to the unit ball with diameter less than 2 if and only if $\mu$ contains atoms and $X$ has it too.

**Proof.** We can assume, without loss of generality, that $\mu(\Omega) = 1$. 

(i) Assume that \( \mu \) is atomless. Let \( W \) be a nonempty weak open subset relative to the unit ball of \( L_\infty(\mu, X) \) and pick \( f_0 \in S_{L_\infty(\mu, X)} \cap W \), since \( L_\infty(\mu, X) \) is infinite-dimensional. Fix \( \delta > 0 \) and consider \( W_0 = \{ t \in \Omega : \| f_0(t) \| > 1 - \delta \} \). Then, there is \( \{ W_n \} \) a decreasing sequence of measurable subsets of \( W_0 \), with \( \mu(W_n) > 0 \) and \( \lim_{n} \mu(W_n) = 0 \), since \( W_0 \) cannot be an atom of \( \Omega \) and \( \mu(W_0) > 0 \). Now \( \{ -\chi_{W_n} + \chi_{\Omega \setminus W_n} \} \) is a sequence of elements in the unit ball of \( L_\infty(\mu) \) weakly converging to \( 0 \), the constant function in \( L_\infty(\mu) \). Then \( \{ \phi_n = (-\chi_{W_n} + \chi_{\Omega \setminus W_n})f_0 \} \) is a sequence of elements in the unit ball of \( L_\infty(\mu, X) \) weakly converging to \( f_0 \) satisfying that
\[
\| f_0 - \phi_n \|_\infty = 2\| \chi_{W_n}f_0 \|_\infty > 2(1 - \delta).
\]

Thus \( \text{diam}(W) > 2(1 - \delta) \), since \( \phi_n \in W \) for enough large \( n \). As \( \delta > 0 \) is arbitrary, we obtain that \( W \) has diameter 2.

In the case \( \mu \) has some atom, it is well known that we have, for some set \( I \), the decomposition \( L_\infty(\mu, X) = L_\infty(\nu, X) \oplus_\infty \ell_\infty(I, X) \), where \( \nu \) is a measure atomless. By Lemma 2.2, the only possibility to obtain nonempty weak open subsets relative to the unit ball of \( L_\infty(\mu, X) \) with diameter less than 2 is that \( \mu \) be purely atomic and, in this case, \( L_\infty(\mu, X) \) is not but an \( \ell_\infty \)-sum of copies of \( X \). The same argument used in the proof of Theorem 2.11 shows, that if a \( \ell_\infty \)-sum of copies of a Banach space \( X \) has a nonempty weak open subset relative to the unit ball with diameter less than 2, then the \( \ell_\infty \)-sum must be finite. The Lemma 2.2 concludes the proof.

(ii) Assume that \( \mu \) is atomless. For every \( A \in \Sigma \) with \( \mu(A) > 0 \), we call \( \{ r_n^A \} \) the sequence of Rademacher functions supported on \( A \) (see [1, 11.55, 11.56]). We recall that \( \{ r_n^A / \mu(A) \} \) is a sequence of functions in the sphere of \( L_1(\mu) \) with support contained into \( A \) which converges weakly to zero in \( L_1(\mu) \). From the construction of the sequence \( \{ r_n^A \} \), it is clear that \( \{ \chi_A + r_n^A / \mu(A) \} \) and \( \{ \chi_A - r_n^A / \mu(A) \} \) are sequences of functions in the sphere of \( L_1(\mu) \), where \( \chi_A \) is the characteristic function of \( A \). We put \( \phi_n^A = \chi_A + r_n^A / \mu(A) \) and \( \psi_n^A = \chi_A - r_n^A / \mu(A) \) for every \( n \). Then \( \{ \phi_n^A \} \) and \( \{ \psi_n^A \} \) are sequences in the unit sphere of \( L_1(\mu) \) weakly converging to \( \chi_A / \mu(A) \).

Furthermore, one has, for every \( n \)
\[
\| \phi_n^A - \psi_n^A \| = 2 \| r_n^A / \mu(A) \| = 2.
\]

Then we have proved the following statement: for every \( A \in \Sigma \), \( \mu(A) > 0 \), there are \( \{ \phi_n^A \} \), \( \{ \psi_n^A \} \) sequences in the unit sphere of \( L_1(\mu) \) supported on \( A \), weakly converging to \( \chi_A / \mu(A) \), and verifying \( \| \phi_n^A - \psi_n^A \| = 2 \) for every \( n \).

Pick \( W \) a nonempty weak open subset relative to the unit ball of \( L_1(\mu, X) \). By the infinite-dimensionality of \( L_1(\mu, X) \) and the density of simple functions in \( L_1(\mu, X) \), there is a simple function \( \varphi \in W \cap S_{L_1(\mu, X)} \). We can write \( \varphi \) in the way
\[
\varphi = \sum_{i=1}^{p} \frac{\chi_{A_i}}{\mu(A_i)} x_i,
\]
where \( p \) is a natural number, \( \{ A_i \} \) are mutually disjoint measurable subsets of \( \Omega \) with positive measure, and \( \{ x_i \} \) are vectors in \( X \) satisfying \( \sum_{i=1}^{p} \| x_i \| = 1 \).
For every $1 \leq i \leq p$ let $\{\phi_i^n = \phi_i^A_n\}$ and $\{\psi_i^n = \psi_i^A_n\}$ be sequences in the unit sphere of $L_1(\mu)$ supported on $A_i$, weakly converging to $\frac{x_i^A}{\mu(A_i)}$ and satisfying $\|\phi_i^n - \psi_i^n\| = 2$, for every $n$.

We do $f_n = \sum_{i=1}^p \phi_i^n x_i$ and $g_n = \sum_{i=1}^p \psi_i^n x_i$, for every $n$. Then $\{f_n\}$ and $\{g_n\}$ are sequences in the unit sphere of $L_1(\mu, X)$ weakly converging to $\varphi$. Hence $f_n, g_n \in W$ for enough large $n$ and for this $n$, we deduce that

$$\text{diam}(W) \geq \|f_n - g_n\| = \int_{\Omega} \left\| \sum_{i=1}^p (\phi_i^n - \psi_i^n) x_i \right\| d\mu = \sum_{i=1}^p \|x_i\| \int_{A_i} \|\phi_i^n - \psi_i^n\| d\mu = \sum_{i=1}^p \|x_i\| \|\phi_i^n - \psi_i^n\| = 2.$$ 

So, we conclude that every nonempty weakly open subset relative to the unit ball of $L_1(\mu, X)$ has diameter 2, when $\mu$ is atomless.

It is well known that $L_1(\mu, X) = L_1(\nu, X) \oplus \ell_1(I, X)$, where $\nu$ is atomless and $I$ is a convenient set. By Lemma 2.1 and the preceding paragraph, we conclude that $L_1(\mu, X)$ has some nonempty weak open set relative to the unit ball with diameter less than 2 if and only if $I$ is nonempty and $X$ has it too. 

Now, one obtains easily the following consequence.

**Corollary 2.14.** Let $(\Omega, \Sigma, \mu)$ a positive measure space and let $X$ be a Banach space.

(i) The unit ball of $L_\infty(\mu, X)$ has some denting (respectively weak-norm continuity) point if and only if $L_\infty(\mu)$ is finite-dimensional and $B_X$ has it too.

(ii) The unit ball of $L_1(\mu, X)$ has some denting (respectively weak-norm continuity) point if and only if $\mu$ contains atoms and $B_X$ has it too.

We always have considered in this note, spaces with its natural norm. It would be excessive to think that the above results hold for equivalent norms. However it seems natural pose the following question: is possible renorming every Banach space failing Radon–Nikodym (respectively point of continuity) property so that every slice (respectively nonempty weak open subset) of the new unit ball has diameter 2? In the affirmative case, it would be a characterization of Radon–Nikodym property, by improving the known ones up to now.

**References**

