The Schrödinger–Poisson equation under the effect of a nonlinear local term

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Abstract

In this paper we study the problem

\[
\begin{cases}
-\Delta u + u + \lambda \phi u = u^p, \\
-\Delta \phi = u^2, \quad \lim_{|x| \to +\infty} \phi(x) = 0,
\end{cases}
\]

where \( u, \phi : \mathbb{R}^3 \to \mathbb{R} \) are positive radial functions, \( \lambda > 0 \) and \( 1 < p < 5 \). We give existence and nonexistence results, depending on the parameters \( p \) and \( \lambda \). It turns out that \( p = 2 \) is a critical value for the existence of solutions.

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1. Introduction

This paper has been motivated by the problem

\[
\begin{cases}
i \psi_t - \Delta \psi + \phi(x) \psi = |\psi|^{p-1}\psi, \\
-\Delta \phi = |\psi|^2, \quad \lim_{|x| \to +\infty} \phi(x) = 0,
\end{cases}
\]  

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where \( 1 < p < 5 \) and \( \psi : \mathbb{R}^3 \times [0, T] \to \mathbb{C} \). We are interested in the existence of stationary solutions \( \psi(x,t) = e^{-it}u(x) \), where \( u : \mathbb{R}^3 \to \mathbb{R} \) is a positive function to be found. Thus, \( u \) must verify

\[
\begin{aligned}
-\Delta u + u + \phi(x)u &= u^p, \\
-\Delta \phi &= u^2, \\
\lim_{|x| \to +\infty} \phi(x) &= 0.
\end{aligned}
\] (2)

Similar equations have been very studied in the literature, see [6,7,19,20]. Those papers, and after them many others, consider the von Weizsäcker correction to the Thomas–Fermi model. The Thomas–Fermi–von Weizsäcker model revealed very useful in the quantum description of the ground states of nonrelativistic atoms and molecules. This model considered a local nonlinearity \( u^p \) with the a sign minus and \( p = 5/3 \).

If the term \( u^p \) is replaced with 0, problem (2) becomes the Schrödinger–Poisson equation (also called Schrödinger–Maxwell equation). This type of equation appeared in semiconductor theory and has been studied in [4,5,21,22], and many others.

In some recent works [4,8,10–15,24,25], a local nonlinear term \( u^p \) (or, more generally, \( f(u) \)) has been added to the Schrödinger–Poisson equation, giving rise to Eq. (2). Those nonlinear terms have been traditionally used in the Schrödinger equation to model the interaction among particles.

In this paper we study the existence of positive radial solutions of the following problem, depending on the parameter \( \lambda > 0 \):

\[
\begin{aligned}
-\Delta u + u + \lambda \phi(x)u &= u^p, \\
-\Delta \phi &= u^2, \\
\lim_{|x| \to +\infty} \phi(x) &= 0.
\end{aligned}
\] (3)

Apart from the nonexistence result given in Theorem 4.1, we always study positive radial solutions. Let us denote by \( H^1_r \) the Sobolev space of radial functions \( u \) such that \( u, \nabla u \) are in \( L^2(\mathbb{R}^3) \). Our approach is variational, and we look for solutions of (3) as critical points of the associated energy functional \( I = I_\lambda : H^1_r \to \mathbb{R} \) (see (6)).

We point out that in spite of the amount of papers dealing with (2), the geometric properties of \( I \) have not been studied in detail. One of the motivations of this paper is to shed some light on the behavior of \( I \). We will study whether \( I \) is bounded below or not and, if it is, we prove the existence of a minimizer. We will also be concerned with critical points of mountain-pass type. As we shall see, the case \( p = 2 \) turns out to be critical.

We have included a parameter \( \lambda \) in the problem motivated by the works [11,14,24], where (3) is studied with \( \lambda \to 0^+ \). In so doing, they prove the existence of the so-called semiclassical states. For more general information on semiclassical states in nonlinear Schrödinger equations, we refer the reader to the monograph [2].

Let us briefly comment the known results for problem (2). In [25] the case \( p = 5/3 \) is studied. The authors use a minimization procedure in an appropriate manifold to find a positive solution (possibly nonradial) for the system

\[
\begin{aligned}
-\Delta u + \beta u + \phi(x)u &= u^p, \\
-\Delta \phi &= u^2, \\
\lim_{|x| \to +\infty} \phi(x) &= 0
\end{aligned}
\] (4)

for some \( \beta > 0 \) (the Lagrange multiplier). By defining \( \psi(x,t) = e^{-i\beta t}u(x) \) they obtain a solution of (1) with frequency \( \beta \). Moreover, in [25] the evolution in time of Eq. (1) is studied.
In our work, though, we look for solutions with a fixed frequency (assumed to be equal to one, for clarity).

In [10,12] a radial positive solution of (2) is found for $3 \leq p < 5$. To do that they use the mountain pass theorem of [3]. It is easy to show that $I_1$ attains a local minimum at zero. Moreover, in [12] it is pointed out that $I_1$ is unbounded below even for $p > 2$.

In order to use the mountain-pass theorem, the (PS) condition is needed. We recall that the functional $I$ satisfies the (PS) condition if for any sequence $\{u_n\} \in H^1_r$ such that $I'(u_n) \to 0$, $I(u_n)$ is bounded, there exists a convergent subsequence. In [12] the (PS) condition is proved only for $p \in [3, 5)$, and then their existence result is restricted to that case.

Furthermore, in [13] a related Pohozaev equality is found. With this equality in hand, the authors can prove that there does not exist nontrivial solutions of (2) for $p \leq 1$ or $p \geq 5$.

As we can see, the previous results leave a gap, say, the case $p \in (1, 3)$. We remark that the most important case in applications, $p = 5/3$, is included in this gap.

One of the main features of this work is to fill this gap. As a consequence of our results, we will prove the following.

**Theorem 1.1.** If $p \leq 2$, problem (2) does not admit any nontrivial solution. Moreover, if $2 < p < 5$, there exists a positive radial solution of (2).

Some differences show up in problem (3) if $\lambda$ is small. To start with, (3) has a solution for any $p \in (1, 5)$ if $\lambda$ is small enough, see Proposition 2.3. Moreover, in [14,24] it is proved that there exists a second positive radial solution of (3) for $\lambda$ small and $p < 11/7$. In this paper we will extend this statement to values $p < 2$. In [14,24] the asymptotic behavior of the solutions is also studied, which is out of the scope of this work.

The following table sums up the main results of this paper:

<table>
<thead>
<tr>
<th>$\lambda$ small</th>
<th>$\lambda \geq 1/4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 &lt; p &lt; 2$</td>
<td>Two solutions</td>
</tr>
<tr>
<td>$p = 2$</td>
<td>One solution</td>
</tr>
<tr>
<td>$2 &lt; p &lt; 5$</td>
<td>One solution</td>
</tr>
</tbody>
</table>

In the above table, “Two solutions” (respectively “One solution”) means that there exist at least two (respectively one) positive radial solutions. On the other hand, “No solution” means that there are no nontrivial solutions.

The paper is organized as follows. In Section 2 we define the functional $I$ and state some known properties of problem (3): compactness, regularity of solutions and Pohozaev equality.

In Section 3 we deal with the case $p > 2$. For $p \in (2, 3)$, whether the (PS) property holds or not remains an incognita; so, we cannot simply apply the mountain pass theorem.

Our approach consists of minimizing $I$ on a certain manifold $M$. We point out that the Nehari manifold is not a good choice for it. Here $M$ is defined by a condition which is a combination of the equation $I'(u)(u) = 0$ and the Pohozaev equality. To the best of our knowledge, this approach is entirely new in the literature.

Section 4 is devoted to the case $p \leq 2$. We use an inequality which dates back to [21] to prove that 0 is the unique solution if $\lambda \geq 1/4$. Again this inequality will be the key to study the (PS) condition and the boundedness of $I$. From this we easily obtain existence results for $\lambda$ small.
In Section 5 we give a result regarding bifurcation of solutions depending on \( \lambda \) for the case \( p < 2 \). We also expose some open problems that, in our opinion, are interesting issues to study in the future.

2. Preliminaries

Let us fix some notations. We will write \( H^1 = H^1(\mathbb{R}^3) \), \( D^{1,2} = D^{1,2}(\mathbb{R}^3) = \{ u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3) \} \) as the usual Sobolev spaces, and \( H^1_r, D^{1,2}_r \) the corresponding subspaces of radial functions. The norm of \( H^1 \) will be simply denoted by \( \| \cdot \| \), while \( \| \cdot \|_D \) is the norm of the space \( D^{1,2} \). Other norms used in the paper will be clear from the notation.

Sometimes we will simply write \( \int f \) to mean the Lebesgue integral of \( f( x) \) in \( \mathbb{R}^3 \).

Throughout the paper we will be interested in weak positive solutions \((u, \phi) \in H^1 \times D^{1,2} \) for the problem (3), where \( 1 < p < 5 \).

Clearly, we can write an integral expression for \( \phi \) in the form:

\[
\phi(x) = \int u^2(y) G(x, y) \, dy,
\]

where \( G(x, y) \) is the Green function of the Laplacian in \( \mathbb{R}^3 \). Note that \( \phi \) is always a positive function. Moreover, if \( u \) is a radial function, then \( \phi \) is also radial and has the following expression:

\[
\phi(r) = \frac{1}{r} \int_0^{+\infty} u^2(s) s \min\{r, s\} \, ds.
\]

First of all, let us study the variational setting of the problem. For any \( u \in H^1 \), the linear operator \( T : D^{1,2}(\mathbb{R}^3) \to \mathbb{R} \) defined as

\[
T(\nu) = \int u^2 \nu \leq \| u^2 \|_{L^{6/5}} \| \nu \|_{L^6} \leq C \| u \|_2^2 \| \nu \|_D
\]

is continuous. Then, by the Lax–Milgram theorem there exists \( \Phi[u] = \phi_u \in D^{1,2}(\mathbb{R}^3) \) such that for any \( \nu \in D^{1,2}(\mathbb{R}^3) \),

\[
\int \nabla \phi_u \nabla \nu = \int u^2 \nu.
\]

Therefore, \( -\Delta \phi_u = u^2 \) in a weak sense. Moreover, \( \| \phi_u \|_D = \| T \| \leq C \| u \|^2 \). Hence, we obtain that

\[
\int \phi_u u^2 \leq \| u \|_{L^{6/5}}^2 \| \phi_u \|_{L^6} \leq C \| u \|^4.
\]

The following compactness result is known: we include its statement and proof for the sake of completeness.
Lemma 2.1. Consider the operator $\Phi : H^1_r \rightarrow D^{1,2}_r$, $\Phi[u] = \phi u$, that is, the solution in $D^{1,2}$ of the problem $-\Delta \phi u = u^2$. Let $u_n$ be a sequence satisfying $u_n \rightharpoonup u$ in $H^1_r$. Then, $\Phi[u_n] \rightarrow \Phi[u]$ in $D^{1,2}$ and, as a consequence,

$$\int \Phi[u_n]u_n^2 \rightarrow \int \Phi[u]u^2.$$  

(5)

Proof. Define the linear operators $T_n, T : D^{1,2} \rightarrow \mathbb{R},$

$$T_n(v) = \int v u_n^2, \quad T(v) = \int v u^2.$$

Recall that the inclusion $H^1_r \hookrightarrow L^q$ is compact for $2 < q < 6$ (see [26]). In particular, $u_n^2 \rightarrow u^2$ in the norm of $L^{6/5}$. Note that

$$|T_n(v) - T(v)| \leq \left( \int |u_n^2 - u^2|^{6/5} \right)^{5/6} \left( \int v^6 \right)^{1/6}.$$

This implies that $T_n$ converges strongly (as a linear operator) to $T$. Hence, $\Phi[u_n] \rightarrow \Phi[u]$ in $D^{1,2}$. To conclude (5) it suffices to observe that $\Phi[u_n] \rightarrow \Phi[u]$ in $L^6$ and $u_n^2 \rightarrow u^2$ in the norm of $L^{6/5}$. □

From the above computations, the functional $I : H^1_r \hookrightarrow \mathbb{R},$

$$I(u) = \int \left( \frac{1}{2} (|\nabla u(x)|^2 + u(x)^2) + \frac{\lambda}{4} \phi u(x) u^2(x) - \frac{1}{p+1} |u(x)|^{p+1} \right) dx$$  

(6)

is well defined. Furthermore, it is known that $I$ is a $C^1$ functional with derivative given by

$$I'(u)[v] = \int \nabla u \nabla v + uv + \lambda \phi u uv - |u|^{p-1} uv.$$

It is also known that the principle of symmetric criticality holds for this functional: in other words, a critical point of $I$ is a true solution for (3). See, for instance, [5,12].

2.1. Regularity of solutions

At this point, let us say a few words about the regularity of solutions. Because of the definition, $u \in H^1 \subset L^6(\mathbb{R}^3)$. By using [1], we deduce that $\phi$ belongs to $W^{2,3}_{loc}(\mathbb{R}^3)$ and from Sobolev inclusions it is a $C^{0,\alpha}_{loc}(\mathbb{R}^3)$. By using a bootstrap argument we easily conclude that $u \in C^{2,\alpha}_{loc}$. As a consequence, also $\phi$ is in $C^{2,\alpha}_{loc}$. So, in the end, the weak solutions of (3) will be solutions in the classical sense.

2.2. A Pohozaev equality

In [13] (see equality (4.35) of that paper), a Pohozaev equality is proved for the problem (3). We state it in a way that may seem strange; however, we will need it in this form.
Theorem 2.2. Let \((u, \phi) \in H^1 \times D^{1,2}\) be a weak solution of the problem

\[
\begin{aligned}
- a \Delta u + bu + c \phi u &= d u^p, \\
- \Delta \phi &= u^2,
\end{aligned}
\]

where \(a, b, c\) and \(d\) are real constants. Then, there holds:

\[
\int \frac{a}{2} |\nabla u|^2 + \frac{3b}{2} u^2 + \frac{5c}{4} \phi u^2 - \frac{3d}{p+1} u^{p+1} = 0.
\]

In particular, the solutions of (3) satisfy

\[
\int \frac{1}{2} |\nabla u|^2 + \frac{3}{2} u^2 + \frac{5\lambda}{4} \phi u^2 - \frac{3}{p+1} u^{p+1} = 0. \tag{7}
\]

2.3. A positive solution for \(\lambda\) small

As a direct application of the Implicit Function theorem, the following result is proved in [24, Proposition 2.1]:

Proposition 2.3. Let \(p \in (1, 5)\) and \(u_0 \in H^1\) be the unique positive radial solution in \(\mathbb{R}^3\) for the problem (see [18])

\[- \Delta u + u = u^p.\]

Then, for \(\lambda\) small enough there exists a radial solution \(u_\lambda\) of the problem (3) such that \(u_\lambda \to u_0\) in \(H^1\) as \(\lambda \to 0\).

3. The case \(p > 2\)

Here we deal with the case \(p > 2\). The first result we state was obtained in [12]: we include it for the sake of completeness.

Proposition 3.1. Let \(\lambda > 0\) and \(p \in (2, 5)\). Then, \(I\) is not bounded below.

Proof. Let \(u \in H^1\) any radial and positive function, and \(v_t(x) = t^2 u(tx)\). In order to estimate \(I(v_t)\), we compute:

\[
\int |\nabla v_t(x)|^2 = \int t^6 |\nabla u(tx)|^2 dx = t^3 \int |\nabla u(y)|^2 dy,
\]

where we have made the change \(y = tx\).

Reasoning in the same way, we get:

\[
\int v_t^2 = t \int u^2, \quad \int v_t^{p+1} = t^{2p-1} \int u^{p+1}.
\]
We now focus on the term $\int \phi_{v_t} v_t^2$. A direct computation gives that 
\[
\phi_{v_t}(x) = t^2 \phi_u(tx).
\]
Then, 
\[
\int \phi_{v_t}(x)v_t^2(x)dx = t^6 \int \phi_u(tx)u^2(tx)dx = t^3 \int \phi_u(y)u^2(y)dy.
\]
So, we have that 
\[
I(v_t) = \int \frac{1}{2} t^3 |\nabla u|^2 + \frac{1}{2} tu^2 + \frac{\lambda}{4} t^3 \phi_u u^2 - \frac{1}{p+1} t^{2p-1}|u|^3.
\]
Since $2p - 1 > 3$, we have that $I(v_t) \to -\infty$ as $t \to +\infty$. $\square$

Recall that the trivial solution $0$ is a local minimum of $I$; so, we are under the geometric conditions of the mountain-pass theorem. If $p \geq 3$, the (PS) condition has been proved in [10, 12], and hence the existence of a solution is guaranteed. But, for $p \in (2, 3)$, it is not known whether the (PS) condition holds or not.

As mentioned in the introduction, we will use a different procedure. Basically, we try to minimize $I$ on a certain manifold. We point out that the usual Nehari manifold does not happen to be a good choice.

Let us justify the choice of the manifold. Suppose that $u$ is a critical point of $I$. Define, as above, 
\[
v_t(x) = t^2 u(tx),
\]
and consider 
\[
\gamma(t) = I(v_t) = \int \frac{1}{2} t^3 |\nabla u|^2 + \frac{1}{2} tu^2 + \frac{\lambda}{4} t^3 \phi_u u^2 - \frac{1}{p+1} t^{2p-1}|u|^p.
\]
Clearly, $\gamma(t)$ is positive for small $t$ and tends to $-\infty$ if $t \to +\infty$. Moreover, $\gamma$ has a unique critical point, corresponding to its maximum (see Lemma 3.3). But, since $u$ is a solution, this critical point should be achieved at $t = 1$.

The equation $\gamma'(1) = 0$ can be written as 
\[
\int \frac{3}{2} |\nabla u|^2 + \frac{1}{2} u^2 + \frac{3\lambda}{4} \phi_u u^2 - \frac{2p-1}{p+1} |u|^{p+1} = 0. \tag{8}
\]

Moreover, the curve $\Gamma = \{v_t\}_{t \in \mathbb{R}}$ crosses the manifold defined by (8), and we have that $I|_{\Gamma}$ attains a maximum at $u$ along $\Gamma$. If $u$ is a solution of the mountain-pass type, it seems natural to look for it as a minimum of $I$ on that manifold.

Of course the previous arguments are not rigorous; we just wanted to shed some light on our procedure.

Note that (8) is nothing but the equation $2I'(u)(u) = 0$ minus the Pohozaev equation (7). Hence, any solution must verify the equality (8).

Define $J : H^1_t \to \mathbb{R}$ as 
\[
J(u) = \int \frac{3}{2} |\nabla u|^2 + \frac{1}{2} u^2 + \frac{3\lambda}{4} \phi_u u^2 - \frac{2p-1}{p+1} |u|^{p+1}.
\]

We shall study the functional $I$ on the manifold $M$ defined as 
\[
M = \{ u \in H^1_t -\{0\} : J(u) = 0 \}.
\]
The above discussion implies that $M$ is not empty (actually, given any $u \neq 0$, there exists $t > 0$ so that $u_i^t \in M$). In the subsequent theorem we state the main result of this section.

**Theorem 3.2.** Suppose that $p \in (2, 5)$ and $\lambda > 0$. Then, there exists a minimizer $u$ of $I|_M$. Moreover, $u$ is positive and $I'(u) = 0$.

**Proof.** The proof will be developed in several steps.

**Step 1.** $0 \notin \partial M$.
This is important, for instance, to show that $M$ is complete. Note that

$$\int |u|^{p+1} \leq c_0 \|u\|^{p+1}.$$ 

Therefore, we get

$$\int \frac{3}{2} |\nabla u|^2 + \frac{1}{2} u^2 + \frac{3\lambda}{4} \phi_u u^2 - \frac{2p-1}{p+1} |u|^{p+1} \geq \frac{1}{2} \|u\|^2 - c_0 \frac{2p-1}{p+1} \|u\|^{p+1},$$

which is strictly positive for $\|u\|$ small.

**Step 2.** $\inf I|_M > 0$.
Take an arbitrary $u \in M$, and define $k = I(u)$. Define also:

$$a = \int |\nabla u|^2, \quad b = \int u^2, \quad c = \lambda \int \phi_u u^2, \quad d = \int |u|^{p+1}.$$ 

Clearly, $a$, $b$, $c$, and $d$ are positive constants satisfying

$$\begin{align*}
\frac{1}{2} a + \frac{1}{2} b + \frac{1}{4} c - \frac{1}{p+1} d &= k, \\
\frac{3}{2} a + \frac{1}{2} b + \frac{3}{4} c - \frac{2p-1}{p+1} d &= 0.
\end{align*}$$

We now solve the above system of equations; for $a$, $b$ and $k$ arbitrary, $c$ and $d$ are given by

$$c = -2 \frac{a(p-2) + b(p-1) + k(1-2p)}{p-2}, \quad d = \frac{6a + 4b + 3c - 6k}{4}.$$ 

Since $c$ must be a positive quantity, we have

$$(a + b)(p - 2) < a(p - 2) + b(p - 1) < k(2p - 1). \quad (9)$$

Recall now that, from Step 1, there exists $\varepsilon > 0$ such that $a + b > \varepsilon$. So, $k = I(u)$ must be above certain positive constant.

**Step 3.** Let $\{u_n\} \in M$ so that $I(u_n) \to \inf I|_M$. We claim that $\{u_n\}$ is bounded.
The proof follows from the argument used to demonstrate Step 2. Actually, from Eq. (9) we have that
\[(p - 2) \int |\nabla u_n|^2 + u_n^2 < (2p - 1) I(u_n).\]

Then, if \(I(u_n) \to \inf I|_M\), \(u_n\) must be bounded in norm.

**Step 4.** We can assume, passing to a subsequence, that \(u_n \rightharpoonup u\). We claim that \(u \in M\) and \(u_n \to u\) strongly in \(H^1_0\). Thus, \(I|_M\) attains its minimum at \(u\).

Define:
\[
\begin{align*}
a_n &= \int |\nabla u_n|^2, & b_n &= \int u_n^2, & c_n &= \lambda \int \phi u_n^2, & d_n &= \int |u_n|^{p+1}, \\
a &= \int |\nabla u|^2, & b &= \int u^2, & c &= \lambda \int \phi u^2, & d &= \int |u|^{p+1}, \\
\bar{a} &= \lim_{n \to \infty} a_n, & \bar{b} &= \lim_{n \to \infty} b_n, & \bar{c} &= \lim_{n \to \infty} c_n, & \bar{d} &= \lim_{n \to \infty} d_n.
\end{align*}
\]

We can assume that the above limits exist, passing to an appropriate subsequence. Because of the compactness of the embedding \(H^1_0 \hookrightarrow L^{p+1}\) (see [26]) and Lemma 2.1, we have that \(c = \bar{c}\), \(d = \bar{d}\). Moreover, from the weak convergence we have that \(a \leq \bar{a}\) and \(b \leq \bar{b}\). We claim that indeed both previous inequalities are equalities. In such case, \(u_n \to u\) and then \(u \in M\).

Suppose, reasoning by contradiction, that \(a + b < \bar{a} + \bar{b}\). Observe that in such case \(I(u) < \inf I|_M\); however, this does not contradict anything, since \(J(u) < 0 \Rightarrow u \notin M\). Our procedure can be sketched as follows: we will find \(t_0 > 0\) such that the function \(v_0(x) = t_0^2 u(t_0 x)\) belongs to \(M\). Then, we will see that \(I(v_0) < \inf I|_M\), getting the desired contradiction.

Since \(I(u_n) \to \inf I|_M\) and \(J(u_n) = 0\), we have that
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{1}{2} \bar{a} + \frac{1}{2} \bar{b} + \frac{1}{4} \bar{c} - \frac{1}{p+1} \bar{d} = \inf I|_M, \\
\frac{3}{2} \bar{a} + \frac{3}{2} \bar{b} + \frac{3}{4} \bar{c} - \frac{2p-1}{p+1} \bar{d} = 0.
\end{array} \right.
\end{align*}
\]

We first show that \(u \neq 0\). Observe that, because of Step 1, \(\bar{a} + \bar{b} > \varepsilon\); in view of the second equation of (12), this implies that \(\bar{d} > 0\). But recall that \(\bar{d} = d = \int |u|^p\), and hence \(u\) cannot be identically equal to zero. As a consequence, \(a > 0\), \(b > 0\), \(c > 0\).

At this point we need an easy technical lemma.

**Lemma 3.3.** Let \(\alpha\), \(\beta\), \(\gamma\), \(\delta\) be positive constants, and \(p > 2\). Define \(f: \mathbb{R}_{0}^+ \to \mathbb{R}, f(t) = \alpha t^3 + \beta t + \gamma t^3 - \delta t^{2p-1}\), for \(t \geq 0\). Then \(f\) has a unique critical point which corresponds to its maximum.

**Proof.** Since \(p > 2\), it is easy to check that \(f\) has a maximum. We now show that this is the only critical point of \(f\).

Let us compute some derivatives of \(f\):
\[f'(t) = 3at^2 + \beta + 3\gamma t^2 - \delta(2p - 1)t^{2p-2},\]
f''(t) = 6\alpha t + 6\gamma t - \delta(2p - 1)(2p - 2)t^{2p-3},

f'''(t) = 6\alpha + 6\gamma - \delta(2p - 1)(2p - 2)(2p - 3)t^{2p-4}.

Clearly, f''' is positive for small values of t, tends to $$-\infty$$ as t → +∞ and is strictly decreasing. Then, there exists $$t_3 > 0$$ such that $$f'''(t_3) = 0$$, $$f'''(t_3 + t) > 0$$ for $$t \neq t_3$$.

We now study f'''. Since f''' is increasing for $$t < t_3$$ and $$f'''(0) = 0$$, f''' takes positive values at least for $$t \in (0, t_3)$$. For $$t > t_3$$, f''' decreases, tending to $$-\infty$$. Then, there exists $$t_2 > t_3$$ such that $$f''(t_2) = 0$$ and $$f''(t)(t_2 - t) > 0$$ for $$t \neq t_2$$.

We now repeat the argument to f' to conclude the existence of $$t_1 > t_2$$ such that $$f'(t_1) = 0$$ and $$f'(t)(t_1 - t) > 0$$ for $$t \neq t_1$$. Therefore, $$t_1$$ is the unique critical point of f; the lemma is proved. □

Define

$$f(t) = \frac{1}{2}a t^3 + \frac{1}{2}b t + \frac{1}{4}c t^3 - \frac{1}{p+1}d t^{2p-1},$$

$$f(t) = \frac{1}{2}a t^3 + \frac{1}{2}b t + \frac{1}{4}c t^3 - \frac{1}{p+1}d t^{2p-1}.$$

From Lemma 3.3, both functions have a unique critical point, corresponding to their maxima. From (12), we conclude that the maximum of f is equal to inf I|_{M} and is achieved at t = 1. Since $$a + b < \bar{a} + \bar{b}$$, then $$f(t) < \bar{f}(t)$$ for all $$t > 0$$. Let $$t_0 > 0$$ be the point where the maximum of f is achieved. Then $$f'(t_0) = 0$$ and $$f(t_0) < \max \bar{f} = \inf I|_{M}$$.

Define $$v_0(x) = t_0^2 u(t_0x)$$. Clearly, we have:

$$I(v_0) = \frac{1}{2}a t_0^3 + \frac{1}{2}b t_0 + \frac{1}{4}c t_0^3 - \frac{1}{p+1}d t_0^{2p-1} = f(t_0) < \inf I|_{M},$$ (13)

$$J(v_0) = \int \frac{3}{2} |\nabla v_0|^2 + \frac{1}{2} v_0^2 + \frac{3\lambda}{4} \phi v_0 v_0 - \frac{2p-1}{p+1} |v_0|^{p+1}$$

$$= \frac{3}{2} a t_0^3 + \frac{1}{2} b t_0 + \frac{3}{4} c t_0^3 - \frac{(2p-1)}{p+1} d t_0^{2p-1} = t_0 f'(t_0) = 0.$$ (14)

Then, $$v_0 \in M$$ and $$I(v_0) < \inf I|_{M}$$, which is a contradiction.

**Step 5.** $$J'(u) \neq 0$$, where u is the minimizer found above.

This condition is necessary in order to use the Lagrange multiplier rule. Again reasoning by contradiction, suppose that $$J'(u) = 0$$. We still use the numbers a, b, c, d as defined in (10), and $$k = \inf I|_{M} > 0$$. In a weak sense, the equation $$J'(u) = 0$$ can be written as

$$-3\Delta u + u + 3\lambda \phi u(x)u - (2p-1)u^p = 0.$$ (15)
Then, there holds:

\[
\begin{aligned}
\frac{1}{2}a + \frac{1}{4}b + \frac{1}{4}c - \frac{1}{p+1}d &= k, \\
\frac{3}{2}a + \frac{1}{2}b + \frac{3}{4}c - \frac{2p-1}{p+1}d &= 0, \\
3a + b + 3c - (2p-1)d &= 0, \\
3\frac{1}{2}a + \frac{3}{2}b + \frac{5}{4}c - (2p-1)\frac{3}{p+1}d &= 0.
\end{aligned}
\]

The first equation comes from the fact that \( u \) minimizes \( I|_M \). The second one holds since \( J(u) = 0 \). The third one follows by multiplying Eq. (15) by \( u \) and integrating. The fourth one is the Pohozaev equality (see Theorem 2.2) applied to (15).

It can be checked out that for any \( p \neq 1, p \neq 2 \), the above system admits one unique solution on \( a, b, c, \) and \( d \), given by

\[
\begin{aligned}
a &= -k\frac{2p-1}{4(p-2)}, \\
b &= 3k\frac{2p-1}{2(p-1)}, \\
c &= -k\frac{2p-1}{2(p-2)}, \\
d &= -3k\frac{p+1}{4(2-3p+p^2)}.
\end{aligned}
\]

Note that, since \( p > 2 \) and \( k > 0 \), \( a, c, \) and \( d \) happen to be negative, what is not possible.

**Step 6.** \( I'(u) = 0 \).

Thanks to the Lagrange multiplier rule, there exists \( \mu \in \mathbb{R} \) so that \( I'(u) = \mu J'(u) \). We claim that \( \mu = 0 \).

As above, the equation \( I'(u) = \mu J'(u) \) can be written, in a weak sense, as

\[
-\Delta u + u + \lambda \varphi_u(x)u - u^p = \mu\left[-3\Delta u + u + 3\lambda \varphi_u(x)u - (2p-1)u^p\right].
\]

So, \( u \) solves the equation

\[
-(3\mu - 1)\Delta u + (\mu - 1)u + (3\mu - 1)\lambda \varphi_u(x)u - [(2p-1)\mu - 1]u^p = 0. \tag{16}
\]

Recall the definitions of \( a, b, c, d \); arguing as above, we have:

\[
\begin{aligned}
\frac{1}{2}a + \frac{1}{4}b + \frac{1}{4}c - \frac{1}{p+1}d &= k, \\
\frac{3}{2}a + \frac{1}{2}b + \frac{3}{4}c - \frac{2p-1}{p+1}d &= 0, \\
(3\mu - 1)a + (\mu - 1)b + (3\mu - 1)c - [(2p-1)\mu - 1]d &= 0, \\
(3\mu - 1)\frac{1}{2}a + (\mu - 1)\frac{3}{2}b + (3\mu - 1)\frac{5}{4}c - \frac{3(2p-1)\mu - 1}{p+1}d &= 0.
\end{aligned}
\]

We now deal with the above system. Considering \( a, b, c, d \) as incognita, the coefficient matrix is

\[
A = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{p+1} \\
\frac{3}{2} & \frac{1}{2} & \frac{3}{4} & -\frac{2p-1}{p+1} \\
3\mu - 1 & \mu - 1 & 3\mu - 1 & -[(2p-1)\mu - 1] \\
(3\mu - 1)\frac{1}{2} & (\mu - 1)\frac{3}{2} & (3\mu - 1)\frac{5}{4} & -\frac{3(2p-1)\mu - 1}{p+1}
\end{pmatrix}.
\]
We compute its determinant:

$$\det A = \frac{\mu(1 - 3\mu)(p - 1)(p - 2)}{p + 1} = 0 \iff p = 1, \quad p = 2, \quad \mu = 0, \quad \mu = \frac{1}{3}.$$  

We will show that $\mu$ must be equal to zero by excluding the other possibilities:

1. Assume $\mu \neq 0, \mu \neq 1/3$. Then, $\det A \neq 0$, and hence the linear system has a unique solution (depending on the parameters $\mu, p, k$). We use Cramer rule to find the value of $d$:

$$d = -\frac{3k(1 + p)}{4(p - 1)(p - 2)}.$$  

But this is not possible, since $d$ must be positive.

2. Assume now that $\mu = 1/3$. In such case, the third equation reads as follows:

$$-\frac{2}{3}b - \frac{2p - 4}{3}d = 0,$$

which is also impossible, since both $b$ and $d$ must be positive.

To end the proof, let us say a word about the positiveness of $u$. Observe that $u$ has been obtained as a minimizer of $I$ restricted to $M$. So, also $|u|$ is a minimizer, and we can apply the arguments of Steps 5 and 6 to $|u|$, instead. In this way we have that $|u|$ is a solution. Because of the maximum principle, $|u|$ must be strictly positive, and hence also $u$ is positive.  

4. The case $p \leq 2$

In this section we will consider the problem

$$\begin{cases}
-\Delta u + u + \lambda \phi(x)u = |u|^{p-1}u, \\
-\Delta \phi = u^2
\end{cases} \quad (17)$$

for $p \in (1, 2]$ and $(u, \phi) \in H^1 \times D^{1,2}$. First of all, we give the following nonexistence result.

**Theorem 4.1.** Assume that $\lambda \geq 1/4$. Then, $u = 0$ is the unique solution of problem (17).

**Proof.** Suppose that $(u, \phi) \in H^1 \times D^{1,2}$ is a solution of (17). Multiply the first equation by $u$ and integrate, to obtain

$$\int |\nabla u|^2 + u^2 + \lambda \phi u^2 - |u|^{p+1} = 0. \quad (18)$$

By the definition of $\phi$, we have that

$$\int \phi u^2 = \int \phi(-\Delta \phi) = \int |\nabla \phi|^2.$$
On the other hand, we deduce that

$$\int |u|^3 = \int (-\Delta \phi)|u| = \int \langle \nabla \phi, \nabla |u| \rangle.$$ 

We can easily conclude

$$\int |u|^3 \leq \int |\nabla u|^2 + \frac{1}{4}|\nabla \phi|^2. \quad (19)$$

Inserting this inequality into (18), we obtain

$$0 = \int |\nabla u|^2 + u^2 + \lambda |\nabla \phi|^2 - |u|^p + 1 \geq \int u^2 + |u|^3 - |u|^{p+1}.$$ 

But it is easy to check that, if $1 < p \leq 2$, the function $f(u) = u^2 + |u|^3 - |u|^{p+1}$ is nonnegative and vanish only at zero. Therefore, $u$ must be equal to zero. \(\square\)

**Remark 4.2.** The previous proposition is the unique result in this paper that does not involve only radial functions. It states nonexistence of any solution, radial or not, positive or not.

Inequality (19) was first obtained by Lions, see [21]. We include its proof since we need to choose the constants in a certain way.

In order to obtain more information, we will have to distinguish between the cases $p < 2$ and $p = 2$. For $p < 2$ we get a very complete description of the problem.

**Theorem 4.3.** Suppose $1 < p < 2$. Then, for any $\lambda$ positive, there holds:

1. $\inf I > -\infty$;
2. $I$ satisfies the (PS) condition.

**Proof.** First of all, some estimates are needed. By the same arguments as in the proof of Theorem 4.1, we get

$$c_\lambda \int |u|^3 = c_\lambda \int (-\Delta \phi)|u| = c_\lambda \int \langle \nabla \phi, \nabla |u| \rangle \leq \int \frac{1}{4}|\nabla u|^2 + \frac{\lambda}{8}|\nabla \phi|^2, \quad (20)$$

where $c_\lambda = \sqrt{\lambda/8} > 0$.

By using this inequality into the definition of $I$, we obtain

$$I(u) \geq \int \frac{1}{4}|\nabla u|^2 + \frac{1}{2}u^2 + \frac{\lambda}{8}\phi u^2 + c_\lambda |u|^3 - \frac{1}{p+1}|u|^{p+1}.$$ 

Define

$$f : \mathbb{R}_0^+ \to \mathbb{R}, \quad f(u) = \frac{1}{4}u^2 + c_\lambda u^3 - \frac{1}{p+1}u^{p+1}.$$
Observe that, for \( \lambda \) greater than a certain constant, \( f(u) \geq 0 \), and hence \( I(u) \geq 0 \). A sharper result in this direction will be given in the end of this section: now we are dealing with any \( \lambda \) positive.

Since \( p \in (1, 2) \), \( f \) is positive for \( u \to 0^+ \) or \( u \to +\infty \). Define \( m = \min f \); if \( m = 0 \), we are done. We assume, in what follows, that \( m < 0 \). Then, the set \( \{ u > 0 : f(u) < 0 \} \) is of the form \( (\alpha, \beta) \), with \( \alpha > 0 \). Note that \( \alpha, \beta, m \), are constants depending only on \( p, \lambda \). Thus,

\[
I(u) \geq \int \frac{1}{4} |\nabla u|^2 + \frac{1}{4} u^2 + \frac{\lambda}{8} \phi u^2 + f(u)
\]

\[
\geq \int \frac{1}{4} |\nabla u|^2 + \frac{1}{4} u^2 + \frac{\lambda}{8} \phi u^2 + \int_{u \in (\alpha, \beta)} f(u)
\]

\[
\geq \int \frac{1}{4} |\nabla u|^2 + \frac{1}{4} u^2 + \frac{\lambda}{8} \phi u^2 + m|A|,
\]

where \( A = \{ x \in \mathbb{R}^3 : u(x) \in (\alpha, \beta) \} \), and \( |A| \) is its Lebesgue measure.

With this inequality in hand, we can begin the proof of (i). Reasoning by contradiction, suppose that there exists a sequence \( \{ u_n \} \in H^1 \) such that \( I(u_n) \to -\infty \). Clearly, \( u_n \) must be unbounded in norm. For each function \( u_n \), define \( A_n = \{ x \in \mathbb{R}^3 : u_n(x) \in (\alpha, \beta) \} \). Note that \( A_n \) is spherically symmetric, and define \( \rho_n = \sup \{|x| : x \in A_n\} \). Since \( I(u_n) < 0 \), we have that

\[
|m||A_n| > \frac{1}{4} \|u_n\|^2, \tag{21}
\]

which, in particular, implies that \( |A_n| \to +\infty \).

We now recall the following general result due to Strauss [26]:

\[
|u(x)| \leq c_0 |x|^{-1} \|u\| \quad \forall u \in H^1_t \tag{22}
\]

for some \( c_0 > 0 \).

Take \( x \in \mathbb{R}^3, |x| = \rho_n \). Clearly, \( u_n(x) = \alpha > 0 \). We use inequalities (21), (22) to obtain

\[
0 < \alpha = u_n(x) \leq c_0 \rho_n^{-1} \|u_n\| \leq 2 c_0 \rho_n^{-1} (|m||A_n|)^{1/2} \Rightarrow c_1 \rho_n \leq |A_n|^{1/2} \tag{23}
\]

for some \( c_1 > 0 \).

On the other hand, since \( I(u_n) < 0 \), we have that \( \frac{8}{\lambda} \int \phi_n u_n^2 \leq |m||A_n| \), where \( \phi_n = \Phi[u_n] \). But

\[
\frac{8}{\lambda} |m||A_n| \geq \int \phi_n u_n^2 = 4\pi \int \int_{A_n} u_n^2(x) u_n^2(y) \frac{dx \, dy}{|x - y|}
\]

\[
\geq 4\pi \int \int_{A_n} u_n^2(x) u_n^2(y) \frac{dx \, dy}{|x - y|} \geq 4\pi \alpha^4 \frac{|A_n|^2}{2 \rho_n}
\]

\[
\Rightarrow c_2 \rho_n \geq |A_n|
\]

for some \( c_2 > 0 \), which is a contradiction with (23).
We now use the same ideas to prove the (PS) condition. Let \( \{u_n\} \) be a sequence in \( H^1_r \) so that \( I'(u_n) \to 0 \): we shall prove that \( u_n \) converges strongly.

We have

\[
I'(u_n)(u_n) = \int |\nabla u_n|^2 + u_n^2 + \lambda \phi_n u_n^2 - |u_n|^{p+1} \leq \|u_n\|,
\]

where \( \phi_n = \Phi[u_n] \).

As in (20), we deduce that

\[
\int \frac{1}{2} |\nabla u_n|^2 + \frac{\lambda}{2} \phi_n u_n^2 \geq \sqrt{\lambda} |u_n|^3.
\]

We then conclude that

\[
\|u_n\| \geq I'(u_n)(u_n) \geq \int \frac{1}{2} |\nabla u_n|^2 + \frac{1}{2} u_n^2 + \frac{\lambda}{2} \phi_n u_n^2 + g(u_n)
\]

for \( g(u) = \frac{1}{2} u^2 + \sqrt{\lambda} |u|^3 - |u|^{p+1} \).

First, we claim that \( u_n \) is bounded. If not, the following inequality must hold for \( n \) large enough:

\[
\frac{1}{3} \|u_n\|^2 + \int \frac{\lambda}{2} \phi_n u_n^2 + g(u_n) \leq 0.
\]

We can argue exactly in the same way as in the first part of the proof to arrive to a contradiction.

Being \( \{u_n\} \) bounded, we can assume that \( u_n \to u_0 \). Then, by taking into account Lemma 2.1, we conclude:

\[
0 \leftarrow I'(u_n)(u_0) = \int \nabla u_n \nabla u_0 + u_n u_0 + \lambda \phi_n u_n u_0 - |u_n|^p u_0
\]

\[
\Rightarrow \int |\nabla u_0|^2 + u_0^2 + \lambda \phi_0 u_0^2 - |u_0|^{p+1} = 0,
\]

where \( \phi_0 = \Phi[u_0] \). On the other hand,

\[
0 \leftarrow I'(u_n)(u_n) = \int |\nabla u_n|^2 + \int u_n^2 + \lambda \int \phi_n u_n^2 - \int |u_n|^{p+1}
\]

\[
\downarrow \quad \downarrow
\]

\[
\lambda \int \phi_0 u_0^2 - \int |u_0|^{p+1}.
\]

We then conclude that \( \|u_n\| \to \|u_0\| \), and then \( u_n \to u_0 \) strongly. \( \square \)

As a consequence of the previous theorem, we can conclude the existence of two solutions for \( \lambda \) small.
Corollary 4.4. Suppose that $p \in (1, 2)$ and $\lambda$ is small enough. Then there are at least two different positive solutions of (17).

Proof. Observe that the solution $0$ is an isolated local minimum of the functional $I$ and $I(0) = 0$. Note also that if $\lambda = 0$, $I$ is not bounded below: so, there exists $\lambda_0$ such that if $0 < \lambda < \lambda_0$, then $\inf I < 0$. Due to Theorem 4.3, $I$ is bounded below and verifies the (PS) condition. Because of the Ekeland variational principle [16], the infimum is attained. This yields a nontrivial solution.

On the other hand, we have an isolated minimum at zero and also an absolute minimum. So, we are in the conditions of the mountain-pass theorem of Ambrosetti and Rabinowitz [3], and then there exists another solution (with positive critical level).

The solutions found in this way need not to be positive, a priori. However, let us consider

$$I_+(u) = \int_{\mathbb{R}^3} \frac{1}{2} \left( |\nabla u(x)|^2 + u(x)^2 \right) + \frac{\lambda}{4} \phi_u(x)u^2(x) - \frac{1}{p+1} u^+(x)^{p+1} \, dx.$$

Due to the maximum principle, the critical points of $I_+$ are positive solutions of (17). By repeating all the reasonings of this section to $I_+$, we obtain two nontrivial critical points of it. □

In the case $p = 2$ we do not have a so complete description of the problem. For instance, we do not know if the (PS) condition holds or not.

However, we can say something about the boundedness of the functional $I$ for $p = 2$. We first claim that for $\lambda$ small, $I$ is not bounded below. As we have previously seen, this is not the case when $p < 2$.

Take $u \in H^1$ fixed, $M > 0$ a constant to be fixed, and $t$ a real positive parameter. We estimate $I(v_t)$, where $v_t(x) = Mt^2u(tx)$, as in Proposition 3.1:

$$I(v_t) = \int_{\mathbb{R}^3} \frac{1}{2} M^2 t^3 |\nabla u|^2 + \frac{1}{2} M^2 t u^2 + \frac{\lambda}{4} M^4 t^3 \phi_u u^2 - \frac{1}{3} M^3 t^3 |u|^3.$$

Fix $M > 0$ so that

$$\frac{1}{2} M^2 \int |\nabla u|^2 \leq \frac{1}{6} M^3 \int |u|^{p+1},$$

and consider $\lambda$ sufficiently small so that

$$\frac{\lambda}{4} M^4 \int \phi_u u^2 < \frac{1}{6} M^3 \int |u|^3.$$

Under the above conditions on $M$ and $\lambda$, it is clear that $I(v_t) \to -\infty$ as $t \to +\infty$. Then, $\inf I = -\infty$.

The last result of this section completes the diagram showed in the introduction for the case $p \leq 2$.

Proposition 4.5. Suppose that $p \in (1, 2]$ and $\lambda \geq 1/4$. Then, $\inf I = 0$. 

Proof. Consider first the case \( p \in (1, 2) \), and suppose, reasoning by contradiction, that \( \inf I < 0 \).
As in the Corollary 4.4, we can prove the existence of two nontrivial solutions. But this contradicts Theorem 4.1; hence, \( \inf I = 0 \).

In order to study the case \( p = 2 \), we argue by continuity on the exponent \( p \). Suppose that \( \inf I < 0 \); this means that there exists \( u \in H^1_0 \) so that

\[
I(u) = \int \frac{1}{2} |\nabla u|^2 + \frac{1}{2} u^2 + \frac{\lambda}{4} \phi u^2 - \frac{1}{3} |u|^3 < 0.
\]

We claim that

\[
\lim_{q \to 2} \int |u|^{q+1} = \int |u|^3.
\]

This follows from the Dominated Convergence theorem. Clearly, \( |u|^{q+1} \) converges to \( |u|^3 \) pointwise, and \( |u|^{q+1} \leq |u|^2 + |u|^4 \in L^1(\mathbb{R}^3) \) for \( q \in (1, 3) \).

So, we can choose \( q < 2 \) sufficiently close to 2 so that

\[
\int \frac{1}{2} |\nabla u|^2 + \frac{1}{2} u^2 + \frac{\lambda}{4} \phi u^2 - \frac{1}{q + 1} |u|^{q+1} < 0.
\]

This contradicts the first case and finishes the proof. \( \square \)

5. Some remarks and open problems

In this section we show some possible branches of future study regarding the problem:

\[
\begin{aligned}
-\Delta u + u + \lambda \phi(x)u &= (u^+)^p, \\
-\Delta \phi &= u^2, \\
\lim_{|x| \to +\infty} \phi(x) &= 0.
\end{aligned}
\] (24)

5.1. The bifurcation diagram

It would be interesting to study problem (24) from the point of view of bifurcation, by using degree theory.

Consider the operator \( N_\lambda : H^1_0 \to (H^1_0)' \), \( N_\lambda(u) = (u^+)^p - \lambda \phi u^2 \), where this expression is understood as a linear functional on \( H^1_0 \), that is

\[
\langle N_\lambda(u), v \rangle = \int [(u^+)^p - \lambda \phi u^2] v,
\]

where \( \langle \cdot, \cdot \rangle \) is the duality of \( H^1_0 \). Because of Lemma 2.1, \( N_\lambda \) is compact.

For any \( \xi \in (H^1_0)' \), define by \( K(\xi) \) the unique \( u \in H^1_0 \) such that \(-\Delta u + u = \xi \) in a weak sense. It is well known that \( K \) is an isometry of Hilbert spaces.

Define the operator \( Q_\lambda : H^1_0 \to H^1_0 \), \( Q_\lambda(u) = u - K \circ N_\lambda(u) \). Clearly \( Q_\lambda \) is a compact perturbation of the identity map, and the zeroes of \( Q_\lambda \) are the solutions of (24).

It is easy to check that 0 is always an isolated zero of \( Q_\lambda \), and 0 is an nondegenerate local minimum of the functional associated. Then, it follows (see [9], for example) that
\[ \deg(Q_\lambda, B(0, \varepsilon), 0) = 1. \] Moreover, for \( \lambda = 0 \) there is a unique radial positive solution, see [18]. It is also known that \( \deg(Q_0, B(u_0, \varepsilon), 0) = -1 \), see [23].

Then, from \( u_0 \) must depart a connected branch of solutions \((u, \lambda)\). This branch cannot join the solution 0, since it is isolated. The behavior of such branch of solutions is, in our opinion, an interesting question to be treated.

We have a complete answer only for \( 1 < p < 2 \). In this case, the branch of solutions cannot intersect the line \( \lambda = 1/4 \), recall Theorem 4.1.

The following a priori estimates will be of use.

**Proposition 5.1.** Assume \( 1 < p < 2 \) and let \( \{u_n\} \) be an unbounded sequence in \( H^1_r \) such that \( Q_{\lambda_n}(u_n) = 0 \) for \( \lambda_n > 0 \). Then, \( \lambda_n \to 0 \).

**Proof.** Note that the functions \( u_n \) must be positive. Suppose that \( \lambda_n > \lambda_0 > 0 \) for all \( n \); we claim that \( u_n \) must be bounded.

Multiplying Eq. (24) by \( u_n \) and integrating, we have

\[
\int |\nabla u_n|^2 + u_n^2 + \lambda_0 \phi_n u_n^2 - u_n^{p+1} \leq 0. \tag{25}
\]

As in (20), the following inequality holds:

\[
\frac{1}{2} \int |\nabla u_n|^2 + \frac{\lambda_0}{2} \int \phi_n u_n^2 \geq \lambda_0 \int u_n^3.
\]

By inserting the previous inequality into (25), we have that

\[
\frac{1}{2} \|u_n\|^2 + \frac{\lambda_0}{2} \int \phi_n u_n^2 + h(u_n) \leq 0,
\]

where

\[
h(u) = \frac{1}{2} u^2 + \sqrt{\lambda_0} u^3 - \frac{1}{p+1} u^{p+1}.
\]

We can now simply follow the reasonings of the proof of Theorem 4.3 to demonstrate that \( \{u_n\} \) must be bounded. \( \square \)

As a consequence of the previous result, we conclude the existence of a branch of solutions as depicted in Fig. 1.

Observe that also in this way we obtain two solutions for small \( \lambda \). The solutions which blow up have degree one, and hence could correspond to the minima of \( I_\lambda \).

**5.2. The (PS) condition for \( p \in [2, 3) \)**

As we have already mentioned, the (PS) condition was proved in [12] for \( p \geq 3 \). For \( p < 2 \), it has been proved in Theorem 4.3. For the rest of values of \( p \), it is still an open question.

We recall that in Theorem 3.2 we proved the existence of solution for \( p \in [2, 3) \). However, if the (PS) holds, the proof could be made in a much easier and standard way.
5.3. Solutions for \( \lambda \) small

In this kind of problems, the study of semiclassical states leads to Eq. (24) with \( \lambda \to 0 \), see [11,14,24].

In [14,24] it is proved the existence of a certain solution \( w_\lambda \) of (24) for \( \lambda \) small and \( p < 11/7 \). The asymptotic behavior of \( w_\lambda \) (as \( \lambda \to 0 \)) is well understood in those papers.

Moreover, in [24] it is shown that \( w_\lambda \) corresponds to a local minimum of the functional \( I_\lambda \). So, one question arises: is the minimum of \( I_\lambda \) achieved at \( w_\lambda \) for \( \lambda \) small?

A negative answer would imply the existence of more solutions when \( p < 11/7 \) and \( \lambda \) small. On the other hand, a positive answer would lead to another interesting question: what happens when \( p \) is close to 11/7? The solution \( w_\lambda \) tends to disappear in this case, but the minimum of \( I_\lambda \) must be attained anyway.

5.4. Existence of nonradial positive solutions

In [25] a solution for (4) is found in \( H^1 \); however, it is not known if that solution is radial or not. On the other hand, in [15] nonradial solutions of (2) are found, but they change sign.

It is well known that for certain nonlinear elliptic equations the only positive solutions are radial, see [17]. To prove that, the moving plane method is used, which does not work if nonlocal terms are involved.

In general, we do not know if there may be nonradial positive solutions of (24).

5.5. The exact number of radial positive solutions

Here we raise the question of the exact number of positive radial solutions of (24). Since in some cases we have found two solutions, uniqueness is not expected in general. This is a striking difference respect to the classical nonlinear Schrödinger equation (see [18]).

To the best of our knowledge, no result has been given in this direction so far.
References


