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CONSISTENCY IN THE ANALYTIC HIERARCHY PROCESS: A NEW APPROACH

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In this paper, we present a statistical criterion for accepting/rejecting the pairwise reciprocal comparison matrices in the analytic hierarchy process. We have studied the consistency in random matrices of different sizes. We do not agree with the traditional criterion of accepting matrices due to their inflexibility and because it is too restrictive when the size of the matrix increases. Our system is capable of adapting the acceptance requirements to different scopes and consistency necessities. The advantages of our consistency system are the introduction of adaptability in the acceptance criterion and the simplicity of the index we have used, the eigenvalue (λ_{max}) and the simplicity of the criterion.

Keywords: Analytic hierarchy process; consistency; eigenvalue, judgement matrix; random index.

1. Introduction

Over the last three decades, a number of methods have been developed which use pairwise comparisons of the alternatives and criteria for solving multi-criteria decision-making (MCDM)[1] between finite alternatives. The analytic hierarchy process (AHP) proposed by Saaty[2,3] is a very popular approach to multi-criteria decision-making (MCDM) that involves qualitative data. It has been applied during the last twenty-five years in many decision-making situations and has been used on a wide range of applications in many different fields. The method uses a reciprocal decision matrix obtained by pairwise comparisons so that the information is given in a linguistic form.

The pairwise comparison method was introduced by Fechner[4] in 1860 and developed by Thurstone[5] in 1927. Based on pairwise comparison, Saaty proposes the analytic hierarchy process (AHP)[2,3] as a method for multi-criteria decision-making. It provides a way of breaking down the general method into a hierarchy of sub-problems, which are easier to evaluate.

In the pairwise comparison method, criteria and alternatives are presented in pairs of one or more referees (e.g. experts or decision makers). It is necessary to evaluate individual alternatives, deriving weights for the criteria, constructing the overall rating of the alternatives and identifying the best one.

Let us denote the alternatives by $\{A_1, A_2, ..., A_n\}$ (n is the number of compared alternatives), their current weights by $\{w_1, w_2, ..., w_n\}$, and the matrix of the ratios of all weights by

$$W = [\, w_i / w_j \,] = \begin{pmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \dots & w_2/w_n \\ \vdots & \vdots & & \vdots \\ w_n/w_1 & w_n/w_2 & \dots & w_n/w_n \end{pmatrix} \tag{1}$$

The matrix of pairwise comparisons $A = [\, a_{ij} \,]$ represents the intensities of the expert's preference between individual pairs of alternatives (A_i versus A_j, for all *i,j=1,2,..,n*). They are usually chosen from a given scale (1/9,1/8,…,8,9). Given n alternatives $\{A_1, A_2, ..., A_n\}$, a decision maker compares pairs of alternatives for all the possible pairs, and a comparison matrix A is obtained, where the element a_{ij} shows the preference weight of A_i obtained by comparison with A_j.

$$A = [a_{ij}] = \begin{pmatrix} 1 & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ 1/a_{12} & 1 & \dots & a_{2j} & \dots & a_{2n} \\ \cdot & & & & & \\ 1/a_{1j} & 1/a_{2j} & \dots & a_{ij} & \dots & a_{in} \\ \cdot & & & & & \\ 1/a_{1n} & 1/a_{2n} & \dots & 1/a_{in} & \dots & 1 \end{pmatrix} \tag{2}$$

The a_{ij} elements estimate the ratios w_i/w_j where w is the vector of current weights of the alternative (which is our goal).

If a matrix A is absolutely consistent, we notice that $A=W$ and in the ideal case of total consistency, the principal eigenvalue (λ_{max}) is equal to n, i.e *"$\lambda_{max} = n$"*, the relations between the weights and the judgements will be given by $w_i / w_j = a_{ij}$ for *i,j = 1,2,…n*. The weights w_i, *i=1,2,...,n*, were obtained using the eigenvector method, they are positive and normalized, and satisfy the reciprocity property.

Let $A = [\, a_{ij} \,]$ for all *i,j=1,2,…,n* denote a square pairwise comparison matrix, where a_{ij} gives the relative importance of the elements *i* and *j*. Each entry in the matrix A is positive ($a_{ij} > 0$) and reciprocal ($a_{ij} = 1/a_{ji}$ $\forall i,j=1,2,..,n$). Our goal is to compute a vector of weights $\{w_1, w_2, ..., w_n\}$ associated with A. According to the Perron-Frobenius Theorem[6], if A is an *nxn,* non-negative, primitive matrix[a] , then one of its eigenvalues λ_{max}, is positive

[a] *An nxn nonnegative matrix A is primitive if $A^k > 0$ for some power k*

and greater than or equal to (in absolute value) all other eigenvalues, and there is a positive eigenvector *w* corresponding to that eigenvalue, and that eigenvalue is a simple root (matrix Frobenius root) of the characteristic equation

$$Aw = \lambda_{\max} w \tag{3}$$

In the eigenvector method, *w* is the weight vector that is our goal.

The traditional eigenvector method for estimating weights in the analytic hierarchy process yields a way of measuring the consistency of the referee's preferences arranged in the comparison matrix. If a square pairwise comparison matrix is not absolutely consistent, two different situations may be considered. The first situation is a contradictory matrix; in this case, we can find some cycles in the associated graph of this matrix[6] i.e. for n=3 if $a_{ij} > 0$, $a_{jk} > 0$ and $a_{ik} < 0$, or the opposite (and essentially similar) situation $a_{ij} < 0$, $a_{jk} < 0$ and $a_{ik} > 0$. A different situation appears when the matrix is neither totally consistent nor contradictory. In this case, Saaty defined the consistency index (CI) as follows:

$$CI = \frac{\lambda_{max} - n}{n - 1} \tag{4}$$

It is well known that small changes in a_{ij} imply small changes in λ_{max}, with the difference between this and *n* being a good measure of consistency. Saaty[2] has shown that if the referee is completely consistent then,

- $a_{ij} \cdot a_{jk} = a_{ik} (\forall\ i,j,k)$,
- $\lambda_{max} = n$ and
- CI = 0.

In this exceptional case, the two different matrices of judgements (*A*) and weights (*W*) are equal. However, it would be unrealistic to require these relations to hold in the general case. For instance, it is known that the number of totally consistent different matrices (using the Saaty scale) for n=3 is 13 or only 4 depending on whether the indifference in the relation of preference is accepted or not, for n=4 these values are 13 and 1, respectively, for n=5 is 14 and none, and so on. Otherwise, if the referee is not absolutely consistent then $\lambda_{max} > n$, and we need to measure this level of inconsistency. For this purpose, Saaty defined the consistency ratio (CR) as

$$CR = \frac{CI}{RI} \tag{5}$$

where RI is the average value of CI for random matrices using the Saaty scale obtained by Forman[8] and Saaty only accepts a matrix as a consistent one iff CR < 0.1.

If (and only if) the decision-makers generate "perfect" judgements (absolutely consistent judgements) for arbitrary *i, j* and *k*, $a_{ij} \cdot a_{jk} = a_{ik}$ $(i, j, k=1,\ldots, n)$, the comparison matrix determinant is null (Lamata et al)[9], the matrix Frobenius root (λ_{max}) is always equal to *n*, and the remaining eigenvalues are all 0 for any a_{ij} . Thus, the eigenvector corresponding to the Frobenius root is always non-negative, and each element of the eigenvector standardized by normalization can be interpreted as the degree of importance of each alternative. In this situation, the comparison matrix obviously satisfies the transitivity property for all pairwise comparisons.

Unfortunately, decision-makers do not normally make "perfect" judgements, and therefore the Frobenius root of such an inconsistent pairwise comparison matrix is always greater than n, and the difference between the root and n is equal to the sum of the remaining eigenvalues (Aupetit and Genest[10], 1993). Consequently, the smaller the difference, the more consistent the decision maker's judgement would be (Murphy[11], 1993). In the AHP, the quotient of this difference divided by (n-1) is defined as the consistency index (CI), which is the index of the consistency of judgements across all pairwise comparisons (Lootsma[12], 1991). When this situation appears, the transitivity property is not always satisfied, and this causes serious problems when ranking the alternatives (one of the goals in MCDM methods).

The problem of accepting/rejecting matrices has been greatly discussed, especially the relation between the consistency and the scale used to represent the decision maker's judgements. Lane and Verdini[13] (1989) have shown that by using a 9-point scale, Saaty's CR threshold is too restrictive due to the standard deviation of CI for randomly generated matrices being relatively small. Murphy[11], on the other hand, has shown that the 9-point scale proposed by Saaty gives results which are outside the accepted consistency when n increases. Salo and Hämäläinen[14] (1993), meanwhile, have shown that the CR threshold depends on the granularity of the scale which is being used.

While there are many other prioritization procedures in the literature, only a few of these present their corresponding indicators to evaluate inconsistency. Furthermore, when these consistency indexes have been proposed (Crawford and Williams[15] (1985); Harker[16] (1987); Golden and Wang[17] (1989); Wedley[18] (1991); Takeda[19] (1993); Takeda and Yu[20] (1995); Monsuur[21] (1996), they lack a meaningful interpretation due to the absence of the corresponding thresholds. If the prioritization procedure is not the eigenvalue method (EVM), because of the way that the Saaty approach is constructed, it is not suitable for evaluating consistency, and therefore new consistency measures relating to the prioritization procedure are required.

There has been a recent significant increase in the use of the row geometric mean method (RGMM) or the logarithmic least squares method, as a prioritization procedure in AHP (Ramanathan[22] (1997); Van den Honert[23] (1998)) due fundamentally to its psychological (Lootsma[24] (1993); Barzilai and Lootsma[25] (1997)) and mathematical (Narasimhan[26] (1982); Barzilai[27] (1997); Aguarón and Moreno-Jiménez[28] (2000); Escobar and Moreno-Jiménez[29], (2000)) properties. Using these alternative approaches of the prioritization procedure, we can highlight Gass and Rapcsák[30] (2004) and Aguaron and Moreno Jimenez[31] (2003), Stein and Mizzy[32] (2006) and Ahull-Hyde et al.[33] (2006). Lamata and Peláez[9,34]. (2002, 2003) use the EVM but with a different definition of the consistency index.

By taking these ideas into account, our aim is to introduce a relative criterion of matrix acceptance. The system can be adapted to whatever scale is needed.

This paper is divided into four sections. The first section introduces the AHP method and shows the traditional way of measuring consistency and accepting/rejecting matrices in the AHP and the drawbacks we found in this approach studied in literature. In the second

section, we study the random index and we propose a method for estimating it. Our consistency criterion is based on the largest eigenvalue (λ_{max}) of this matrix, and the relation between the Saaty approach and our proposal is developed in the third section. Finally, we present the conclusions of the paper.

2. Random Index Study

A historical study of several RIs used and a way of estimating this index can be seen in Alonso and Lamata[35]. The main idea is that the CR is a normalized value since it is divided by an arithmetic mean of random matrix consistency indexes (RI). Various authors have computed and obtained different RIs depending on the simulation method and the number of generated matrices involved in the process.

Saaty (at Wharton) and Uppuluri (at Oak Ridge) simulated the experiment with 500 and 100 runs[2], respectively. Lane and Verdini[13] (1989), Golden and Wang [36](1990), and Noble[37] (1990) carried out 2500, 1000, and 5000 simulation runs. Forman[8] (1990) also provided values for matrices of size 3 through 7 using examples from 17672 to 77487 matrices. Tumala and Wan[38] (1994) subsequently performed the experiment with samples ranging from 4600 to 470000, and they obtained the values shown in Table 1.

Table 1. RI(n) values from various authors.

	Oak Ridge	Wharton	Golden Wang	Lane, Verdini	Forman	Noble	Tumala, Wan	Aguaron et al	Alonso, Lamata
	100	500	1000	2500		500		100000	100000
3	0.382	0.58	0.5799	0.52	0.5233	0.49	0.500	0.525	0.5245
4	0.946	0.90	0.8921	0.87	0.8860	0.82	0.834	0.882	0.8815
5	1.220	1.12	1.1159	1.10	1.1098	1.03	1.046	1.115	1.1086
6	1.032	1.24	1.2358	1.25	1.2539	1.16	1.178	1.252	1.2479
7	1.468	1.32	1.3322	1.34	1.3451	1.25	1.267	1.341	1.3417
8	1.402	1.41	1.3952	1.40		1.31	1.326	1.404	1.4056
9	1.350	1.45	1.4537	1.45		1.36	1.369	1.452	1.4499
10	1.464	1.49	1.4882	1.49		1.39	1.406	1.484	1.4854
11	1.576	1.51	1.5117			1.42	1.433	1.513	1.5141
12	1.476		1.5356	1.54		1.44	1.456	1.535	1.5365
13	1.564		1.5571			1.46	1.474	1.555	1.5551
14	1.568		1.5714	1.57		1.48	1.491	1.570	1.5713
15	1.586		1.5831			1.49	1.501	1.583	1.5838

These results show that the values can change between different experiments. The values obtained by Golden and Wang[36], Lane and Verdini[13], and Forman[8] are closer, whereas the values obtained by Saaty and Uppuluri[2] seem to be higher. On the other hand, Noble, Tumala and Wan produced lower RI values. In recent years, authors such as Aguaron[31] et al, Ozdemir[39], Alonso and Lamata[35] have obtained different RI values but they are all very close (as we can see in Table 1).

2.1. *Our RI study*

2.1.1. *Traditional way of computing RI.computational problems*

Alonso and Lamata[35] have estimated the RI using 100000 matrices of each dimension. For this article, the authors calculated the RI using 500,000 matrices. We use the same number of each dimension to simplify our program. We know that in order to make a better estimation, we might use a growing number of matrices when n increases. We have used the same scale {1/9, 1/8, …,1/2,1,2,...8,9} that Saaty[2] and Forman used in the traditional RI estimation.

The steps of the algorithm were

- Random matrix generation (Saaty scale. Uniform distribution)
- Calculating corresponding CIs (for each matrix).
- Obtaining the mean of these values for each size (RI of each size).

We can see that there are no important differences between the two situations (100000 vs. 500000 matrices) in Table 2 below.

Table 2. Alonso-Lamata RI values and standard deviation (for 100000 and 500000 matrices).

	100000 matrices		500000 matrices	
n	RI	std	RI	std
3	0.5245	0.6970	0.5247	0.6973
4	0.8815	0.6277	0.8816	0.6277
5	1.1086	0.5087	1.1086	0.5087
6	1.2479	0.4071	1.2479	0.4071
7	1.3417	0.3312	1.3417	0.3310
8	1.4056	0.2779	1.4057	0.2777
9	1.4499	0.2383	1.4499	0.2381
10	1.4854	0.2076	1.4854	0.2074
11	1.5141	0.1847	1.5140	0.1844
12	1.5365	0.1670	1.5365	0.1667
13	1.5551	0.1516	1.5551	0.1514
14	1.5713	0.1383	1.5713	0.1380
15	1.5838	0.1279	1.5838	0.1276

We want to calculate good RI values to matrices with a great number of alternatives. The differences we have found between the various authors might be due to the fact that the number of matrices used in their experiments is very small compared to the number of different matrices that we can obtain for each size, $17^{\sum_{i=1}^{(i-1)}}$. The usual calculus of RI involves calculating the CI of a certain number of matrices of each dimension, and estimating the RI as the arithmetic mean of these CIs. In this case, there is a serious problem since we will need to calculate an exponentially growing number of matrices. Additionally, the calculus of each CI of a single matrix is more computationally expensive when the dimension of the matrix increases. The combination of these two problems will

mean that in practice it is not really possible (on this type of simulation) to generate a good RI estimation when *n* increases.

2.1.2. *First attempt to estimate RI. Adjustment curve. Fail*

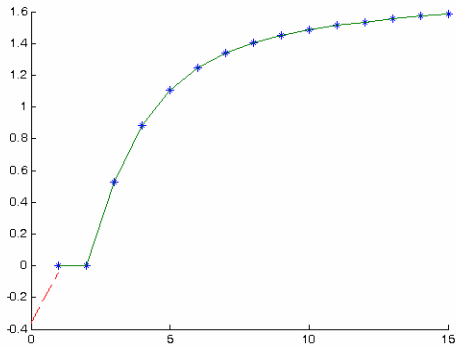

Fig. 1. Plot of the Alonso-Lamata random index.

This plot shows that RI(n) is an increasing and convergent function on the X-axis when $n \rightarrow \infty$.

The plot in Figure 2, representing RI(n+1) – RI(n), shows that $\text{Lim}_{n \rightarrow \infty}$ [RI(n+1) - RI(n)]=0, and we can therefore say that RI(n) is a convergent function.

Due to the computational problems of obtaining a good RI(n) estimation when n increases, we try to find an adjustment straight curve (in a least-square sense) in order to use that curve as an estimator of RI values.

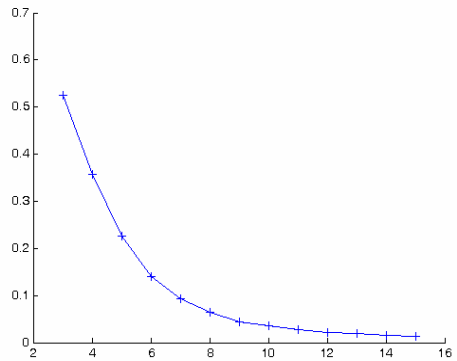

Fig 2. Plot of the RI(n+1) – RI(n) differences in the Alonso-Lamata random index.

We tried to use the RI squared adjustment curve as an RI estimator. This function is

$$RI(n) = -0.0021\ n^2 + 0.1183\ n - 0.0001 \qquad (6)$$

It can be seen that graphically this function is not a good RI estimator and subsequently decreases after reaching its maximum. This function is obviously not a good choice. In such a situation, we test a cube function adjustment curve. This function is

$$\mathrm{RI}\,(n) = 0.00149\,n^3 - 0.05121\,n^2 + 0.59150\,n - 0.79124 \qquad (7)$$

It can be observed that this function is not a good estimator either as it increases exponentially when *n* increases.

2.1.3. *Second attempt to estimate RI. Adjustment curve using* $\overline{\lambda}_{\max}$

We will estimate the RI using $\overline{\lambda}_{\max}$. We attempt to find a function that could be a good estimator of $\overline{\lambda}_{\max}\,(n)$. In a similar way to the RI procedure, we experimentally calculated the values of $\overline{\lambda}_{\max}\,(n)$ to matrix sizes from 3 to 15 (Table 3).

Table 3. Table of $\overline{\lambda}_{\max}$ and RI for several (3 to 15) matrix dimensions. Alonso and Lamata (500000 matrices).

n	**3**	**4**	**5**	**6**	**7**	**8**	**9**
$\overline{\lambda}_{\max}$	4.0486	6.6531	9.4383	12.2394	15.0476	17.8336	20.6045
Dif(n)		2.6045	2.7859	2.8011	2.8082	2.786	2.7709
	10	**11**	**12**	**13**	**14**	**15**	
	23.3723	26.1317	28.9002	31.6552	34.4170	37.1737	
	2.7678	2.7594	2.7685	2.755	2.7618	2.7567	

$Dif\,(n) = \overline{\lambda}_{\max}\,(n) - \overline{\lambda}_{\max}\,(n+1)$.

We can see that $\overline{\lambda}_{\max}\,(n)$ increases constantly. In this situation, a line might be a good estimator.

We will only take eight points (sizes 3 to 10) in order to obtain the adjustment line (in a least-square sense) and we calculate the estimated values from 11 to 15 using this function. As we have already experimentally calculated the $\overline{\lambda}_{\max}$ values for sizes 11 to 15, we can test this function as a $\overline{\lambda}_{\max}\,(n)$ estimator. The function we will use is

$$\overline{\lambda}_{\max}(n) = 2.7740\,n - 4.3764 \qquad (8)$$

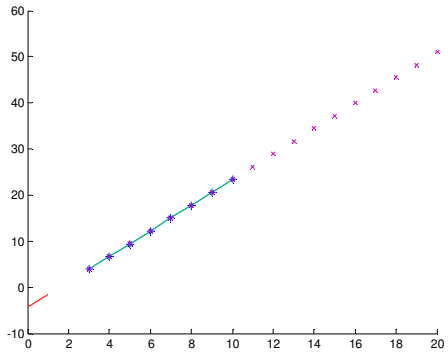

Fig 5. Values of $\overline{\lambda}_{\max}$ (sizes 3 to 8) and least-square adjustment straight line $\overline{\lambda}_{\max}$ (n) = 2.7740 n -4.3764.

We can see that the least-square line that we obtain is a good estimator of the values of $\overline{\lambda}_{\max}$ (n) and we decide to improve the least-square line by using the whole points which we calculated experimentally (13 points, sizes 3 to 15) in order to obtain the best adjustment line possible. The function is slightly different and is shown in Figure 6.

$$\overline{\lambda}_{\max}(n) = 2.7699n - 4.3513 \tag{9}$$

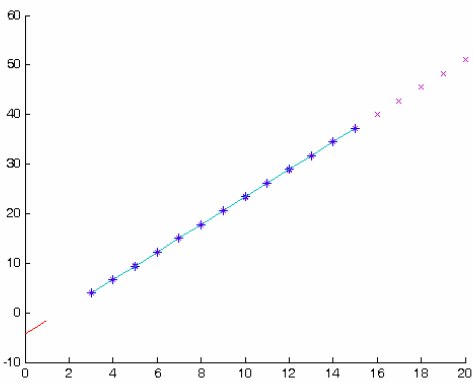

Fig 6. Values of $\overline{\lambda}_{\max}$ (sizes 3 to 15) and least-square adjustment straight line $\overline{\lambda}_{\max}$ = 2.7699 n - 4.3513.

As we can see in Figure 6, the least-square adjustment line is very accurate with a correlation coefficient of 0.99. The x-axis represents the matrix size and the *y*-axis presents the values of the corresponding values of $\overline{\lambda}_{\max}$.

Using the calculated values of $\overline{\lambda}_{\max}$, the values of the corresponding RIs become trivial. We can use the estimated $\overline{\lambda}_{\max}$ to calculate estimated RI values to greater dimensions. It is possible to see our results in Table 4.

Table 4. Table of the $\overline{\lambda}_{\max}$ and random index for dimensions greater than 15.

n	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**
λ_{max}	39.9676	42.7375	45.5074	48.2774	51.0473	53.8172	56.5872	59.3571
RI	1.5978	1.6086	1.6181	1.6265	1.6341	1.6409	1.6470	1.6526
	24	**25**	**26**	**27**	**28**	**29**	**30**	**31**
	62.1270	64.8969	67.6669	70.4368	73.2067	75.9767	78.7466	81.5165
	1.6577	1.6624	1.6667	1.6706	1.6743	1.6777	1.6809	1.6839
	32	**33**	**34**	**35**	**36**	**37**	**38**	**39**
	84.2864	87.0564	89.8263	92.5962	95.3662	98.1361	100.9060	103.6759
	1.6867	1.6893	1.6917	1.6940	1.6962	1.6982	1.7002	1.7020

These estimated RI values are plotted in Figure 7.

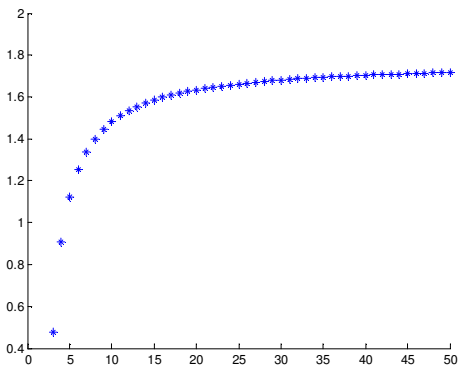

Fig 7. Plot of estimated RI(n) values.

3. An Adaptable and Simpler Criterion of Matrix Acceptance

In this paper, we present a new criterion for acceptance and a new index for representing consistency in pairwise reciprocal comparison matrices. This index and criterion allows the decision maker to study the consistency of each matrix in an adaptable way. Using the index and criterion that we present, the user can decide about the matrix consistency using not only the matrix entries but also the level of consistency that the decision maker needs in this particular case.

We will use the maximum right eigenvalue (λ_{max}) of each studied matrix as a consistency index, and this index is simpler than Saaty´s (CI). The main idea is that a matrix is consistent or not depending on the scope. In different situations, the decision maker might need different levels of consistency and he/she can represent these levels

using percentages. One specific matrix is therefore either consistent or not (i.e. either accepted or not as a consistent matrix) depending on two different factors:

a) A consistency index (λ_{max})

b) The level of consistency needed (α), $0 < \alpha \leq 1$. This level α provides adaptability to different scopes.

In this case, we can decide if a specific matrix is a sufficiently consistent matrix (or not) as a Boolean function with two parameters, λ_{max} and α.

$$F\left(\lambda_{\max}, \alpha\right) \tag{10}$$

What is the real meaning of this α? This number relates the value of the error (consistency error) which we calculate in our matrix to the "average error" of the matrices with the same dimension as our matrix.

We can define the "average error" (difference between the mean of λ_{max} and the "perfect" one *n*) of the matrices of a specific dimension *n* as

$$Avg\ Err\left(n\right) = \overline{\lambda}_{\max}\left(n\right) - n \tag{11}$$

and we can define the error of a specific matrix (*mat*) of this dimension *n* as

$$Err\left(mat\right) = \lambda_{\max}\left(mat\right) - n \tag{12}$$

In this situation, we consider the matrix *mat* to be consistent if and only if

$$\frac{Err\left(mat\right)}{Avg\ Err\left(mat\right)} \leq \alpha \tag{13}$$

We can compute this criterion easily. We will consider a matrix to be sufficiently consistent if and only if

$$\lambda_{\max} - n \leq \alpha\left(\overline{\lambda}_{\max}\left(n\right) - n\right) \tag{14}$$

and using the adjustment line (Eq. 9)

$$\lambda_{\max} \leq n + \alpha\left(1.7699n - 4.3513\right) \tag{15}$$

We can therefore easily compute our criterion in this way. We only accept a matrix of a specific dimension *n*, and with a certain level of consistency needed α if and only if Equation 15 is satisfied. In our system, the simplicity of our criterion and its adaptability is guaranteed. In Table 5, we can see the maximum λ_{max} values that are accepted by our system for different dimensions and levels of consistency.

Table 5. Showing the accepted maximum λ_{max} for various dimensions and α.

n	**3**	**5**	**10**	**15**	**20**	**50**	**100**	**500**
α								
0.01	3.0096	5.0450	10.1335	15.2220	20.3104	50.8414	101.7264	508.8060
0.05	3.0478	5.2248	10.6673	16.1098	21.5523	54.2071	108.6319	544.0299
0.08	3.0765	5.3597	11.0677	16.7756	22.4836	56.7314	113.8110	570.4478
0.10	3.0957	5.4497	11.3346	17.2196	23.1045	58.4142	117.2637	588.0597
0.20	3.1913	5.8993	12.6692	19.4391	26.2090	66.8284	134.5274	676.1194
0.50	3.4784	7.2483	16.6730	26.0978	35.5225	92.0710	186.3185	940.2985

3.1. *Relation between our acceptance system and Saaty's acceptance system*

Using the definition for the consistency index and consistency ratio (Eq. 5), and the traditional acceptance criterion

$$CR = \frac{CI}{RI} < 0.1 \tag{16}$$

and taking into account the RI definition as a mean of CI [2]

$$CI = \frac{\lambda_{\max} - n}{n-1} \quad \text{and} \quad RI = \frac{\overline{\lambda}_{\max} - n}{n-1} \tag{17}$$

we can infer that

$$CR = \frac{\lambda_{\max} - n}{\overline{\lambda}_{\max} - n} < 0.1 \tag{18}$$

Saaty therefore only considers a matrix to be consistent if and only if

$$\lambda_{\max} < n + 0.1\left(\overline{\lambda}_{\max} - n\right) \tag{19}$$

Using the previously calculated least-square adjustment straight line

$$\overline{\lambda}_{\max}(n) = 2.7699\, n - 4.3513 \tag{20}$$

we can conclude that Saaty's consistency criterion using eigenvalue can be expressed as

$$\lambda_{\max} < n + 0.1\left(1.7699n - 4.3513\right) \tag{21}$$

and thus

$$\max \lambda_{\max}\, err = 0.1\left(1.7699n - 4.3513\right) \tag{22}$$

which represents the maximum error that Saaty´s accepts in λ_{max}

Table 6. Showing the maximum λ_{max} accepted by Saaty depending on the number of alternatives (3 to 15).

n	**3**	**4**	**5**	**6**	**7**	**8**	**9**
λ_{max}	3.0957	4.2727	5.4497	6.6266	7.8036	8.9806	10.1576
	10	**11**	**12**	**13**	**14**	**15**	
	11.3346	12.5116	13.6886	14.8656	16.0426	17.2196	

5. Conclusions

In this paper, we have presented an estimation method for RI values when n increases which is easily computable and generates a good estimation of RI. We have used this estimation to propose a new system and criterion of accepting/rejecting matrices (based on their inconsistency) in the analytic hierarchy process. Our system uses a consistency index which is simpler than Saaty´s (λ_{max}) and a very simple criterion for accepting or rejecting matrices (Eq 9).

Furthermore, our system is able to accept different levels of consistency required (to adapt the criterion to more or less restrictive situations), and uses a relative criterion to decide whether a matrix should be accepted or not as consistent. Our system compares the matrix level of consistency to the consistency level of the remaining matrices (random matrices) of the same dimension. Taking into account the dimension of the matrices (structural inconsistency), and the different levels of consistency required, our (relative) system offers clear advantages over the traditional system. Additionally, our method for calculating the RI for very large dimension matrices (with an insignificant computational cost) enables the acceptance criterion to be easily used with this type of matrix, something which was previously not possible and this is an innovation.

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