Postnikov invariants of crossed complexes

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Received 11 June 2004
Available online 15 January 2005
Communicated by William Crawley-Boevey

Abstract

We determine the Postnikov tower and Postnikov invariants of a crossed complex in a purely algebraic way. Using the fact that crossed complexes are homotopy types for filtered spaces, we use the above “algebraically defined” Postnikov tower and Postnikov invariants to obtain from them those of filtered spaces. We argue that a similar “purely algebraic” approach to Postnikov invariants may also be used in other categories of spaces.

Keywords: Crossed complexes; Postnikov towers; Postnikov invariants; Filtered spaces

1. Introduction

The theory of Postnikov towers provides both, a way of analyzing a space $X$ from the point of view of its homotopy groups, and a prescription for the construction of spaces with
specified homotopy groups in each dimension. The required data for this construction is the information contained in the Postnikov tower of the space: a diagram of spaces

$$\cdots \to X_{n+1} \xrightarrow{\eta_{n+1}} X_n \xrightarrow{\eta_n} X_{n-1} \to \cdots \to X_0,$$

whose inverse limit is a space with the same homotopy type as the given space and where each map $\eta_n$ is a fibration whose fibers are Eilenberg–Mac Lane spaces of the type of a $K(\Pi, n)$.

The Postnikov invariants of the space $X$ are cohomology invariants, denoted $k_n$, $n \geq 1$, which provide the necessary information in order to build the Postnikov tower of $X$ floor by floor. The Postnikov invariant $k_n$, associated to the fibration $\eta_n$, says how to glue $K(\Pi, n)$ spaces into the space $X_{n-1}$ to form $X_n$.

The purpose of this paper is to present a purely algebraic approach to the calculation of the Postnikov invariants of a space in the sense that it avoids the use of topological tools such as universal covering; these are replaced by algebraic tools such as free resolutions.

One of the motivations for such an algebraic approach is the fact that it offers the possibility of applying it to more complicated contexts such as categories of diagrams of spaces. Previous work in this direction can be seen in [8], where the third equivariant Postnikov invariant of a $G$-space is calculated by purely algebraic methods of the same nature as those presented here.

The topological approach to the construction of a Postnikov tower of a given space $X$ relies on the process of building spaces $P_n(X)$ with the same homotopy $n$-type as $X$ by the method of killing the higher homotopy groups by adding cells. This process is not functorial “on the nose,” although it can be made so by passing to certain quotient categories (see [2, p. 56]). On the other hand, in the algebraic case a similar process can be devised which is already functorial, this being one of the advantages of this approach.

Our approach to Postnikov towers and Postnikov invariants of the spaces in a given category of spaces $\mathcal{T}$ is based in the existence of an algebraic category $\mathcal{S}$ with a Quillen model structure, together with a pair of functors $\Pi : \mathcal{T} \to \mathcal{S}$, $B : \mathcal{S} \to \mathcal{T}$ which induce an equivalence in the corresponding homotopy categories. Hence $\mathcal{S}$ is a category of algebraic models for the homotopy types of the spaces in $\mathcal{T}$. In this situation, the calculation of the Postnikov towers of the spaces in $\mathcal{T}$ can be reduced to calculating Postnikov towers in $\mathcal{S}$ provided that the “classifying space” functor $B$ preserves fibrations as well as the homotopy type of their fibers. This is the case for the functors which are the object of this paper. These are, on the one hand, the functor $\Pi =$ “Fundamental crossed complex of the singular complex of a space,” and on the other hand, the functor $B =$ “Geometric realization of the nerve of a crossed complex” (see below).

The calculation of Postnikov towers of the algebraic models which are the objects of $\mathcal{S}$ is based in the following general scheme: For every non-negative integer $n$ we fix a full, reflective subcategory $i_n : \mathcal{S}_n \to \mathcal{S}$ whose objects model all “homotopy $n$-types” in $\mathcal{S}$, and such that $\mathcal{S}_n$ is contained in $\mathcal{S}_{n+1}$ in such a way that the inclusions $j_n : \mathcal{S}_n \to \mathcal{S}_{n+1}$ satisfy $i_{n+1}j_n = i_n$. Then if $\tilde{P}_n$ is the left adjoint to $i_n$, the identity $\tilde{P}_{n+1}i_{n+1} = 1_{\mathcal{S}_{n+1}}$ implies
\[ \tilde{P}_{n+1}i_n = j_n \text{ and the composites } P_n = i_n\tilde{P}_n \text{ are idempotent endofunctors of } \mathcal{S} \text{ verifying } P_{n+1}P_n = P_n \simeq P_nP_{n+1}, \text{ and related by a chain of natural transformations} \\
\cdots \longrightarrow P_{n+1} \xrightarrow{\eta_{n+1}} P_n \xrightarrow{\eta_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0, \quad (1) \]

(where \( \eta_{n+1} \) is the image by \( P_{n+1} \) of the unit of the adjunction \( \tilde{P}_n \dashv i_n \)). Then we prove that this chain is the "universal Postnikov tower" in \( \mathcal{S} \), in the sense that for any object \( C \in \mathcal{S} \) the evaluation of the above chain (1) in \( C \) yields the Postnikov tower of \( C \).

The sequence of functors \( j_n \) reminds us of the concept of tower of categories defined by Baues in [1] (and used there to obtain Postnikov towers), except that the functors in a tower of categories go in the opposite direction. Actually, by the above properties, the functors \( j_n \), are all reflective. If we denote \( \lambda_n = \tilde{P}_n i_{n+1} \) the left adjoint to \( j_n \), we obtain a sequence of functors

\[ \cdots \longrightarrow \mathcal{S}_{n+1} \xrightarrow{\lambda_{n+1}} \mathcal{S}_n \xrightarrow{\lambda_n} \mathcal{S}_{n-1} \longrightarrow \cdots, \]

which seems to be part of the structure of a tower of categories in Baues’ sense. It would be interesting to confirm this to be so, and to verify that it leads, by Baues’ methods, to the same results we obtain here. This is something beyond the scope of this work, which we must leave for a separate paper.

With regards to the Postnikov invariants, it is noteworthy the simple form taken by the fibrations of the Postnikov towers of the objects of \( \mathcal{S} \), making it easy to analyze them and to show that the component of \( \eta_{n+1} \) in each object can be interpreted as a 2-extension, a 2-torsor, and then it gives rise to an element \( k_{n+1} \) of a 2-dimensional algebraic (cotriple) cohomology in (a slice of) the category \( \mathcal{S}_n \) of algebraic \( n \)-types. We regard such 2-dimensional cohomology element as a sort of "algebraic Postnikov invariant," the "topological" one residing in a \((n+2)\)-dimensional singular cohomology.

The last step in our approach consists in obtaining the topological Postnikov invariants from the algebraic ones. This is achieved by showing the existence of a natural map from the algebraic 2-cohomology of an algebraic \( n \)-type in \( \mathcal{S}_n \) to the singular \((n+2)\)-cohomology of its corresponding classifying space.

The ideal scenario offering the necessary tools to apply the algebraic approach just described is that in which \( T \) is the category of CW-complexes and \( \mathcal{S} \) is the category of simplicial groupoids. The main interest of this context lies, of course, in the fact that simplicial groupoids model all homotopy types and, therefore, a procedure to calculate the algebraic Postnikov invariants of simplicial groupoids could be used to obtain the Postnikov invariants of any space. The work presented here is, however, more modest in scope and it is, in fact, a preliminary step in that direction. We carry out the general method described above in the category of crossed complexes, a category which does not model all homotopy types. However, although our present results cannot be used to obtain the Postnikov invariants of all spaces, they are, of course, sufficient to obtain the Postnikov invariants of any space having the homotopy type of a crossed complex.

The general plan of the paper is as follows: Section 2 serves to set-up our notation and to introduce the definition and main facts about crossed modules that are used in the paper.
Everything here is review material which can be found elsewhere in the literature, except that it is presented in a, perhaps, slightly non-conventional way, with an emphasis in the functorial aspect of the definitions. We apologize for any distraction this may cause to those readers who are already familiar with the subject. Section 3 introduces crossed complexes, the categories of $n$-types in crossed complexes and the Postnikov towers they give rise to. Crossed complexes are again introduced in a slightly non-conventional way, being defined in terms of crossed modules instead of in terms of groupoids, as it is customary. Choosing a definition which is based on a more elaborate concept not only simplifies the definition itself but, more important, it allows simpler and clearer reasonings and proofs. We also show in this section that the geometric realizations of these Postnikov towers are the Postnikov towers of spaces. Section 4 is the main section of the paper. Here the fibrations in the Postnikov towers of crossed complexes are analyzed and interpreted as extensions, torsors, and therefore, as a consequence of Duskin’s interpretation theorem, as cohomology elements in a cotriple cohomology. Finally a general theorem is proved showing how to map the cotriple cohomology of crossed complexes to the singular cohomology of its classifying spaces. Appendix A contains the basic definitions and results about torsors and their role in Duskin’s interpretation theorem of cotriple cohomology. This material, essential for the main results of the paper, is well-known to the specialist but it is not so well-known in larger circles. It has been put in an appendix in order not to break the discourse and to allow the reader to focus on the main line of reasoning.

2. Crossed modules

We denote $\text{Gr}$ and $\text{Gpd}$ the categories of groups and small groupoids, that is, the category of internal groupoids in the category $\text{Set}$ of sets. By $\text{TdGpd}$ we denote the full subcategory of $\text{Gpd}$ determined by the totally disconnected groupoids. If $X$ is a set, $\text{Gpd}_X$ denotes the subcategory of $\text{Gpd}$ whose objects are all groupoids with set of objects $X$ and whose arrows are functors which are the identity on objects. Similarly, $\text{TdGpd}_X$ denotes the full subcategory of $\text{Gpd}_X$ determined by the totally disconnected groupoids. Clearly, $\text{TdGpd}_X$ can be identified with the category $\text{Gr}(\text{Set}/X)$ of internal group objects in the slice category $\text{Set}/X$. For a given groupoid $G$ we denote $\text{obj}(G)$ its set of objects, and $\text{arr}(G)$ its set of arrows. It is clear that $\text{obj}$ determines a functor $\text{obj} : \text{TdGpd} \to \text{Set}$ whose fiber over a set $X$ is the category $\text{TdGpd}_X$.

If $G$ is a groupoid, a (left) $G$-group is a functor from $G$ to $\text{Gr}$. We will use exponential notation to denote functor categories, so that the category of (always left-) $G$-groups will be denoted $\text{Gr}^G$. An important example of a $G$-group is the functor $\text{End}_G : G \to \text{Gr}$ taking each object of $G$ to its group of endomorphisms and each arrow $u$ in $G$ to the group homomorphism (really an iso) given by conjugation by $u$. This $G$-group is often referred to as the groupoid $G$ acting on itself by conjugation.

A given $G$-group $C : G \to \text{Gr}$ is often determined in terms of an action of (the arrows of) $G$ on (the arrows of) a totally disconnect groupoid, $\hat{C}$, whose set of object is $\text{obj}(G)$ and whose endomorphism groups are $\text{End}_{\hat{C}}(x) = C(x)$. This action of $G$ on $\hat{C}$ is traditionally denoted

$$t^*u = C(t)(u),$$
for $t : x \rightarrow y$ an arrow in $G$ and $u$ an element in $C(x)$. This description of $G$-groups is
objectified by a full and faithful functor $(\cdot): \text{Gr} G \rightarrow \text{TdGpd}_{\text{obj}(G)}$ which reflects zero objects and zero maps, and therefore not only preserves but also reflects chain complexes.

Obviously, $\text{End}_G = \text{End}(G)$, the subcategory of $G$ consisting of just its endomorphisms.

For a given groupoid $G$ we denote $\text{Pxm}_G$ the category of pre-crossed modules over $G$, which we define as the slice category $\text{Pxm}_G = \text{Gr} G / \text{End}_G$. The initial and terminal objects in $\text{Pxm}_G$ are denoted $0_G$ and $1_G$, respectively, so that $0_G$ is a constant zero functor $G \rightarrow \text{Gr}$ together with the unique natural transformation from it to $\text{End}_G$, while $1_G$ is the functor $\text{End}_G$ together with its identity map. Note that $0_G = 1_G$ if and only if $G$ is discrete as category (that is, all arrows in $G$ are identities).

If $G$ and $G'$ are any two groupoids and $(C, \delta), (C', \delta')$ are pre-crossed modules respectively over $G$ and $G'$, a morphism of pre-crossed modules from $(C, \delta)$ to $(C', \delta')$ is a pair $(f, \alpha)$ where $f: G \rightarrow G'$ is a change-of-base functor and $\alpha: C \rightarrow C' \circ f$ is a natural transformation such that $(\delta' \ast f) \circ \alpha = \tilde{f} \circ \delta$, where $\tilde{f}$ is the same functor $f$ but regarded as natural transformation from $\text{End}_G$ to $\text{End}_{G'} \circ f$.

For an object $x \in G$ and an element $u \in C(x)$, this condition reads $f(\delta_x(u)) = \delta'_f(x)(\alpha_x(u))$. The general morphisms of pre-crossed modules just defined are the arrows of the category of pre-crossed modules, denoted $\text{Pxm}$. The structure of an object in $\text{Pxm}$ can be described as a triple $(G, C, \delta)$ where $G$ is a groupoid and $(C, \delta)$ is a pre-crossed module over $G$. By a reduced pre-crossed module we mean one in which $G$ is just a group.

The following proposition provides the definition of the fundamental groupoid of a pre-crossed module. Note that if $(C, \delta)$ is a pre-crossed module over $G$, by applying the functor $(\cdot): C \rightarrow \text{End}_G$, we obtain a functor $\tilde{\delta}: \tilde{C} \rightarrow \text{End}(G)$ which ($\tilde{C}$ being totally disconnected) is equivalent to a functor $\tilde{C} \rightarrow G$. The latter will not be distinguished from $\tilde{\delta}$.

Proposition 2.1. The categories $\text{Pxm}_G$ are the fibres of a fibration “base groupoid of a pre-crossed module,” $\text{base}: \text{Pxm} \rightarrow \text{Gpd}$. This functor has both adjoints $\text{discr} \dashv \text{base} \dashv \text{codiscr}$ given by the initial (left) and terminal (right) objects in the corresponding fibres. Furthermore, the left adjoint $\text{discr}$ has a further left adjoint “fundamental groupoid” $\pi_1 \dashv \text{discr}$.

Proof. Everything is quite standard; we just comment on the last statement. The fundamental groupoid of a pre-crossed module $C = (G, C, \delta)$ is calculated by the coequalizer

$$
\begin{array}{ccc}
C & \xrightarrow{\alpha} & C' \circ f \\
\downarrow & & \downarrow \delta' \ast f \\
\text{End}_G & \xrightarrow{f} & \text{End}_{G'} \circ f.
\end{array}
$$

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Proof. Everything is quite standard; we just comment on the last statement. The fundamental groupoid of a pre-crossed module $C = (G, C, \delta)$ is calculated by the coequalizer

$$
\begin{array}{ccc}
\tilde{C} & \xrightarrow{\delta} & G \\
\downarrow & & \downarrow \pi_1(C),
\end{array}
$$
that is, the fundamental groupoid is given by the quotient \( \pi_1(C) = G / \operatorname{im}(\hat{\delta}) \). Note that all functors in the above diagram are the identity on objects. \( \square \)

For a given pre-crossed module \( C = (G, C, \delta) \) the functors \( C \circ \hat{\delta} \) and \( \operatorname{End}_C \hat{\delta} \) agree on objects but, in general, not on arrows. Therefore, that these two functors be equal is a special property a pre-crossed module may have.

**Definition 2.2.** A crossed module is a pre-crossed module \( C = (G, C, \delta) \) such that \( C \circ \hat{\delta} = \operatorname{End}_C \hat{\delta} \). The category of crossed modules, denoted \( \mathbf{Xm} \), is the corresponding full subcategory of \( \mathbf{Pxm} \). For a given groupoid \( G \), the category of \( G \)-crossed modules, denoted \( \mathbf{Xm}_G \), is the obvious full subcategory of \( \mathbf{Pxm}_G \).

In terms of elements, the condition that \( C \circ \hat{\delta} \) and \( \operatorname{End}_C \hat{\delta} \) agree on arrows reads:

\[
\delta_x(u)v = uvu^{-1},
\]
for all objects \( x \in G \) and elements \( u, v \in C(x) \). This is the well-known Peiffer identity. This property implies that \( \ker \delta_x \) is contained in the center of \( C(x) \), and this, in turn, has the following important consequences:

1. For any given \( G \)-group \( C \), the \( G \)-pre-crossed module \( (C, 0) \) is a crossed module if and only if every group \( C(x) \) is abelian, that is, if \( C \) is a \( G \)-module.
2. For any \( G \)-crossed module \( (C, \delta) \), the kernel of \( \delta \) (calculated in \( \mathbf{Gr}^G \)) is a \( G \)-module (that is, \( (\ker \delta)(x) = \ker \delta_x \) is an abelian group).
3. The action of \( \operatorname{im} \delta \) on \( \ker \delta \) is trivial, that is, the following diagram commutes:

\[
\begin{array}{ccc}
\hat{C} & \overset{\delta}{\longrightarrow} & G \\
\vline & \vline & \vline \\
& \ker \delta & \longrightarrow \Ab
\end{array}
\]

where we have denoted “0” the functor which is the identity on objects and sends every map to an identity; this functor, if regarded as a map in \( \mathbf{Gpd}_{\operatorname{Adj}(G)} \), is indeed a zero map.

**Examples.**

1. For any \( G \)-module \( A : G \rightarrow \Ab \), the pre-crossed module \( \text{zero}(A) = (G, A, 0) \) is a crossed module.
2. Any pre-crossed module \( (G, C, \delta) \) with \( \delta \) a monomorphism is a crossed module.
3. In particular, for any groupoid \( G \), the pre-crossed modules \( 0_G \) and \( 1_G \) are crossed modules.

As a consequence of the above Example 3, the right and left adjoints to “base groupoid of a pre-crossed module” are also right and left adjoints to “base groupoid of a crossed module” and furthermore “fundamental groupoid of a crossed module” is left adjoint to
“discrete crossed module on a groupoid.” From now on we will regard the functors in the
sequence of adjunctions \( \pi_1 \dashv \text{discr} \dashv \text{base} \dashv \text{codiscr} \) of Proposition 2.1 as defined/taking
values in \( \mathbb{X}m \).

There is an important forgetful functor defined on the category of \( G \)-crossed modules,
which will be used later on. This is the functor
\[
\text{ceil}_2 = \text{ceil} : \mathbb{X}m_G \to \text{Gr}^G,
\]
(3)
taking a \( G \)-crossed module \( C = (G, C, \delta) \) to the \( G \)-group \( C \) and each map \((1_G, \alpha): (G, C, \delta) \to (G, C', \delta')\) of \( G \)-crossed modules to the natural transformation \( \alpha : C \to C' \).

It is an important property of this functor the fact that it preserves finite limits and coequalizers.

Our next objective is to establish the tripleability of the category of crossed modules
over a certain category (see below) so that we can define the cotriple which will be used to
calculate an algebraic cohomology of crossed modules.

**Proposition 2.3.** The inclusion functor \( U : \mathbb{X}m \to \mathbb{P}xm \) has a left adjoint which is calculated by factoring out the Peiffer subgroup.

**Proof.** See [7, p. 9], [6], or [16]. \( \Box \)

This inclusion functor \( U : \mathbb{X}m \to \mathbb{P}xm \) is in fact monadic. We will use it to obtain, by
composing it with a certain forgetful functor \( U' : \mathbb{P}xm \to \mathbb{A}Gpd \), another monadic functor
which will determine the cotriple on \( \mathbb{X}m \) by means of which we will calculate the coho-
mology of crossed modules. Let \( \mathbb{A}Gpd \) be the category of “arrows to groupoids” whose
objects are triples \((X, f, G)\) where \( X \) is a set, \( G \) is a groupoid, and \( f : X \to \text{End}(G) \) is
a map from \( X \) to the set of all arrows of \( G \) which are endomorphisms. An arrow from
\((X, f, G)\) to \((X', f', G')\) in \( \mathbb{A}Gpd \) is a pair \((\alpha, \beta)\) where \( \alpha : X \to X' \) is a map of sets, and
\( \beta : G \to G' \) is a functor such that \( f' \alpha = \beta f \). Then we have:

**Proposition 2.4.** The obvious forgetful functor \( U' : \mathbb{P}xm \to \mathbb{A}Gpd \) has a left adjoint.

**Proof.** The forgetful functor \( U' : \mathbb{P}xm \to \mathbb{A}Gpd \) takes a pre-crossed module \((G, C, \delta)\) to the triple \((\text{arr}(\hat{C}), \delta, G)\). We will merely give the definition of its left adjoint \( F : \mathbb{A}Gpd \to \mathbb{P}xm \). This is defined on objects as \( F(X, f, G) = (G, C, \delta) \), where \( C : G \to \text{Gr} \) is defined on objects as
\[
C(x) = F_{\text{gp}} \left( \prod_{z \in \text{obj}(G)} \left( \text{Gr}(z, x) \times \prod_{v \in \text{obj}(G)} \text{fbr}(f, v) \right) \right),
\]
\( F_{\text{gp}} \) is the free group functor, and \( \text{fbr}(f, v) \) is the fiber of the map \( f \) at \( v \). Thus, \( C(x) \) is the free group generated by all pairs \((t, u)\), where \( t : z \to x \) is a map in \( G \) and \( u \in X \) is such that \( f(u) \) is an endomorphism of \( z \) in \( G \). The \( G \)-group \( C \) is defined on arrows \( s : x \to y \) in \( G \) by defining on generators:
\[
C(s)(t, u) = (st, u).
\]
The natural map $\delta : C \to \text{End}_G$ has components $\delta_x : C(x) \to \text{End}_G(x)$ which are defined on generators as

$$\delta_x(t,u) = tf(u)t^{-1}.$$ 

It is easy to show that this defines $F$ on objects. Now on arrows: For an arrow $(\alpha, \beta) : (X, f, G) \to (X', f', G')$ in $\text{AGpd}$, we define $F(\alpha, \beta) = (\beta, \hat{\alpha})$, where $\hat{\alpha} : C \to C' \circ \beta$ has components defined on generators by

$$\hat{\alpha}_x(t,u) = (\beta(t), \alpha(u)).$$

In this way we get a functor $F : \text{AGpd} \to \text{Pxm}$. This is easily verified to be left adjoint to $U'$ (see [13, Proposición 3.1.13] for the details). □

**Proposition 2.5.** The composite functor $U_2 : \text{Xm} \xrightarrow{U} \text{Pxm} \xrightarrow{U'} \text{AGpd}$ is monadic.

**Proof.** We already know that $U_2$ has a left adjoint. By Beck’s tripleability theorem it is sufficient to prove that it reflects isomorphisms and that it preserves coequalizers of $U_2$-contractible pairs. The first thing is easy to see. The second thing requires a careful analysis of coequalizers in $\text{Xm}$ and in $\text{AGpd}$. It is proved in this way: on the coequalizer of the $U_2$-image of a $U_2$-contractible pair one can build in a natural way a structure of crossed module together with a map in $\text{Xm}$ from the codomain of the given contractible pair to this crossed module. After some tedious calculations one verifies that this map is a coequalizer in $\text{Xm}$ and that its image is the coequalizer in $\text{AGpd}$ from which it was built, proving the desired property. See [13, Proposición 3.1.15, p. 148] for the details. □

We denote $G_2$ the cotriple induced on $\text{Xm}$ by the monadic functor $U_2$, that is, $G_2 = F_2U_2$.

The next proposition reminds us of the well-known fact that crossed modules can be regarded as groupoids (actually, as 2-groupoids or groupoids enriched in the category of groupoids). For some purposes in our context it will be convenient to regard 2-groupoids (i.e., crossed modules) as those special double groupoids (or internal groupoids in the category of groupoids) whose groupoids of objects and of arrows have the same set of objects and whose structural functors (domain, codomain, identity, and composition) are the identity on objects.

**Proposition 2.6.** There is a functor $\text{Xm} : \text{Gpd}(\text{Gpd}) \to \text{Xm}$, from the category of double groupoids to that of crossed modules, which has a pseudo section

$$\text{gpd} : \text{Xm} \to \text{Gpd}(\text{Gpd}),$$

allowing us to regard any crossed module $(G, C, \delta)$ as an internal groupoid in groupoids, having $G$ as groupoid of objects and with groupoid of arrows given by the “semidirect product” or Grothendieck construction $G \ltimes C = \int G C$. Furthermore, the above functors establish an isomorphism between the category of crossed modules and the category of 2-groupoids.
Proof. Let \((G_0, G_1, s, t, i)\) be the underlying reflexive graph of \(G\), a double groupoid. We define a pre-crossed module \(\text{xm}(G) = (C, \delta)\) over the groupoid \(G_0\) of objects of \(G\) by

\[
C = (\ker \tilde{s}) \circ i \quad \text{and} \quad \delta = (\tilde{f} \circ j) \ast i,
\]

where \(j : \ker \tilde{s} \rightarrow \text{End}_{G_1}\) is the canonical inclusion, and we use again the notation of tilde to denote the natural transformation \(\tilde{f} : \text{End}_{G_1} \rightarrow \text{End}_{G_1} \circ f\) induced by a functor \(f : G \rightarrow G'\). It is immediate to verify that \(\delta\) satisfies Peiffer's identity and therefore \(\text{xm}(G)\) is a crossed module. This defines the functor \(\text{xm}\) on objects. On arrows \((f_0, f_1) : G \rightarrow G'\), \(\text{xm}\) is defined by \(\text{xm}(f_0, f_1) = (f_0, \alpha)\), where \(\alpha = (f_1 \circ j) \ast i\).

Let us now define the functor \(\text{gpd} : \text{Xm} \rightarrow \text{Gpd(Gpd)}\). Given a \(G\)-crossed module \(C = (C, \delta)\) by applying Grothendieck's semidirect product construction to \(C\), we get a groupoid \(G \ltimes C\) together with a canonical split projection

\[
G \ltimes C \xrightarrow{i} G,
\]

which is the identity on objects. Then the underlying reflexive graph of \(\text{gpd}(C)\) is \((G, G \ltimes C, s, t, i)\), were the functor \(i\) is the identity on objects and takes any arrow \((u, a) : x \rightarrow y\) in \(G \ltimes C\) \((u : x \rightarrow y\) an arrow in \(G\) and \(a \in C(y)\)) to the composition \(\delta_y(a) \circ u\). The composition map making this graph into an internal groupoid in groupoids is the only possible one, which on the arrows \(x \rightarrow y\) of \(G \ltimes C\) is given by the formula

\[
(v, b) \circ (u, a) = (u, ba) \quad (\text{supposed } v = \delta_y(a)u).
\]

This double groupoid is in fact a 2-groupoid. To an arrow \((f, \alpha) : (G, C, \delta) \rightarrow (G', C', \delta')\) \(\text{gpd}\) associates the map of crossed modules \(\text{gpd}(f, \alpha) = (f, \alpha')\) where \(\alpha' : G \ltimes C \rightarrow G' \ltimes C'\) is the functor defined by

\[
\alpha'(u, a) = (f(u), \alpha_y(a)),
\]

for each \(u : x \rightarrow y\) and \(a \in C(y)\). It is easy to verify that the crossed module corresponding to \(\text{gpd}(C)\) is isomorphic to \(C\) and also that for any 2-groupoid \(G\) the 2-groupoid \(\text{gpd}(\text{xm}(G))\) is isomorphic to \(G\). □

Note that for any groupoid \(G\) the functors \(\text{xm}\) and \(\text{gpd}\) induce an equivalence between the category \(\text{Xm}_G\) and the subcategory of those 2-groupoids determined by those 2-groupoids having \(G\) as groupoid of objects and by those functors which are the identity on objects. We note also that, the fundamental groupoid of a crossed module \(C\) is equal to the groupoid of connected components of the 2-groupoid \(\text{gpd}(C)\).
By the set of connected components of a (pre-) crossed module, \( \pi_0(C) \), it is understood
the set of connected components of its base groupoid. Besides \( \pi_0(C) \) and \( \pi_1(C) \), the commutativity of diagram (2) allows us to define the second “homotopy group,” \( \pi_2(C) \), of the
crossed module \( C \), as the unique \( \pi_1(C) \)-module such that \( \pi_2(C) \circ q = \ker \delta \),

\[
\begin{array}{ccc}
\widehat{C} & \xrightarrow{\delta} & G \\
\downarrow{\delta} & & \downarrow{q} \\
\ker \delta & \rightarrow & \pi_1(C) \\
& \xleftarrow{\pi_2(C)} & \leftarrow \Ab
\end{array}
\]

(5)

3. Crossed complexes and their Postnikov towers

As indicated in the Introduction, we give a definition of crossed complex which rests on
the concept of crossed module instead of (the usual) building crossed complexes all the way
from groupoids. Crossed complexes over a fixed groupoid are very easy to define as special
types of chain complexes in the category of crossed modules over the given groupoid.
Having done that, it is evident how to define morphisms between crossed complexes over
different groupoids to get the full category of crossed complexes. Standard references for
crossed complexes are [3,4,18].

If \( G \) is a fixed groupoid, a chain complex in \( \Xm_G \) is a diagram

\[
\cdots \rightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \cdots \rightarrow C_2 \xrightarrow{\partial_2} C_1,
\]

of \( G \)-crossed modules whose underlying diagram of \( G \)-groups is a chain complex in \( \Gr^G \).
In such a diagram, the fact that (for \( n > 1 \)) there is a zero map from \( C_{n+1} \) to \( C_{n-1} \), in \( \Xm_G \), implies that in the crossed module \( C_{n+1} = (C_{n+1}, \delta_{n+1}) \), \( \delta_{n+1} = 0 \) and therefore \( C_{n+1} \) is
abelian, meaning that it is not just a \( G \)-group but a \( G \)-module, \( C_{n+1} : G \rightarrow \Ab \). Thus, in
a chain complex in \( \Xm_G \) such as the above one, for all \( n \geq 3 \), \( C_n \) is an abelian crossed
module.

Definition 3.1. If \( G \) is a groupoid, a \( G \)-crossed complex is a chain complex in \( \Xm_G \) of the form

\[
\mathcal{C} : \cdots \rightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \cdots \rightarrow C_2 \xrightarrow{\partial_2} 1_G,
\]

(6)
such that for \( n \geq 3 \) the action of \( \text{im}(\partial_2) \) on \( \widehat{C}_n \) is trivial. In other words, for every \( n \geq 3 \) the
following diagram commutes:

\[
\begin{array}{ccc}
\widehat{C}_2 & \xrightarrow{\delta_2} & G \\
\downarrow{0} & & \downarrow{q} \\
& \rightarrow & \Ab
\end{array}
\]

(7)
Here we are using the notation $C_n = (C_n, \delta_n)$, $n \geq 2$, for the crossed modules in $C$. The groupoid $G$ is called the base groupoid of the crossed complex, and $C_2$ is called the base crossed module.

For $n \geq 3$, by the commutativity of (7), $C_n$ induces a $\pi_1(C_2)$-module $\hat{C}_n$.

$$
\begin{array}{ccc}
\hat{C}_2 & \xrightarrow{\hat{\delta}_2} & G \\
\downarrow & & \downarrow q \\
C_2 & \xrightarrow{\partial_2} & \pi_1(C_2)
\end{array}
$$

and the crossed complex (6) induces a chain complex of $\pi_1(C_2)$-modules of the form

$$
\cdots \longrightarrow \overline{C}_{n+1} \xrightarrow{\partial_{n+1}} \overline{C}_n \xrightarrow{\partial_n} \overline{C}_{n-1} \longrightarrow \cdots \longrightarrow \overline{C}_3 \xrightarrow{\partial_3} \pi_2(C_2) \longrightarrow 0,
$$

which will be used in the definition of the higher “homotopy groups” of a crossed complex. (We name the natural maps in this chain complex after their corresponding maps in (6) because they are essentially the same, having the same components in $\text{Ab}$).

If $C$ is a $G$-crossed complex and $C'$ is a $G'$-crossed complex, a morphism $f : C \to C'$ is just a chain map, that is, a family $f = \{f_n : C_n \to C'_n\}_{n \geq 1}$ of maps of crossed modules such that for $n \geq 1$, $f_n \partial_{n+1} = \partial_{n+1}' f_n$.

Note that this condition implies that all maps $f_n$ have the same change-of-base functor, which is equal to $f_1$. The resulting category of crossed complexes will be denoted $\text{Crs}$.

A morphism of crossed complexes $f : C \to C'$ is a fibration if each component $f_n$, is a fibration of crossed modules, that is, if the functor $f_1 : G \to G'$ is a fibration of groupoids and the natural map of $G$-groups underlying each $f_n$ is surjective. The fibrations in $\text{Crs}$ are part of a Quillen model structure in this category (see [5]).

An $n$-truncated crossed complex or crossed complex of rank $n$ is a crossed complex such as (6) in which all crossed modules $C_m$ for $m > n$ are equal to $0_G$. The full subcategory of $\text{Crs}$ determined by the $n$-truncated crossed complexes will be denoted $\text{Crs}_n$. For a $G$-crossed complex to be of rank 0 it is necessary that $G$ be a discrete groupoid, that is, just a set. Conversely, associated to a discrete groupoid $G$ there is precisely one 0-truncated crossed complex over $G$. Thus, $\text{Crs}_0$ can be identified with the category of sets and we will put $\text{Crs}_0 = \text{Set}$. Similarly, since a map between two 1-truncated crossed complexes is completely determined by the change-of-base functor, which may be arbitrary, we will identify $\text{Crs}_1$ with the category of groupoids and we write $\text{Crs}_1 = \text{Gpd}$. Finally, we will also write $\text{Crs}_2 = \text{Xm}$ for similar reasons.
The objects in $\text{Crs}_n$ are homotopy $n$-types of crossed complexes in the following sense:

**Definition 3.2.** An object in $\text{Crs}_n$ is said to be a homotopy $n$-type if it has trivial “homotopy groups” in dimensions greater than $n$.

The homotopy groups of a crossed complex are defined as follows: $\pi_0(C)$ is the set of connected components of the base groupoid, so $\pi_0(C) = \pi_0(G)$. Similarly, $\pi_1(C) = \pi_1(C_2) = G / \text{im} (\partial_2)$, the fundamental groupoid of the base crossed module of $C$. For $n \geq 2$, $\pi_n(C)$ is defined as the “homology group” $H_n(C) : \pi_1(C) \to \text{Ab}$ of the induced chain complex of $\pi_1(C)$-modules (9). Note that if we consider $\pi_n(C)$ as a $G$-module via the canonical projection $q : G \to \pi_1(C)$, for $n \geq 2$, the $G$-crossed module $(\pi_n(C), 0)$ is the kernel of the induced map $\tilde{n}_n : C_n / \text{im} (\partial_{n+1}) \to C_{n-1}$ (see (10) below).

In the same way that the discrete inclusion of sets into groupoids is both reflexive and coreflexive, Proposition 2.1 tells us that the “discrete” inclusion $G \hookrightarrow I_G$ of groupoids into crossed modules is both reflexive and coreflexive. These are particular cases of a general situation. For every $n \geq 0$, the subcategory $\text{Crs}_n$ of $\text{Crs}$ is both reflexive and coreflexive.

We are mainly interested in the reflector $\bar{P}_n : \text{Crs} \to \text{Crs}_n$, left adjoint to the inclusion $i_n : \text{Crs}_n \to \text{Crs}$. For $n = 0, 1$, we have $\bar{P}_0 = \pi_0 \circ \text{base}$ (“set of connected components of the base groupoid”) and $\bar{P}_1 = \pi_1 \circ \text{base}$ (“fundamental groupoid of the base crossed module”). For higher $n$, $\bar{P}_n$ is calculated in terms of the following coequalizer in $\text{Xm}$:

$\begin{array}{c}
C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{q_n} C_n / \text{im} \partial_{n+1}.
\end{array}$

Thus $\bar{P}_n$ associates to the crossed complex $\mathcal{C}$ given in (6) the following $n$-truncated crossed complex:

$\bar{P}_n(\mathcal{C}) : \cdots \to \mathbf{0}_G \to C_n / \text{im} \partial_{n+1} \xrightarrow{\tilde{n}_n} C_{n-1} \to \cdots \to C_2 \xrightarrow{\delta_2} I_G,$

(10)

where $\tilde{n}_n$ is the unique map of crossed modules such that $\tilde{n}_n = \bar{n}_n \circ q_n$, induced by the fact that $\partial_{n} \cdot \partial_{n+1} = 0$.

In addition to the fact that the objects in $\text{Crs}_n$ are homotopy $n$-types, all homotopy $n$-types of crossed complexes are represented in $\text{Crs}_n$. That is, if $\mathcal{C} \in \text{Crs}$ is any crossed complex which is a homotopy $n$-type, there exists an object $\mathcal{C}_n \in \text{Crs}_n$ which is weak equivalent to $\mathcal{C}$. That object can be taken to be $\mathcal{C}_n = \bar{P}_n(\mathcal{C})$.

Regarding the reflectors $\bar{P}_n$ as endofunctors, $P_n$, of $\text{Crs}$, we have a situation as described in the introduction. We have idempotent endofunctors $P_n : \text{Crs} \to \text{Crs}$ such that $P_n = P_{n+1} P_n$, and $\eta_{n+1} = P_{n+1} \circ \delta^{(n)}$ is the composition of $P_{n+1}$ with the unit $\delta^{(n)}$ of the adjunction $\bar{P}_n \dashv i_n$. Working out the components of the unit $\delta^{(n)}$, for $n > 1$, one finds that for a given $G$-crossed complex $\mathcal{C}$, the components of the map $(\eta_{n+1}) \mathcal{C} : P_{n+1}(\mathcal{C}) \to P_n(\mathcal{C})$ are: the trivial map $C_{n+1} / \text{im} \partial_{n+2} \to \mathbf{0}_G$ at dimension $n+1$, the “projection to the quotient,” $q_n : C_n \to C_{n-1} / \text{im} \partial_{n+1}$, at dimension $n$, and an identity map at all other dimensions. The cases of $\eta_1$ and $\eta_2$ are little different since for these maps the change-of-base
functor is not an identity. For \( \eta_1 \) the change of base functor is the canonical projection \( q_0 : \pi_1(\mathcal{C}) \to \pi_0(\mathcal{C}) \), while for \( \eta_2 \) it is the canonical projection \( q_1 : \mathcal{G} \to \pi_1(\mathcal{C}) \).

**Proposition 3.3.** For every crossed complex \( \mathcal{C} \in \text{Crs} \), and every \( n \geq 0 \) the map \( (\eta_{n+1})_\mathcal{C} : P_{n+1}(\mathcal{C}) \to P_n(\mathcal{C}) \) is a fibration with fibers of the type of \( K(\Pi, n + 1) \). For \( n > 1 \), the fiber of \( (\eta_{n+1})_\mathcal{C} \) over \( x \in \mathcal{G} \) has the homotopy type of \( K(\pi_{n+1}(\mathcal{C})(x), n + 1) \).

**Proof.** Let us first consider \( (\eta_1)_{\mathcal{C}} \), which is \( q_0 : \pi_1(\mathcal{C}) \to \pi_0(\mathcal{C}) \), a surjective map to a discrete groupoid and therefore it is a fibration of groupoids. The fiber over a given connected component \( x \in \pi_0(\mathcal{C}) \) is a connected groupoid and therefore has the homotopy type of a \( K(\Pi, 1) \) (taking for \( \Pi \) any of the groups of endomorphisms of any object in that connected groupoid).

Next, we look at \( (\eta_2)_{\mathcal{C}} : P_2(\mathcal{C}) \to P_1(\mathcal{C}) \). The fiber of this map over an object \( x \in \mathcal{G} \) is the reduced 2-truncated crossed complex

\[
(C_2/\text{im} \partial_3)(x) : C_2(x)/\text{im}(\partial_3)_x \to \text{im}(\partial_2)_x.
\]

This is a crossed module over the group \( \text{im}(\partial_2)_x \) and therefore it has \( \pi_0 = 0 \). Since the above map is surjective, this crossed module has \( \pi_1 = 0 \), while \( \pi_2 \) is precisely the abelian group \( \pi_2(\mathcal{C})(x) \). For higher \( n \), \( \pi_n = 0 \), thus \( \eta_2 \) is a fibration with fiber over \( x \) of the type \( K(\pi_1(\mathcal{C})(x), 2) \).

For \( n > 2 \) all the \( \eta_n \) are morphisms of crossed complexes whose change of base functor is the identity on objects (as in the case \( n = 2 \)). Therefore, their fiber on an object \( x \in \mathcal{G} \) is a reduced crossed complex. The special thing for \( n > 2 \) is that the base groupoid of the fiber is trivial and therefore the fiber is just a chain complex of abelian groups. In general, the fiber of \( (\eta_n)_{\mathcal{C}} \) (for \( n > 2 \)) over an \( x \in \mathcal{G} \) is a crossed complex with trivial components below the \( n - 1 \) an a surjective morphism in dimension \( n \). Therefore, all homotopy groups of the fiber are trivial in dimensions other than \( n \), and it is equal to \( \pi_n(\mathcal{C})(x) \) in dimension \( n \). \( \square \)

**Proposition 3.4.** For every crossed complex \( \mathcal{C} \), the chain of fibrations

\[
\cdots \to \eta_{n+2} P_{n+1}(\mathcal{C}) \xrightarrow{\eta_{n+1}} P_n(\mathcal{C}) \xrightarrow{\eta_n} \cdots \xrightarrow{\eta_2} P_0(\mathcal{C})
\]

is a Postnikov tower for \( \mathcal{C} \).

**Proof.** By Proposition 3.3, it is sufficient to prove that the limit of diagram (11) is \( \mathcal{C} \). We know that the morphisms \( \delta^{(n)} : \mathcal{C} \to P_n(\mathcal{C}) \) determined by the units of the adjunctions \( P_n \dashv i_n \) constitute a cone over (11). Given any other cone \( \{\phi(n) : \mathcal{C}' \to P_n(\mathcal{C})\} \) over (11) there is a unique way of defining a map of crossed complexes \( f : \mathcal{C}' \to \mathcal{C} \) such that for all \( n \), \( \delta^{(n)} f = \phi^{(n)} \). One just needs to take into account that \( \delta^{(n)}_m \) is an identity map for all \( m < n \) and define \( f_n = \phi^{(n+1)}_n \). \( \square \)

The subcategories \( \text{Crs}_n \) of \( \text{Crs} \) are not only reflexive, but also coreflexive, the right adjoint to the inclusion being “simple truncation,” \( T_n : \text{Crs} \to \text{Crs}_n \), so that \( T_0 \) is essentially
the set of objects of the base groupoid, $T_1$ is "base groupoid," $T_2$ is "base crossed module." Furthermore, $T_n$ has itself a right adjoint denoted $\text{cosk}^n$. For $n = 0$ and $n = 1$ this further right adjoint is "codiscrete groupoid on a set" ($n = 0$) and "trivial crossed module on a groupoid" (so that $\text{cosk}^1(G) = 1_G$ or $\text{cosk}^1(G) = (\cdots 0_G \rightarrow \cdots 0_G \rightarrow 1_G)$.

For $n > 1$, the right adjoint to $T_n$, $\text{cosk}^n : \text{Crs}_n \rightarrow \text{Crs}$, assigns to an $n$-truncated crossed complex

$$C : \cdots \rightarrow 0_G \rightarrow 0_G \xrightarrow{\partial_0} C_n \xrightarrow{\partial_n} C_{n-1} \cdots \rightarrow C_2 \xrightarrow{\partial_2} 1_G,$$

the following $(n + 1)$-truncated crossed complex:

$$\text{cosk}^n(C) : \cdots \rightarrow 0_G \rightarrow \ker \partial_n \xrightarrow{\partial_0} C_n \xrightarrow{\partial_n} C_{n-1} \cdots \rightarrow C_2 \xrightarrow{\partial_2} 1_G.$$

For $n > 2$, the functor

$$\text{ceil}_n : \text{Crs}_n, G \rightarrow \text{Ab}^G,$$

takes each $n$-truncated crossed complex $C$ having $G$ as base groupoid, to the $G$-module $\text{ceil}_n(C) = \text{ceil}(C_n)$. Note that, since finite limits and coequalizers in $\text{Crs}_n, G$ are calculated componentwise, and since the functor $\text{ceil} = \text{ceil}_2$ preserves finite limits and coequalizers, for $n > 2$ the functor $\text{ceil}_n$ also preserves finite limits and coequalizers.

We end this section with a higher dimensional analog of Proposition 2.6. The idea is to regard the $(n + 1)$-truncated crossed complexes as some kind of internal groupoids in the category of $n$-truncated crossed complexes. For $n > 1$ there is a difficulty we do not have in Proposition 2.6. For example, (in case $n = 2$) it is possible to carry out a construction analogous to the one defining the functor $x_n$, but starting with a groupoid internal in crossed modules: $(G_0, C_1, s, t, i, \gamma)$. Let the crossed modules of objects and arrows of this groupoid be $C_i = (G_i, C_i, \delta_i)$, $i = 0, 1$, and let the domain, codomain and identity maps be $s = (f_i, \alpha_i)$, etc. We can define $C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} 1_{G_0}$ where $C_3 = (G_0, \ker(\alpha_0, f_0), \delta_3)$, $\delta = \delta_0, \delta_3 = \alpha_0 \circ f_0$, and $C_2 = C_0$. One does not, however, obtain directly from this construction a 3-crossed complex unless the base groupoids of the crossed modules $C_0, C_1$ are the same and the structural maps ($s$, $t$, $i$ and $\gamma$) have trivial change of base (i.e., $f_i = 1_{G_0}$ etc.). It is of course possible to force the result to be a 3-truncated crossed complex by making the appropriate quotients, but this would introduce unnecessary complication. Thus, we shall appropriately restrict the categories of internal groupoids in crossed complexes so that, for example, in the case $n = 2$ we will only consider those internal groupoids in crossed modules satisfying the conditions said above. In general, for each $n > 0$ let $G\text{Crs}_n$ be the full subcategory of $\text{Gpd}(\text{Crs}_n)$ (internal groupoids in $\text{Crs}_n$) determined by those groupoids $G \in \text{Gpd}(\text{Crs}_n)$ whose $n$-truncated crossed complex of objects has the same $(n - 1)$-truncation as its $n$-truncated crossed complex of arrows and whose structural maps
(domain, codomain, identity, and composition) have the identity map as \((n-1)\)-truncation. Thus, an object \(G \in GCrs_n\) gives rise to a diagram in \(Xm\) of the form

\[
\begin{array}{cccc}
C_n^1 \times C_n^0 & \xrightarrow{\circ} & C_n^1 & \xrightarrow{s} C_n^0 \\
\downarrow & & \downarrow & \\
C_{n-1} & \xrightarrow{r} & C_{n-2} & \\
\end{array}
\]

(13)

We can now state:

**Proposition 3.5.** For each \(n > 1\) there is a functor \(crs_n: GCrs_n \rightarrow Crs_{n+1}\), which has a section

\[
gpd_n: Crs_{n+1} \rightarrow GCrs_n,
\]

allowing us to regard any \((n+1)\)-truncated crossed complex

\[
\mathcal{C}: \cdots \rightarrow 0_G \rightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow \cdots \rightarrow C_2 \xrightarrow{\partial_2} 1_G,
\]

as an internal groupoid in \(n\)-truncated crossed complexes, having \(\mathcal{F}_0 = T_n(\mathcal{C})\) as \(n\)-truncated crossed object of whose and whose \(n\)-truncated crossed complex of arrows is given by

\[
\begin{array}{cccc}
\mathcal{F}_1: C_{n+1} \times C_n & \xrightarrow{\partial_n p_0} & C_{n-1} & \xrightarrow{\partial_{n-1}} \cdots & C_2 & \xrightarrow{\partial_2} 1_G \\
\end{array}
\]

where \(C_{n+1} \times C_n\) is a cartesian product in \(Xm_G\) and \(p_0: C_{n+1} \times C_n \rightarrow C_n\) is the corresponding canonical projection. The functors \(crs_n\) and \(gpd_n\) establish an isomorphism between \(Crs_{n+1}\) and \(GCrs_n\).

**Proof.** Let us complete the definition of \(gpd_n\). The domain map \(s: \mathcal{F}_1 \rightarrow \mathcal{F}_0\) is induced by the projection \(p_0: C_{n+1} \times C_n \rightarrow C_n\), and the codomain map \(t: \mathcal{F}_1 \rightarrow \mathcal{F}_0\) is induced by the morphism of \(G\)-groups \(C_{n+1} \times C_n \rightarrow C_n\) (actually \(G\)-modules except in the case \(n = 2\)) defined on \((u, v) \in C_{n+1}(x) \times C_n(x)\) as \((u, v) \mapsto \partial_{n+1}(u) v \in C_n(x)\) (note that even
in the case $n = 2$, in which $C_2(x)$ may not be abelian, $\partial_{n+1}$ takes its values in the center of $C_n$ and thus this is a homomorphism. Clearly the canonical map $C_n \hookrightarrow C_{n+1} \times C_n$ is a common section for $s$ and $t$, so that we get an internal graph

\[
\begin{array}{ccc}
\mathcal{F}_1 & \xrightarrow{id} & \mathcal{F}_0 \\
\downarrow_t & & \downarrow_s \\
\end{array}
\]

(15)

in the category $(\text{Crs}_n)_{T_{n-1}(\mathcal{C})}$ of $n$-truncated crossed complexes whose $(n-1)$-truncation is $T_{n-1}(\mathcal{C})$.

We want to endow this graph with a structure of internal groupoid in $\text{Crs}_n$. For this it is sufficient to do it at the highest dimension, only place where the graph structure is not trivial. In this dimension we have the internal graph

\[
\begin{array}{ccc}
C_{n+1} \times C_n & \xrightarrow{s} & C_n \\
\downarrow_t & & \downarrow_s \\
\end{array}
\]

(16)

in the category $\text{Gr}^G$ of $G$-groups, which admits a unique structure of internal groupoid in $\text{Gr}^G$.

The structure of internal groupoid in $\text{Gr}^G$ of the graph (16) determines a structure of internal groupoid in $(\text{Crs}_n)_{T_{n-1}(\mathcal{C})}$ on the graph (15), this internal groupoid in $\text{Crs}_n$ will be denoted $\text{gpd}_n(\mathcal{C})$. It is easy to show that this construction is functorial and thus we have a functor

\[
\text{gpd}_n : \text{Crs}_{n+1} \rightarrow \text{GCrs}_n.
\]

(17)

In order to define its quasi-inverse $\text{crs}_n : \text{GCrs}_n \longrightarrow \text{Crs}_{n+1}$, we consider an object

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{id} & \mathcal{G} \\
\downarrow_t & & \downarrow_s \\
\end{array}
\]

in $\text{GCrs}_n$, as in (13), and we apply $\text{ceil}_n$ to the morphism $s : \mathcal{C}^1 \rightarrow \mathcal{C}^0$. We then obtain a morphism of $G$-groups

\[
s = \text{ceil}_n(s) : \text{ceil}_n(\mathcal{C}^1) = C_n^1 \longrightarrow C_n^0 = \text{ceil}_n(\mathcal{C}^0),
\]

whose kernel is a $G$-module $K = \ker(s) : G \rightarrow \text{Ab}$ associating with each object $x \in G$ the subgroup $K(x)$ of $C_n^1(x)$ consisting of the elements $u \in C_n^1(x)$ such that $s_x(u) = 0_{C_n^0(x)}$, with an action which is induced by the action of $C_n^1$. Evidently, this $G$-module determines a
crossed module $K = (G, K, 0)$ and $\partial_2$ acts trivially on the totally disconnected groupoid $\hat{K}$. As a result we have a $(n+1)$-truncated crossed complex

$$crs_n(G) = (K \xrightarrow{\partial_{n+1}} C_n^0 \xrightarrow{\partial_0} C_{n-1} \xrightarrow{} \cdots \xrightarrow{} C_2 \xrightarrow{\partial_2} 1_G),$$

where $\partial_{n+1} : K \to C_n^0$ is a morphism of $G$-groups induced by the morphism of $G$-groups $t = \text{cel}_n(t) : C_n^1 \to C_n^0$ associated to the codomain of $G$, that is, for each object $x \in G$, the $x$-component of $\partial_{n+1}$ is given by

$$(\partial_{n+1})_x : K(x) \to C_n^0(x), \quad (\partial_{n+1})_x(u) = t_x(u).$$

Note that $crs_n(G)$ is really a chain complex, that is, $\partial_0 \partial_n + 1 = 0$. This construction of $crs_n(G)$ is also functorial so that we have a functor

$$crs_n : GCrs_n \to Crs_{n+1}. \quad (18)$$

Let us see that it is a quasi-inverse for $gpd_n$. If $G \in GCrs_n$ as in (13), $\text{cel}_n(G)$ is an internal groupoid in the category of $G$-groups, hence for every $x \in \text{obj}(G)$ we have an internal groupoid in the category of groups

$$C_n^1(x) \times C_n^0(x) \xrightarrow{id_x \times s_x} C_n^0(x) \xrightarrow{t_x} C_n^0(x).$$

Thus, we have a group isomorphism

$$G_x : C_n^0(x) \cong N_{(x, id_x)} = K(x) \times C_n^0(x) \xrightarrow{\cong} C_n^1(x); \quad G_x(u, v) = u \text{id}_x(v).$$

Since the above isomorphism is natural, it is immediate that the pair $(G, \text{Id}_{C_n^0})$ is an isomorphism of graphs in $Gr^G$ which induces a graph isomorphism in $Crs_n$, hence an isomorphism in $GCrs_n$ between the groupoids $gpd_n crs_n(G)$ and $G$. Conversely, if $C$ is a $(n+1)$-truncated crossed complex, then

$$crs_n(gpd_n(C)) = (K \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_0} C_{n-1} \xrightarrow{} \cdots \xrightarrow{} C_2 \xrightarrow{\partial_2} 1_G),$$

where $K = (G, K, 0)$ with $K = \ker(C_{n+1} \times C_n \xrightarrow{s} C_n)$. Since $s$ is the canonical projection, it is clear that $K = C_{n+1}$ and $\hat{K} = C_{n+1}$. Looking closely at the connecting morphisms in $crs_n(gpd_n(C))$, one realizes immediately that $crs_n(gpd_n(C)) = C$. \qed
Proposition 3.6. For any \( n > 0 \) the following equations of functors hold:

\[
\begin{align*}
\text{Crs}_{n+1} & \xrightarrow{i_{n+1}} \text{Crs} \\
gpd_n & \xrightarrow{\gamma_{n+1}} \text{Crs}_{n+1} \\
\text{GCrs}_n & \xrightarrow{\pi_0} \text{Crs}_n,
\end{align*}
\]

\[
\begin{align*}
\text{Crs}_{n+1} & \xrightarrow{i_{n+1}} \text{Crs} \\
gpd_n & \xrightarrow{\gamma_{n+1}} \text{Crs}_{n+1} \\
\text{GCrs}_n & \xrightarrow{\pi_0} \text{Crs}_n,
\end{align*}
\]

\[
\tilde{P}_{n+1} = \pi_{0}\text{gpd}_n, \quad \tilde{P}_{n+1} = \pi_{0}.
\]

in other words, for each \((n+1)\)-truncated crossed complex \( \mathcal{C} \) and each groupoid \( \mathcal{G} \in \text{GCrs}_n \):

\[ P_n(\mathcal{C}) = \pi_{0}\text{gpd}_n(\mathcal{C}) \quad \text{and} \quad P_{n+1}(\mathcal{G}) = \pi_{0}\text{crrs}_n(\mathcal{G}). \]

Proof. Since \( \text{crrs}_n(\text{gpd}_n(\mathcal{C})) = \mathcal{C} \), it is sufficient to verify the right hand equation, that is, \( \pi_{0}(\mathcal{G}) = P_{n}\text{crrs}_n(\mathcal{G}) \). These two crossed complexes agree in dimensions less than \( n \). In dimension \( n \), \( \pi_{0}(\mathcal{G}) \) is the coequalizer of \( s \) and \( t \) (see diagram (13)). On the other hand, \( P_{n}\text{crrs}_n(\mathcal{G}) \) is, in dimension \( n \), the quotient of \( C^0 \) by the image of \( \partial_{n+1}(t) : \ker(s) \rightarrow C^0 \).

That this quotient is equal to the previous coequalizer is an immediate consequence of the general fact that the coequalizer of a parallel pair of group homomorphisms, \( s, t : G \rightarrow H \) having a common section is the quotient of \( H \) by \( t(\ker(s)) \).

Proposition 3.7. For each groupoid \( \mathcal{G} \in \text{GCrs}_n \) and each \((n+1)\)-truncated crossed complex \( \mathcal{C} \) we have natural isomorphisms:

\[
\begin{align*}
\text{cell}_n \text{End}(\mathcal{G}) & \cong \text{cell}_n \text{obj}(\mathcal{G}) \times (\pi_{n+1}\text{crrs}_n(\mathcal{G}) \circ q) \\
\text{cell}_n \text{End gpd}_n(\mathcal{C}) & \cong \text{cell}_n \text{T}_{n}(\mathcal{C}) \times (\pi_{n+1}(\mathcal{C}) \circ q).
\end{align*}
\]

where \( \mathcal{G} \) denotes both the base groupoid of \( \text{obj}(\mathcal{G}) \) and that of \( \mathcal{C} \), and \( q \) denotes either the canonical projection \( \mathcal{G} \rightarrow \pi_{1}\text{obj}(\mathcal{G}) \) or \( \mathcal{G} \rightarrow \pi_{1}(\mathcal{C}) \).

Proof. The second isomorphism is a consequence of the first one and of the identities \( \text{obj}(\text{gpd}_n(\mathcal{C})) = \text{T}_{n}(\mathcal{C}) \) and \( \text{crrs}_n \text{gpd}_n(\mathcal{C}) = \mathcal{C} \). For each object \( x \in \mathcal{G} \), the isomorphism

\[
\text{cell}_n \text{End}(\mathcal{G})(x) \cong \text{cell}_n \text{obj}(\mathcal{G})(x) \times (\pi_{n+1}\text{crrs}_n(\mathcal{G}) \circ q(x))
\]

takes each \( u \in \text{cell}_n \text{End}(\mathcal{G})(x) \) to the pair

\[
(s(u) = t(u), u - \text{id}(s(u))) \in \text{cell}_n \text{obj}(\mathcal{G})(x) \times (\pi_{n+1}\text{crrs}_n(\mathcal{G}) \circ q(x)).
\]

\]
4. The Postnikov invariants of a crossed complex

As indicated in the Introduction, we distinguish two types of Postnikov invariants of a crossed complex. On the one hand we have the “algebraic” invariants, which are elements of algebraic (cotriple) cohomologies in the categories $\text{Crs}_n$. On the other hand, corresponding to each algebraic invariant $k_{n+1}$, we have a “topological” invariant, which is an element of a singular cohomology. In this section we first determine the algebraic invariants, characterizing them as extensions and as torsors. Then, we define the topological invariants and the singular cohomologies in which they live. Finally, we show how to map the cotriple cohomologies to the singular ones so that one can obtain the topological invariants from the algebraic ones.

4.1. The algebraic invariants

Let $\mathcal{C}$ be a $G$-crossed complex. For $n \geq 0$, the $(n + 1)$th algebraic Postnikov invariant, $k_{n+1}$, of $\mathcal{C}$ is determined by the fibration $\eta_{n+1} : P_{n+1}(\mathcal{C}) \to P_n(\mathcal{C})$, which is completely described by the following diagram:

\[
\begin{array}{cccccccccc}
\cdots & \longrightarrow & 0_G & \longrightarrow & C_{n+1}/\text{im} \partial_{n+2} & \stackrel{\tilde{\partial}_{n+1}}{\longrightarrow} & C_n & \stackrel{\tilde{\partial}_n}{\longrightarrow} & C_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & 0_G & \longrightarrow & 0_G & \longrightarrow & C_n/\text{im} \partial_{n+1} & \stackrel{\tilde{\partial}_n}{\longrightarrow} & C_{n-1} & \longrightarrow & \cdots \\
\end{array}
\]

(19)

For $n = 0$ we get the first invariant, $k_1$, determined by $\eta_1$:

\[
\begin{array}{cccccccccc}
\cdots & \longrightarrow & 0_{\pi_1(\mathcal{C})} & \longrightarrow & 1_{\pi_1(\mathcal{C})} & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & 0_{\pi_0(\mathcal{C})} & \longrightarrow & 1_{\pi_0(\mathcal{C})} \\
\end{array}
\]

where we have indicated in the vertical arrows the change of base functor $q_0$ since it is not an identity.

This invariant is an element in the topos cohomology of $\text{Crs}_0 = \text{Set}$, which has very little structure and, correspondingly, with a very simple cohomology: it is trivial in dimensions $\geq 1$, so that $H^2(P_0(\mathcal{C}), \pi_1(\mathcal{C}))$ only has one element. Although it is not difficult to see that this element corresponds to $\eta_1$, there is really no need of our machinery to determine it, and we will not discuss here this case any further. For a more complete discussion of this case we refer the interested reader to [13].
For \( n = 1 \) we have the second invariant, \( k_2 \), determined by \( \eta_2 \):

\[
\cdots \rightarrow 0_G \rightarrow \mathcal{C}_2/\text{im}\, \partial_3 \rightarrow \bar{\partial}_2 \rightarrow 1_G \\
\downarrow (q_1,0) \quad \downarrow (q_1,0) \quad \downarrow (q_1,q_1) \\
\cdots \rightarrow 0_{\pi_1(\mathcal{C})} \rightarrow 0_{\pi_1(\mathcal{C})} \rightarrow 1_{\pi_1(\mathcal{C})}.
\]

Note, however, that \( q_1 \) is the cokernel of \( \bar{\partial}_2 \), while \( \ker\bar{\partial}_2 \) is, as noted earlier (see page 248), the \( G \)-crossed module \( \mathcal{A}_2 = (\pi_2(\mathcal{C}), 0) \), where \( \pi_2(\mathcal{C}) \) is considered as \( G \)-module via \( q_1 : G \rightarrow \pi_1(\mathcal{C}) \). Thus \( \eta_2 \) is completely determined by the following sequence of crossed modules

\[
0 \rightarrow \mathcal{A}_2 \rightarrow \mathcal{C}_2/\text{im}\, \partial_3 \rightarrow \bar{\partial}_2 \rightarrow 1_G \\
\downarrow q_1 \quad \downarrow q_1 \quad 1_{\pi_1(\mathcal{C})} \\
0 \rightarrow 0_{\pi_1(\mathcal{C})} \rightarrow 0_{\pi_1(\mathcal{C})} \rightarrow 1_{\pi_1(\mathcal{C})}, (20)
\]

which can be regarded as a genuine exact sequence

\[
0 \rightarrow \pi_2(\mathcal{C}) \rightarrow \mathcal{C}_2/\text{im}\, \partial_3 \rightarrow \bar{\partial}_2 \rightarrow G \rightarrow q_1 \rightarrow \pi_1(\mathcal{C}) \rightarrow 0, (21)
\]

in the category \( \text{Gpd}_{\text{obj}}(G) \).

A sequence such as (20) or (21) is an extension of the groupoid \( \pi_1(\mathcal{C}) \) by the \( \pi_1(\mathcal{C}) \)-module \( \pi_2(\mathcal{C}) \), according to the following definition, of which the “reduced” or pointed case is the well-known definition of 2-extensions of groups by modules:

**Definition 4.1.** A 2-extension of a groupoid \( \Pi \) by a \( \Pi \)-module \( A : \Pi \rightarrow \text{Ab} \) is a crossed module \( \mathcal{C} = (G, C, \delta) \), called the fiber of the extension, together with an exact sequence

\[
0 \rightarrow \mathcal{A} \rightarrow \mathcal{C} \rightarrow \mathcal{G} \rightarrow \Pi \rightarrow 0
\]

(22)

in \( \text{Gpd}_{\text{obj}}(\Pi) \) such that the kernel of \( \delta \) factors through \( \Pi \) as \( A \circ q \),

\[
\begin{array}{ccc}
\mathcal{G} & & \rightarrow \Pi \\
\downarrow \text{ker}\, \delta & & \\
A & \rightarrow & \ \ \ \ \\
\end{array}
\]

(23)

The standard definition of morphism of extensions gives rise to a category, denoted \( \text{Ext}^2(\Pi, A) \), whose objects are the 2-extensions of a groupoid \( \Pi \) by a \( \Pi \)-module \( A \). As usual we denote with brackets, \( \text{Ext}^2[\Pi, A] \), the category of connected components of \( \text{Ext}^2(\Pi, A) \).
It is now easy to show that the extensions just defined represent cohomology elements in a well known cotriple cohomology of groupoids. We use the notation and results of Appendix A.

**Proposition 4.2.** Let \( U : \text{Gpd} \to \text{Gph} \) be the underlying graph functor defined on the category of groupoids. If \( \Pi \) is a groupoid, \( A \) is a \( \Pi \)-module, and \( \tilde{A}_1 \) is the obvious abelian group object in \( \text{Gpd}/\Pi \) given by \( \tilde{A}_1 = (\Pi \ltimes A \cong \Pi) \), there is a full and faithful functor

\[
\text{Ext}^2(\Pi, A) \to \text{Tor}^2_U(\Pi, \tilde{A}_1).
\]

**Proof.** Consider a 2-extension of \( \Pi \) by \( A \), as in Definition 4.1, with \( \tilde{C} = (G, C, \delta) \) being the fiber crossed module. The commutativity of the triangle (23) gives a commutative square of groupoids and functors

\[
\begin{array}{ccc}
G \times \ker \delta & \xrightarrow{\alpha} & \Pi \times A \\
\downarrow & & \downarrow \\
G & \xrightarrow{q} & \Pi,
\end{array}
\]

which is a pullback. Let then \( \tilde{C} \) be the internal groupoid in groupoids corresponding to the crossed module \( C \) by the functor \( \text{gpd} \) of Proposition 2.6. The groupoid of connected components of \( \tilde{C} \) is \( \pi_1(C) = \Pi \), and the groupoid of endomorphisms of \( \tilde{C} \) is \( G \ltimes \ker \delta \). Therefore \( (\Pi, \tilde{C}, \alpha) \) is an \((\tilde{A}_1, 2)\)-torsor which is clearly \( U \)-split. Furthermore, it is a routine straightforward verification to see that the above construction is functorial. That the functor \( \text{Ext}^2(\Pi, A) \to \text{Tor}^2_U(\Pi, \tilde{A}_1) \) so defined is full and faithful is an immediate consequence of Proposition 2.6. \( \square \)

**Proposition 4.3.** For any groupoid \( \Pi \) and \( \Pi \)-module \( A \),

\[
H^2_{G_1}(\Pi, \tilde{A}_1) \cong \text{Ext}^2[\Pi, A],
\]

where \( G_1 \) is the cotriple on \( \text{Gpd}/\Pi \) induced by the underlying graph functor \( U : \text{Gpd} \to \text{Gph} \) and \( \tilde{A}_1 \) is an abelian group object in \( \text{Gpd}/\Pi \) as in Proposition 4.2.

**Proof.** By Proposition A.5, it is sufficient to prove that the inclusion functor \( \text{Tor}^2_U(\Pi, \tilde{A}_1) \hookrightarrow \text{Tor}^2_U(\Pi, \tilde{A}_1) \) factors through the full and faithful functor of Proposition 4.2. This follows from the fact that the free groupoid on a graph has as objects the vertices of the graph and that the counit map for the free adjunction is the identity on objects. This implies that the fiber groupoid of any \( U \)-split 2-torsor in \( \text{Tor}^2_U(\Pi, \tilde{A}_1) \) is actually a 2-groupoid and then, by Proposition 2.6, it is isomorphic to the groupoid associated by the functor \( \text{gpd} \) to a crossed module which is the fiber of a 2-extension of \( \Pi \) by \( A \). \( \square \)
The last results show that the fibration \( \eta_2 : P_2(C) \to P_1(C) \) uniquely corresponds to an element

\[ k_2 \in H^2_{G_1}(P_1(C), \tilde{A}_1). \]

where \( A = \pi_2(P_1(C)) \). This cohomology element will be called the algebraic second Postnikov invariant of the crossed complex \( C \).

For \( n > 1 \), an observation about \( q_n : C_n \to C_n/\text{im} \, \partial_{n+1} \) similar to the one made for \( q_1 \) holds, namely that \( q_n \) is (not only the cokernel of \( \partial_{n+1} \), but also) the cokernel of \( \overline{\partial}_{n+1} : C_{n+1}/\text{im} \, \partial_{n+2} \to C_n \). As a consequence, \( \eta_{n+1} \) (described by diagram (19)) represents a 2-extension of the \( n \)-truncated crossed complex \( P_n(C) \) by the \( \pi_1(C) \)-module \( \pi_{n+1}(C) \), according to the following definition, which extends that of 2-extensions of groupoids:

**Definition 4.4.** If \( C = (C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} 1_G) \) is an \( n \)-truncated crossed complex with \( n > 2 \), and \( A : \Pi \to \text{Ab} \) is a \( \Pi \)-module over the fundamental groupoid \( \Pi = \pi_1(C) \) of \( C \), a 2-extension of \( C \) by \( A \) is an exact sequence in the category of \( G \)-groups:

\[
0 \longrightarrow A \longrightarrow E_1 \overset{\sigma}{\longrightarrow} E_0 \overset{\tau}{\longrightarrow} C_n \longrightarrow 0,
\]

(\( A \) is considered as a \( G \)-module via the canonical projection \( q : G \to \Pi \), such that

\[
E_1 = (G, E_1, 0) \]

is an \((n+1)\) \( G \)-crossed complex, where \( E_i = (G, E_i, 0), i = 0, 1 \). For \( n = 2 \), we define a 2-extension of a crossed module \( C = (G, C, \delta) \) by a \( \Pi \)-module \( A \) as an exact sequence of \( G \)-groups

\[
0 \longrightarrow A \longrightarrow E_1 \overset{\sigma}{\longrightarrow} E_0 \overset{\tau}{\longrightarrow} C \longrightarrow 0,
\]

such that

\[
E_1 = (G, E_1, \delta \tau) \overset{\delta \tau}{\longrightarrow} 1_G
\]

is a 3-truncated crossed complex, where \( E_0 = (G, E_0, \delta \tau) \) and \( E_1 = (G, E_1, 0) \).

Let us note that, as in Definition 4.1, a 2-extension of a \( n \)-truncated crossed complex \( C \) can be seen as an exact sequence

\[
0 \longrightarrow A \longrightarrow E_1 \overset{\sigma}{\longrightarrow} E_0 \overset{\tau}{\longrightarrow} C \longrightarrow 0,
\]
in the category \((\text{Crs}_n)_{T_n-1}(\mathcal{C})\) of \(n\)-truncated crossed complexes with a fixed \((n - 1)\)-truncation, together with an extra structure in the central part that makes (24) or (26) an \((n + 1)\)-truncated crossed complex. The \(n\)-truncated crossed complexes \(\mathcal{A}\) and \(\mathcal{E}_i, i = 0, 1\), have at dimension \(n\) the crossed modules \(\text{zero}(\mathcal{A}) = (G, \lambda, 0)\) and \(\mathcal{E}_i\), respectively.

Using those definitions, our discussion can be summarized in the following statement:

**Proposition 4.5.** For all \(n \geq 1\) the fibration \(\eta_{n+1}\) of the Postnikov tower of \(\mathcal{C}\) provides a \(2\)-extension of \(P_n(\mathcal{C})\) by the \(\pi_1(\mathcal{C})\)-module \(\pi_{n+1}(\mathcal{C})\).

**Proof.** For \(n = 1\), the \(2\)-extension associated to \(\eta_2\) is given by the crossed module \(P_2(\mathcal{C})\) and the sequence (21). For \(n > 1\) the \(2\)-extension associated to \(\eta_{n+1}\) is given by the sequence

\[
0 \longrightarrow \pi_{n+1}(\mathcal{C}) \longrightarrow C_{n+1} / \im \partial_{n+2} \quad \overset{\delta_{n+1}}{\longrightarrow} \quad C_n \quad \overset{\partial_n}{\longrightarrow} \quad C_n / \im \partial_{n+1} \longrightarrow 0.
\]

As suggested above, it is not difficult to extend this proposition to the case \(n = 0\) by giving an appropriate definition of a \(2\)-extension of a set \(X\) by an \(X\)-indexed family of groups. The details are in [13].

As in the case of groupoids (case \(n = 1\)), the \(2\)-extensions of an \(n\)-truncated crossed complex \(\mathcal{C}\) by a fixed \(\pi_1(\mathcal{C})\)-module \(A\) constitute a category, denoted \(\text{Ext}^2(\mathcal{C}, A)\), where morphisms between extensions are defined in the obvious way, that is, as commutative diagrams

\[
\begin{array}{ccccccc}
0 & \longrightarrow & A & \longrightarrow & E_1 & \overset{\sigma}{\longrightarrow} & E_0 & \overset{\tau}{\longrightarrow} & C_n & \longrightarrow & 0, \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & E'_1 & \overset{\sigma'}{\longrightarrow} & E'_0 & \overset{\tau'}{\longrightarrow} & C_n & \longrightarrow & 0.
\end{array}
\]

Again, \(\text{Ext}^2[\mathcal{C}, A]\) denotes the set of connected components of \(\text{Ext}^2(\mathcal{C}, A)\).

It is now necessary to give a more elaborate analysis than the one made for groupoids in order to show that the \(2\)-extensions of crossed modules defined above represent cohomology elements in a cotriple cohomology of crossed modules (this, of course, will be the cohomology corresponding to the cotriple induced by \(U_2\), Proposition 2.5). An essential step in this process consists in determining the coefficients to be used to calculate the cohomology. These coefficients (constituting an internal abelian group object in the category \(\text{Crs}_2/P_2(\mathcal{C})\)) are obtained from the “homotopy group” \(\pi_3(\mathcal{C})\) using the fact that \(\pi_3(\mathcal{C})\) is a module over the fundamental groupoid of \(P_2(\mathcal{C})\) (equal to the fundamental groupoid of \(\mathcal{C}\), in turn equal to \(P_1(\mathcal{C})\)). Since the determination of these coefficients follow the same pattern in dimensions greater than 1, we will now explain the process assuming only \(n \geq 2\).

If \(n \geq 2\) and \(\mathcal{C} \in \text{Crs}_n\) is a \(n\)-truncated crossed complex, pulling back along the canonical map \(\mathcal{C} \rightarrow \Pi = \pi_1(\mathcal{C}) = P_1(\mathcal{C})\) (a finite product-preserving functor) produces an abelian group object in \(\text{Crs}_n/\mathcal{C}\) from any abelian group object in \(\text{Crs}_n/\Pi\). This allows us to reduce the search for our coefficients to obtaining an abelian group object in \(\text{Crs}_n/\Pi\).
For every \( n \geq 2 \) there is a functor \( \text{ins}_n : \Pi \text{-module} A : \Pi \to \text{Ab} \) to the \( n \)-truncated crossed complex over \( \text{ins}_n(A) = (\text{zero}(A) = 0 \to 0 \to \cdots \to 0 \to 1) \), (28)

where \( \text{zero}(A) = (\Pi, A, 0) \) (Example (1), page 243). This functor can be regarded as taking its values in \( \text{Crs}_n/\Pi \) via the obvious map \( (1, 0) : \text{ins}_n(A) \to \Pi \), and when regarded this way it becomes a finite product preserving functor whose value at a \( \Pi \)-module \( A \) will be denoted \( \tilde{A}_n \).

Since (the theory of abelian groups being commutative) \( A \) has the structure of an internal abelian group object in \( \text{Ab}_\Pi \), the fact that \( \text{ins}_n : \text{Ab}_\Pi \to \text{Crs}_n/\Pi \) preserves finite products implies that \( \tilde{A}_n \) has a structure of internal abelian group object in \( \text{Crs}_n/\Pi \).

If \( C \) is fixed by the context and \( \Pi = \pi_1(C) \), the abelian group object in \( \text{Crs}_n/C \) obtained from a \( \Pi \)-module \( A \) after pulling \( \text{ins}_n(A) \to \Pi \) back along the canonical map \( C \to \Pi \) will be denoted \( \tilde{A}_n \), so that we have,

\[
\begin{array}{ccc}
\tilde{A}_n & \text{pb} & \text{can.} \\
\text{ins}_n(A) & \text{C} & \Pi.
\end{array}
\] (29)

Regarding \( \tilde{A}_n \) as an internal abelian group object in \( \text{Crs}_n/C \), it will be taken as a system of global coefficients for calculating the cotriple cohomology.

Coming back to dimension 2, let \( \mathcal{C} = (G, C, \delta) \in \text{Crs}_2 = \text{Xm} \), let \( \Pi = \pi_1(\mathcal{C}) \) be its fundamental groupoid and let \( A : \Pi \to \text{Ab} \) be a system of local coefficients. It is not difficult to prove that the resulting crossed module \( \tilde{A}_2 \) obtained in (29) for \( n = 2 \) is

\[
\tilde{A}_2 = (G, C \times (A \circ q), \delta')
\] (30)

having as structure \( G \)-group the functor \( G \to \text{Gr} \) given by the cartesian product \( C \times (A \circ q) \): that is, on objects it is given by the cartesian product of groups, \( x \mapsto C(x) \times A(x) \), and on arrows it is determined by the action \( u(v, b) = (u^v, u^b) \). Furthermore, the crossed module connecting morphism of \( \tilde{A}_2, \delta' \), has components \( \delta'_x : C(x) \times A(x) \to \text{End}_G(x) \) given by \( \delta'_x((u, a)) = \delta(u) \).

Let us consider now a 2-extension of \( \mathcal{C} \) by \( A \) such as (25). By the condition that diagram (26) be a 3-truncated crossed complex, the action of im \( \delta \tau \) on \( \tilde{E}_2 \) is trivial. The cartesian product \( E_0 \times E_1 \) in \( \text{Xm}_G \) is given by

\[ E_0 \times E_1 = (G, E_0 \times E_1, \delta \tau p_0). \]

This crossed module has two obvious maps of crossed modules to \( E_0 \), namely, the canonical projection \( p_0 : \tilde{E}_0 \times \tilde{E}_1 \to \tilde{E}_0 \) and the map determined by \( t : (x, y) \mapsto x\sigma(y) \) from \( E_0 \times E_1 \).
to $E_0$. These two maps have a common section determined by the map $x \mapsto (x, 0)$ and the resulting internal graph in $Xm$.

$$\mathcal{E} : \mathcal{E}_0 \times \mathcal{E}_1 \xrightarrow{p_0} \mathcal{E}_0$$

admits a unique groupoid structure in which the multiplication is determined by

$$\{ (x, y), (x', y') \} \mapsto (x, yy').$$

Since the crossed module of connected components of $\mathcal{E}$ (the coequalizer of $p_0$ and $t$) is easily verified to be the canonical map $(\tau, 1_G) : \mathcal{E}_0 \to C$ determined by $\tau : E_0 \to C$ (and the identity of $G$ as change of base), we can take the structure of internal groupoid of $E$ as the fiber groupoid of a $\tilde{A}_2$-torsor above $C$. To define such a torsor we just need to give the corresponding cocycle map $\alpha : \text{End}(\mathcal{E}) \to \tilde{A}_2$. The domain of this map is the crossed module obtained as the equalizer of $p_0$ and $t$. Since the condition defining this equalizer is $x = x\sigma(y)$, one quickly finds that $\text{End}(\mathcal{E}) = (G, E_0 \times \ker \sigma, \delta \tau p_0)$. The required morphism $\alpha$ is defined as the map of $G$-crossed modules determined by the following map of $G$-groups:

$$E_0 \times \ker \sigma \xrightarrow{t \times \tilde{\sigma}} C \times (A \circ q),$$

where $\tilde{\sigma}$ is the canonical isomorphism $\ker \sigma \cong A \circ q$ induced by the exactness of $0 \to A \circ q \to E_1 \xrightarrow{\sigma} E_0$.

The above arguments have prepared the ground for the following:

**Proposition 4.6.** For any crossed module $C = (G, C, \delta)$ and any $\pi_1(C)$-module $A : \pi_1(C) \to \text{Ab}$, there is a full and faithful functor

$$\text{Ext}^2(C, A) \to \text{Tor}^2_{U_2}(C, \tilde{A}_2),$$

where $U_2$ is the monadic functor of Proposition 2.5.

**Proof.** It is a simple exercise to verify that the construction given above indeed produces a 2-torsor above $C$ with coefficients in $\tilde{A}_2$ from any 2-extension of $C$ by $A$, and that this construction is functorial. Furthermore, if one examines the correspondence between morphisms of extensions

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{j} & E_1 & \xrightarrow{\sigma} & E_0 & \xrightarrow{\tau} & C & \longrightarrow & 0, \\
\downarrow & \downarrow & f_1 & \downarrow & f_0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & A & \xrightarrow{j'} & E'_1 & \xrightarrow{\sigma'} & E'_0 & \xrightarrow{\tau'} & C & \longrightarrow & 0
\end{array}
\quad \tau' F_0 = \tau,
$$

$$f_0 \sigma = \sigma' f_1,$$

$$f_1 j = j'.$$
and morphisms of 2-torsors

\[
\begin{array}{ccc}
E_0 \times \ker \sigma & \xrightarrow{(1_\tilde{A} \times f_0)\alpha = \alpha'(f_0 \times f_1)} & C \times (A \circ q), \\
E_0 \times f_0 \times f_1 & \xrightarrow{\sigma} & E_0' \times \ker \sigma' \\
E_0 \times E_1 & \xrightarrow{p_0} & E_0 \\
\end{array}
\]

it becomes evident that it is a bijective correspondence and therefore the functor from 2-extensions of \( \tilde{C} \) by \( A \) to \((\tilde{A}_2, 2)\)-torsors above \( \tilde{C} \) is full and faithful.

It only remains to prove that the internal groupoid \( \mathcal{E} \) in \( \text{Xm} \) that we have associated to an extension of \( \tilde{C} \) by \( A \) is \( U_2 \)-split. For that, let us consider \( U_2((p_0, t)) \) has a section. Giving a section for this map is equivalent to giving, for each object \( x \in G \), a map of sets \( K_x : E_0(x) \times C(x) E_0(x) \rightarrow E_1(x) \) such that \( \sigma_x(K_x(u, v)) = u^{-1}v \). Taking into account that \( E_0(x) \times C(x) E_0(x) \) is the set of pairs \( (u, v) \), \( u, v \in E_0(x) \) such that \( \tau_x(u) = \tau_x(v) \), it is clear that for every \( (u, v) \in E_0(x) \times C(x) E_0(x) \) we have \( u^{-1}v \in \ker \tau_x = \im \sigma_x \). We can choose any section \( \beta \) of the set map \( E_1(x) \xrightarrow{\sigma_x} \im \sigma_x \) and define \( K_x(u, v) = \beta(u^{-1}v) \). This proves \( U_2((p_0, t)) \) has a section, and therefore \( \mathcal{E} \) is \( U_2 \)-split. \( \square \)

**Proposition 4.7.** For any crossed module \( C \) and any \( \pi_1(C) \)-module \( A \),

\[
H^2_{\hat{G}_2}(\tilde{C}, \tilde{A}_2) \cong \text{Ext}^2[C, A],
\]

where \( \tilde{A}_2 \) is the abelian group object in \( \text{Xm}/C \) obtained from \( C \) and \( A \) and given by (30).

**Proof.** This is consequence of Proposition A.5 and the fact that the inclusion \( \text{Tor}_{U_2}^2(C, \tilde{A}_2) \hookrightarrow \text{Tor}_{U_2}^2(C, \tilde{A}_2) \) factors through the full and faithful inclusion of Proposition 4.6. To see the last statement, observe that the counit map for the free adjunction \( F_2 : \text{AGpd} \Rightarrow \text{Xm} : U_2 \) is always the identity at the level of base groupoid. Therefore the fiber groupoid (internal in \( \text{Xm} \)) of any 2-torsor in \( \text{Tor}_{U_2}^2(C, \tilde{A}_2) \) lives in \( \text{Xm}_G \). Finally it
is straightforward to see that any internal groupoid in $X_{mG}$ is isomorphic to the groupoid built from a 2-extension as in the proof of Proposition 4.8.

The last results show that the fibration $\eta_2: P_2(\mathcal{C}) \to P_1(\mathcal{C})$ uniquely corresponds to an element

$$k_2 \in H^2_{G_1}(P_1(\mathcal{C}), \tilde{A}_1),$$

where $A = \pi_2(P_1(\mathcal{C}))$. This cohomology element will be called the algebraic second Postnikov invariant of the crossed complex $\mathcal{C}$.

The higher invariants
Our next objective is to establish a general bijection between

$$\text{Ext}^2_{\mathcal{C}}[P_n(\mathcal{C}), \pi_{n+1}(\mathcal{C})]$$

and the set of elements in certain cotriple cohomology in the category of $n$-truncated crossed complexes for $n \geq 3$.

Let $n \geq 3$, $\mathcal{C}$ an $n$-truncated crossed complex, $\Pi = \pi_1(\mathcal{C})$ and $A: \Pi \to \text{Ab}$ a $\Pi$-module. We define $\tilde{A}_n$ by (29) and take the resulting abelian group object in $\text{Crs}_n/\mathcal{C}$ (also denoted $\tilde{A}_n$) as a system of global coefficients for 2-torsors.

**Proposition 4.8.** Let $n \geq 3$, for any $n$-truncated crossed complex $\mathcal{C}$ and any $\pi_1(\mathcal{C})$-module $A$ there is a full and faithful functor

$$\text{Ext}^2_{\mathcal{C}}(\mathcal{C}, A) \to \text{Tor}^2_{\mathcal{C}}(\mathcal{C}, \tilde{A}_n).$$

**Proof.** Let us consider a 2-extension of $\mathcal{C}$ by $A$,

$$0 \longrightarrow A \longrightarrow E_1 \xrightarrow{\sigma} E_0 \xrightarrow{\tau} C_n \longrightarrow 0,$$

let us also denote $\mathcal{E}_i = (G, E_i, 0)$ as in Definition 4.4. The $G$-module $E_0 \oplus E_1$ with the zero map gives a $G$-crossed module $\mathcal{E}_0 \oplus \mathcal{E}_1$ which, substituted for $C_n$ in $\mathcal{C}$ with the boundary map $\partial_n \tau p_0: \mathcal{E}_0 \oplus \mathcal{E}_1 \to C_{n-1}$, gives rise to an $n$-truncated crossed complex over $G$,

$$\mathcal{F}_1: \mathcal{E}_0 \oplus \mathcal{E}_1 \xrightarrow{\partial_n \tau p_0} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow 1_G.$$

We will take this as the object of arrows of an internal groupoid in $\text{Crs}_n$. As the object of objects we take the $n$-truncated crossed complex

$$\mathcal{F}_0: \mathcal{E}_0 \xrightarrow{\partial_n \tau} C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow 1_G.$$

The “source” map $s: \mathcal{F}_1 \to \mathcal{F}_0$ is the obvious map of crossed complexes induced by the projection $p_0: E_0 \oplus E_1 \to E_0$, and the “target” map $t: \mathcal{F}_1 \to \mathcal{F}_0$ is the one induced by the
map \( x \oplus y \mapsto x \sigma(y) \) form \( E_0 \oplus E_1 \) to \( E_0 \). Then, the canonical inclusion \( E_0 \hookrightarrow E_0 \oplus E_1 \) determines a common section for \( s \) and \( t \), and we obtain an internal groupoid in \( \text{Crs}_n \) in which composition is determined by the map 
\[(x \oplus y, x' \oplus y') \mapsto x \oplus yy'.\]

It is a simple matter to show that the \( n \)-truncated crossed complex of endomorphisms of this groupoid (the equalizer of \( s \) and \( t \) in \( \text{Crs}_n \)) is
\[E : E_0 \oplus (A \circ q) \longrightarrow 0_G \longrightarrow \cdots \longrightarrow 0_G \longrightarrow 1_G,
\]
and the \( n \)-truncated crossed complex of connected components of this groupoid (the co-equalizer of \( s \) and \( t \) in \( \text{Crs}_n \)) is \( C \). Thus, the above internal groupoid could be taken as the fiber of a torsor in \( \text{Tor}^2(\mathcal{C}, \hat{\mathcal{A}}_n) \) if a cocycle map \( \alpha : E \rightarrow C \times \hat{\mathcal{A}}_n \) can be given. A simple calculation shows that \( C \times \hat{\mathcal{A}}_n \) is the \( n \)-truncated crossed complex
\[(C_n \oplus (A \circ q), 0) \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow 1_G,
\]
and the cocycle map \( \alpha \) can be defined as the obvious map induced by
\[
\tau \oplus (A \circ q) : E_0 \oplus (A \circ q) \longrightarrow C_n \oplus (A \circ q),
\]

note that the square
\[
\begin{array}{ccc}
E_0 \oplus (A \circ q) & \xrightarrow{\tau \oplus (A \circ q)} & C_n \oplus (A \circ q) \\
\downarrow \quad & & \downarrow \\
E_0 & \xrightarrow{\tau} & C_n \\
\end{array}
\]
is a pullback in the category of \( G \)-modules.

This defines the functor \( \text{Ext}^2(\mathcal{C}, A) \rightarrow \text{Tor}^2(\mathcal{C}, \hat{\mathcal{A}}_n) \) on objects. On morphisms the functor is defined by the obvious map, which establishes a bijection between the morphisms between two extensions and the morphisms of torsors between the corresponding torsors. In this way one gets the desired full and faithful functor. \( \Box \)

We will define now a monadic functor \( U_n \) in \( \text{Crs}_n \) and prove that the torsors obtained by the functor of Proposition 4.8 are \( U_n \)-split. Let \( \text{ACrs}_{n-1} \) be the category whose objects are triples \( (X, f, \mathcal{C}) \) where \( X \) is a set, \( \mathcal{C} \) is an \( (n-1) \)-truncated crossed complex, and \( f : X \rightarrow \text{arr}(\pi_{n-1}(\mathcal{C})) \) is a map from \( X \) to the set of arrows of the totally disconnected groupoid associated to \( \pi_{n-1}(\mathcal{C}) : \pi_1(\mathcal{C}) \rightarrow \text{Ab} \). An arrow \( (X, f, \mathcal{C}) \rightarrow (X', f', \mathcal{C}') \) in \( \text{ACrs}_{n-1} \) is a pair \( (\alpha, \beta) \) where \( \alpha : X \rightarrow X' \) is a map of sets and \( \beta : \mathcal{C} \rightarrow \mathcal{C}' \) is a map in \( \text{Crs}_{n-1} \) such that \( \beta f = f' \alpha \) (where \( \beta : \pi_{n-1}(\mathcal{C}) \rightarrow \pi_{n-1}(\mathcal{C}') \) is the obvious natural map induced by \( \beta \)).

There is an obvious forgetful functor \( U_n : \text{Crs}_n \rightarrow \text{ACrs}_{n-1} \), taking an \( n \)-truncated crossed module \( \mathcal{C} \) to the triple \( (\text{arr}(\hat{\mathcal{C}}_n), \hat{\partial}_n, T_{n-1}(\mathcal{C})) \) (the “simple truncation” functor \( T_{n-1} \) is defined in Section 3).

**Proposition 4.9.** For \( n > 2 \), the forgetful functor \( U_n : \text{Crs}_n \rightarrow \text{ACrs}_{n-1} \) is monadic.
Proof. We begin by defining a left adjoint. This is very similar to the case of Proposition 2.5. The main difference is that in this case we directly get a crossed module with trivial connecting morphism instead of a simple pre-crossed module. Given \((X, f, C)\) in \(\text{ACrs}_{n-1}\), with

\[
\begin{array}{c}
\mathcal{C} : C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \rightarrow \cdots \rightarrow C_2 \xrightarrow{\partial_2} C_1 = G
\end{array}
\]

and \(C_2 = (G, C, \delta)\), we need to define a trivial \(G\)-crossed module \(C_n = (G, C_n, 0)\) on which \(\text{im}(\delta)\) acts trivially. For this, it is sufficient to define a \(\pi_1(C)\)-module \(\overline{C}_n\) over \(\pi_1(C)\) and to put \(C_n = \overline{C}_n \circ q\), where \(q : G \rightarrow \pi_1(C)\) is the canonical map. For each object \(x \in G\) we define the abelian group \(\overline{C}_n(x)\) as the free abelian group generated by all pairs \(\langle t, u \rangle\) where \(t : z \rightarrow x\) is a map in \(\pi_1(C)\) and \(u \in X\) such that \(f(u) \in \pi_{n-1}(C)(z)\).

This defines the functor \(\overline{C}_n\). We now take \(C_n = \overline{C}_n \circ q\) and obtain a \(G\)-crossed module \(\overline{C}_n = (C_n, 0)\) on which (by construction) \(\text{im}(\delta)\) acts trivially. Thus, to obtain an object in \(
\begin{array}{c}
\text{ACrs}_{n-1}
\end{array}
\)

we only need a morphism \(\partial_n : C_n \rightarrow C_{n-1}\) such that \(\partial_{n-1} \partial_n = 0\). Given \(x \in G\) we define \((\partial_n)x : C_n(x) \rightarrow C_{n-1}(x)\) as the group homomorphism defined on the generators of \(C_n(x)\) by

\[
(\partial_n)x(t, u) = tf(u) = C_{n-1}(t)(f(u)).
\]

We have thus defined an \(n\)-truncated crossed complex

\[
\begin{array}{c}
F_n((\mathcal{C}, X, f)) : C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \rightarrow \cdots \rightarrow C_2 \xrightarrow{\partial_2} C_1 = G
\end{array}
\]

It remains to define \(F_n\) on arrows. Given an arrow \((\alpha, \beta) : (X, f, \mathcal{C}) \rightarrow (X', f', \mathcal{C}')\) in \(\text{ACrs}_{n-1}\), a map of crossed complexes is determined by \(\alpha\) with the additional component \(\alpha_n\) being the natural transformation with components defined on the generators of \(C_n(x)\) by

\[
(\alpha_n)x(t, u) = \alpha_0(t), \beta(u),
\]

where \(\alpha_0\) is the “change of base groupoid” part of \(\alpha\). It is now a straightforward exercise to verify that \(F_n\) is a functor \(\text{Crs}_n \rightarrow \text{ACrs}_{n-1}\) and that this functor is left adjoint to the forgetful \(U_n\) (see [13, Proposición 3.2.10, pp. 161–166], for the details).
As in Proposition 2.5, it is easy to see that $U_n$ reflects isomorphisms. The proof is completed by a calculation similar to the one used in the said proposition, showing that $U_n$ preserves coequalizers of $U_n$-contractible pairs.

Note that the counit $\epsilon_C : F_n U_n(C) \to C$ of the adjunction $F_n \dashv U_n$ is an identity at dimensions other than $n$. The same thing occurs with the unit $\eta_C : U_n F_n U_n(C) \to U_n(C)$ and with its image by $F_n$.

**Proposition 4.10.** In the hypothesis of Proposition 4.8, the 2-torsor obtained from any 2-extension of $C$ by $A$ is $U_n$-split. Therefore, the functor defined in Proposition 4.8 actually represents a full and faithful functor

$$\text{Ext}^2(C, A) \to \text{Tor}^2_{U_n}(C, \tilde{A}).$$

**Proof.** Since all components of the coequalizer map $F_0 \to C$ in dimensions $< n$ are identities, that the obtained torsor is $U_n$-split is equivalent to the fact that the $n$-dimensional component $\tau : F \to C_n$ is surjective.

We denote $G_n$ the cotriple induced on $\text{Crs}_n$ by the monadic functor $U_n$, that is, $G_n = F_n U_n$. Given a $n$-truncated crossed complex $C$, the adjoint pair $(F_n, U_n)$ induces an adjoint pair (also denoted $(F_n, U_n)$) between the corresponding slice categories $\text{Crs}_n/C$ and $\text{ACrs}_{n-1}/U_n(C)$. Furthermore, the induced $U_n$ on the slice is again monadic. Thus we obtain again a cotriple on the category $\text{Crs}_n/C$ and this cotriple will also be denoted $G_n$.

**Proposition 4.11.** For any $C \in \text{Crs}_n$ and any $A : \pi_1(C) \to \text{Ab}$, the inclusion functor $\text{Tor}^2_{U_n}(C, \tilde{A}_n) \hookrightarrow \text{Tor}^2_{U_n}(C, \tilde{A}_n)$ factors through the full and faithful functor of Proposition 4.10. As a consequence,

$$H^2_{G_n}(C, \tilde{A}_n) \cong \text{Ext}^2[C, A].$$

**Proof.** Again this proof is based on the fact that the counit of the adjunction $F_n \dashv U_n$ is the identity on the $(n-1)$-truncation, then the groupoid fiber of any 2-torsor in $\text{Tor}^2_{U_n}(C, \tilde{A}_n)$ is isomorphic to one which comes from a 2-extension.

In view of the last results, the fibrations in the Postnikov tower of a crossed complex $C$ can be regarded as 2-extensions, and we have seen they represent a cotriple cohomology of crossed complexes. The fibration $\eta_{n+1} : P_{n+1}(C) \to P_n(C)$ uniquely corresponds to an element

$$k_{n+1} \in H^2_{G_n}(P_n(C), \tilde{A}_1),$$

where $\Lambda = \pi_{n+1}(P_n(C))$. This cohomology element will be called the algebraic $(n + 1)$th Postnikov invariant of the crossed complex $C$. 
4.2. Singular cohomology and the topological invariants

The knowledge of the above algebraic Postnikov invariants is sufficient information to reconstruct the (homotopy type of the) crossed complex $C$. This solves the problem of calculating the Postnikov invariants of any space having the homotopy type of a crossed complex. However, in order to relate the above invariants to the usual calculation of the Postnikov invariants of a space (as elements in the singular cohomology of the space), we would like to show that our algebraically obtained invariants determine, in a natural way, singular cohomology elements.

Remember that the “fundamental crossed complex” functor (see [5]) associates a crossed complex to each simplicial set,

$$\Pi : \mathcal{S} \rightarrow \text{Crs}.$$  

This functor has a right adjoint “nerve of a crossed complex” which associates to each crossed complex $C$ the simplicial set whose set of $n$-simplices is the set of maps of crossed complexes

$$\text{Ner}(C)_n = \text{Crs}(\Pi(\Delta[n]), C).$$

Applying this nerve functor to a fibration of crossed complexes one obtains a fibration of simplicial sets whose fibers have the same homotopy type as those of the initial fibration.

Since the geometric realization functor $\mathcal{S} \rightarrow \text{Top}$ also preserves fibrations and the homotopy type of their fibers, we can define a functor “classifying space of a crossed complex” which again has the same property. If we restrict ourselves to the category $\mathcal{T}_{\text{crs}}$ of those spaces having the homotopy type of a crossed complex we obtain two functors

$$\Pi : \mathcal{T}_{\text{crs}} \rightarrow \text{Crs}, \quad B : \text{Crs} \rightarrow \mathcal{T}_{\text{crs}},$$

which induce an equivalence in the corresponding homotopy categories and which allow us to reduce the calculation of the Postnikov towers of the spaces in $\mathcal{T}_{\text{crs}}$, to the calculation of Postnikov towers in $\text{Crs}$.

We define the singular cohomology of a crossed complex $C$ as the singular cohomology of its geometric realization, $B(C)$, that is, of the geometric realization of its nerve. Thus, if $\Pi = \pi_1(C)$ and $A : \Pi \rightarrow \text{Ab}$, we define

$$H^m_{\text{sing}}(C, A) = H^m_{\text{sing}}(B(\text{Ner}(C)), A),$$

and our aim is to establish a natural map

$$H^2_{\text{Gr}}(P_n(C), A) \rightarrow H^{n+2}_{\text{sing}}(P_n(C), A).$$

In order to establish this map, we will obtain a representation of the singular cohomology of an $n$-truncated crossed complex by homotopy classes of maps in a certain category of simplicial $n$-truncated crossed complexes (denoted $\text{SCrs}_n$) which is naturally associated to the cotriple $\mathcal{G}_n$. Then the desired map will be induced by a natural map from Duskin’s
representation of the cotriple cohomology to our representation in $\text{SCrs}_n$ of the singular cohomology. This representation is actually obtained as a “lifting” of the generalized Eilenberg–MacLane representation in simplicial sets (see [15] and [9]):

$$H^m_{\text{sing}}(C, A) = \left[\text{Ner}(C), L_{\Pi}(A, m)\right]_{\text{SCrs}/\text{Ner}(\Pi)}$$  \hspace{1cm} (32)

where $L_{\Pi}(A, m)$ is the canonical fibration from the homotopy colimit of the functor $K(A(\cdot), m)$ to $K(\Pi, 1)$ (= Ner$(\Pi)$) (see [15] and [9]). The “ladder” through which this lifting is achieved is a chain of adjunctions

$$\xymatrix{ S \ar[r]^{\mathcal{F}_1 = G} \ar@<1ex>[d]_W & \text{SCrs}_1 \ar[r]^{\mathcal{F}_2} \ar@<1ex>[d]_W & \cdots \ar[r]^{\mathcal{F}_{n-1}} \ar@<1ex>[d]_W & \text{SCrs}_n \ar@<1ex>[d]_W \cdots }$$

having sufficiently good properties so as to preserve homotopy classes of certain maps.

We will first establish the adjunctions and their properties in the cases $n = 1$ (essentially done in [11]) and $n = 2$, which are the harder cases. Later, the cases $n \geq 3$ will be dealt with.

Let us begin by introducing the categories $\text{SCrs}_n$, $n \geq 0$, whose definition is motivated by a special property of the simplicial crossed complexes arising from the cotriple $G_n$.

Let $G_n^* : \text{Crs}_n \to \text{Crs}_n^{\Delta^\text{op}}$ denote the functor associating to each $n$-truncated crossed complex $C$ the value of the simplicial resolution of $G_n$ at $C$, so that $G_n^*(C)$ is a simplicial crossed complex which at dimension $k$ is equal to $G_n^k(C) = G_n^{k+1}(C)$ (where $G_n^k$ is the $k$-fold composition $G_n \circ \cdots \circ G_n$). We can state the following (obvious) lemma, which provides the definition of $\text{SCrs}_n$.

**Lemma 4.12.** For all $n \geq 1$ and all $k \geq 0$ the crossed complexes $G_n^k(C)$ have the same $(n - 1)$-truncation and all faces and degeneracies of $G_n^k(C)$ at all dimensions are morphisms of crossed complexes whose $(n - 1)$-truncation is the identity of $T_{n-1}(C)$. The functor $G_n^*$ factors through the subcategory $\text{SCrs}_n$ defined by the following pullback of categories:

$$\xymatrix{ \text{SCrs}_n \ar[r]^-{\mathcal{F}} & \text{Crs}_{n-1} & \text{Crs}_n \ar[r]^-{\mathcal{F}} & \text{Crs}_{n-1}^{\Delta^\text{op}}. }$$

(33)

The category $\text{SCrs}_1$ is the category of those simplicial groupoids having the same set of objects in all dimensions and all faces and degeneracies equal to the identity at the level of objects. This is precisely the category (denoted $Gd$ in [11]) on which Dwyer and Kan define the classifying complex functor, $W$, extending the classical functor $W : \text{Gr}^{\Delta^\text{op}} \to \text{Set}^{\Delta^\text{op}}$ of [12,17]. We will find it convenient to define $\text{SCrs}_0 = \text{Set}^{\Delta^\text{op}} (= \mathcal{S})$ and to refer to $W$ as $W_1 : \text{SCrs}_1 \to \text{SCrs}_0$. Note that $\text{SCrs}_1$ can also be described as the category of groupoids enriched in simplicial sets.
The category $\text{SCrs}_1$ is the subcategory of $\text{Cr}_n \xrightarrow{\Delta^\text{op}} \text{Xm}$ whose objects are simplicial crossed modules $\Sigma: \Delta^\text{op} \to \text{Xm}$ such that for all dimensions the crossed modules $\Sigma_n$ have the same base groupoid and whose morphisms from $\Sigma$ to $\Sigma'$ are simplicial maps $\alpha: \Sigma \to \Sigma'$ having in each dimension the same change-of-base functor.

4.2.1. The lifting to $Gd = \text{SCrs}_1$

As we said above $\tilde{W}_1$ is the functor $\tilde{W}$ defined in [11]. We repeat here the definition in order to have it handy. If $\Sigma \in \text{SCrs}_1$, then the vertices of $\tilde{W}_1(\Sigma)$ are the objects of $\Sigma$, the $n$-simplices ($n > 0$) are the sequences of arrows $(z_n \xrightarrow{u_{n-1}} z_{n-1} \to \cdots \to z_1 \xrightarrow{u_0} z_0)$ where $u_i$ is an arrow in the groupoid $\Sigma_i$ for $i = 0, \ldots, n-1$, and faces and degeneracies given by the formulas

$$d_i(u_0, \ldots, u_{n-1}) = (u_0, \ldots, u_{n-i-2}, u_{n-i-1} \cdot d_iu_{n-i}, d_iu_{n-i+1}, \ldots, d_{i-1}u_{n-1}),$$

$$s_i(u_0, \ldots, u_{n-1}) = (u_0, \ldots, u_{n-i-1}, id, s_iu_{n-i}, \ldots, s_{i-1}u_{n-1}).$$

(Note: this is different from [11, p. 383] where there is an error in the indices.)

Evidently, if $\Pi$ is a groupoid regarded as a constant simplicial groupoid, then the $n$-simplices of $\tilde{W}(\Pi)$ are precisely the $n$-simplices of $\text{Ner}(\Pi)$ and in fact we have $\tilde{W}_1(\Pi) = \text{Ner}(\Pi)$. Thus, $\tilde{W}_1$ can be regarded as a functor

$$\tilde{W}_1: \text{SCrs}_1/\Pi \longrightarrow \text{Set}^{\Delta^\text{op}}/\text{Ner}(\Pi). \tag{34}$$

**Proposition 4.13.** Let $\Pi$ be a groupoid, let $A: \Pi \to \text{Ab}$ be a $\Pi$-module and let $\tilde{A}_1 = \text{in}_A(A) \supseteq \Pi \ltimes A$, seen as an abelian group object in $\text{Gpd}/\Pi$. Then the simplicial object $K(\tilde{A}_1, n) \in (\text{Gpd}/\Pi)^{\Delta^\text{op}} = \text{Gpd}^{\Delta^\text{op}}/\Pi$ actually belongs to $\text{SCrs}_1/\Pi$ (where $\Pi$ is regarded as the trivial “constant” simplicial groupoid) and

$$\tilde{W}_1(K(\tilde{A}_1, n)) = L_{\Pi}(A, n + 1). \tag{35}$$

**Proof.** Most of what is said in the statement is evident, the essential part being the proof of (35). This can actually be deduced from a more general isomorphism between $L_{\Pi}(A, n + 1)$ and the nerve of certain higher dimensional groupoid which is actually equal to $\tilde{W}_1(K(\tilde{A}_1, n))$ (see [13, Proposition 2.4.9]). Here we will not make use of these general facts but will indicate just how to verify the equality at the level of simplices so as to justify the dimensional jump, and will not bore the reader with the tedious verification of the equality of faces and degeneracies. Since the $m$-simplices in both simplicial sets are open horns for every $m > n + 2$, it is sufficient to prove

$$\tilde{W}_1(K(\tilde{A}_1, n))_m = L_{\Pi}(A, n + 1)_m, \quad m = 0, \ldots, n + 2.$$

In the right-hand side we have,

$$L_{\Pi}(A, n + 1)_m = \begin{cases} \text{Ner}(\Pi)_m, & \text{if } m \leq n, \\ \bigsqcup_{\xi \in \text{Ner}(\Pi)_{n+1}} A(x^\xi_0), & \text{if } m = n + 1, \\ \bigsqcup_{\xi \in \text{Ner}(\Pi)_{n+2}} A(x^\xi_0)^{n+2}, & \text{if } m = n + 2, \end{cases}$$
where we used the representation of the generalized Eilenberg MacLane spaces \( L_{\Pi}(A,n) \) given in [9]. On the other hand, it is clear, using the fact that \( K(\widetilde{A},n)_{m} = (\Pi_{n} \xrightarrow{\Pi_{m}} \Pi) \in \text{Gpd}/\Pi, m = 0, \ldots , n - 1 \), and the definition of the \( m \)-simplices of \( \widetilde{W} \) given in [11], that for \( m < n \), \( \widetilde{W}_{1}(K(\widetilde{A},n))_{m} = \text{Ner}(\Pi)_{m} \). By the same token, an \( n \)-simplex in \( \widetilde{W}_{1}(K(\widetilde{A},n)) \) is a sequence \( (z_{n} \xrightarrow{u_{n-1}} z_{n-1} \rightarrow \cdots \rightarrow z_{1} \xrightarrow{u_{0}} z_{0}) \) in \( \Pi \) and we get again \( \widetilde{W}_{1}(K(\widetilde{A},n))_{n} = \text{Ner}(\Pi)_{n} \). Let us now consider an \( n + 1 \) simplex in \( \widetilde{W}_{1}(K(\widetilde{A},n)) \), that is, a sequence

\[
\xi = (z_{n+1} \xrightarrow{(u_{n+1},a)} z_{n} \xrightarrow{(u_{n},a)} z_{n-1} \rightarrow \cdots \rightarrow z_{1} \xrightarrow{u_{0}} z_{0})
\]

where the \( u_{i} \) are arrows in \( \Pi \) and furthermore \( z_{n+1} \xrightarrow{(u_{n+1},a)} z_{n} \) is an arrow in \( \Pi \times A \). This means that \( a \) is an arbitrary element in \( A(z_{n}) \equiv A(z_{n+1}) = A(\xi) \) (using the isomorphism \( A(u_{n}^{-1}) : A(z_{n}) \rightarrow A(z_{n+1}) \)). Thus,

\[
\widetilde{W}_{1}(K(\widetilde{A},n))_{n+1} = \coprod_{\xi \in \text{Ner}(\Pi)_{n+1}} A(\xi) = L_{\Pi}(A,n + 1)_{n+1}.
\]

Finally, let us consider an \( n + 2 \) simplex in \( \widetilde{W}_{1}(K(\widetilde{A},n)) \), that is, a sequence

\[
\xi = (z_{n+2} \xrightarrow{(u_{n+1},a)} z_{n+1} \xrightarrow{(u_{n},a)} z_{n} \rightarrow \cdots \rightarrow z_{1} \xrightarrow{u_{0}} z_{0})
\]

where the \( u_{i} \) are arrows in \( \Pi \) and furthermore \( z_{n+1} \xrightarrow{(u_{n+1},a)} z_{n} \) and \( z_{n+2} \xrightarrow{(u_{n},a)} z_{n+1} \) are arrows in \( \Pi \times A \) and \( (\Pi \times A)^{n+1} \), respectively. This means that \( a \in A(z_{n}) \) and \( \alpha = (a_{1}, \ldots , a_{n+1}) \in A(z_{n+1})^{n+1} \), while

\[
\xi' = (z_{n+2} \xrightarrow{u_{n+1}} z_{n+1} \xrightarrow{u_{n}} z_{n} \rightarrow \cdots \rightarrow z_{1} \xrightarrow{u_{0}} z_{0}) \in \text{Ner}(\Pi)_{n+2}.
\]

Thus, we get an \( n + 2 \) simplex \( (\xi', \alpha') \) in \( L_{\Pi}(A,n+1) \) with \( \alpha' = (a'_{0}, \ldots , a'_{n+1}) \in A(z_{n+2})^{n+2} \)

where

\[
\begin{align*}
a'_{0} &= A(u_{n}u_{n+1})^{-1}(a), \\
a'_{1} &= A(u_{n+1})^{-1}(a_{1}), \\
& \vdots \\
a'_{n} &= A(u_{n+1})^{-1}(a_{n}), \\
a'_{n+1} &= A(u_{n+1})^{-1}\left(\sum_{i=1}^{n+1} (-1)^{n+1-i} a_{i}\right).
\end{align*}
\]

It is now straightforward to verify that the correspondences between simplices in \( L_{\Pi}(A,n + 1) \) and in \( \widetilde{W}_{1}(K(\widetilde{A},n)) \) we have established is a \( (n + 2) \)-truncated bijective simplicial map whose \( (n + 2) \)-component satisfies the cocycle condition, thus determining a simplicial isomorphism. \( \square \)
A simple extension of [11, Theorem 3.3] yields without difficulty the following proposition.

**Proposition 4.14.** The functor $\overline{W}_1$ in (34) preserves fibrations and weak equivalences; it has a left adjoint $F_1$ which also preserves fibrations and weak equivalences and for every pair of objects $X \in \text{SCrs}_1/\Pi$, $Y \in \text{Set}^{\Delta^\text{op}}/\text{Ner}(\Pi)$ a map $Y \to \overline{W}_1 X$ is a weak equivalence if and only if its adjoint $F_1 Y \to X$ is a weak equivalence.

It follows from this that the adjunction goes through to the corresponding homotopy categories and as a consequence the set of classes of homotopic maps $Y \to \overline{W}_1 X$ in $\text{Set}^{\Delta^\text{op}}/\text{Ner}(\Pi)$ is bijective with the set of classes of homotopic maps $F_1 Y \to X$ in $\text{SCrs}_1/\Pi$. Taking $X = K(\tilde{A}_1, m)$ and $Y = \text{Ner}(C)$, we have

**Corollary 4.15.** For any crossed complex $C$, let $\Pi = \pi_1(C)$ be its fundamental groupoid and let $A : \Pi \to \text{Ab}$ be any $\Pi$-module. Then, if $\tilde{A}_1 = \Pi \ltimes A$ is regarded as an abelian group object in $\text{Gpd}/\Pi$,

$$H_{\text{sing}}^{m+1}(C, A) = [F_1 \text{Ner}(C), K(\tilde{A}_1, m)]_{\text{SCrs}_1/\Pi}.$$

In higher dimensions we will avoid trying to generalize Proposition 4.14, proving instead directly the higher dimensional analog of Corollary 4.15.

4.2.2. The lifting to $\text{SCrs}_2$

We now define the functor $\overline{W}_2 : \text{SCrs}_2 \to \text{SCrs}_1$.

Let $\text{SGd} \subseteq \text{Gpd}^{\Delta^\text{op} \times \Delta^\text{op}}$ be the full subcategory of double simplicial groupoids determined by those simplicial groupoids all whose vertical and horizontal faces and degenerations are (functors of groupoids which are) the identity on objects. Equivalently, $\text{SGd}$ is the category of groupoids enriched in double simplicial sets. We first notice that the Artin–Mazur diagonal functor, $\overline{W}_{A,M}$, takes object in $\text{SGd}$ to objects in $\text{SCrs}_1$.

$$\begin{array}{ccc}
\text{SGd} & \xrightarrow{\overline{W}} & \text{SCrs}_1 \\
\downarrow & & \downarrow \\
\text{Gpd}^{\Delta^\text{op} \times \Delta^\text{op}} & \xrightarrow{\overline{W}_{A,M}} & \text{Gpd}^{\Delta^\text{op}}.
\end{array}$$

On the other hand, there is an isomorphism between the category of crossed modules and the subcategory $(\text{SCrs}_1)_2 \subseteq \text{SCrs}_1$ of simplicial groupoids with trivial Moore complex in dimensions $\geq 2$ [13, Corollary 3.1.10] which, composed with the inclusion $(\text{SCrs}_1)_2 \hookrightarrow$
\( \text{SCrs}_1 \) induces a functor \( \text{Xm}^{\Delta^\text{op}} \to \text{SCrs}_1^{\Delta^\text{op}} \) whose restriction to \( \text{SCrs}_2 \) takes its values in \( \text{SGd} \).

\[
\begin{array}{ccc}
\text{SCrs}_2 & \xrightarrow{\theta} & \text{SGd} \\
\downarrow & & \downarrow \\
\text{Xm}^{\Delta^\text{op}} & \xrightarrow{} & (\text{SCrs}_1)^{\Delta^\text{op}}.
\end{array}
\]

We define \( \overline{W}_2 \) as the composition

\[
\overline{W}_2 : \text{SCrs}_2 \xrightarrow{\theta} \text{SGd} \xrightarrow{\overline{\iota}} \text{SCrs}_1.
\]

We next describe the action of \( \overline{W}_2 \) on objects. Let \( (G, \Sigma) \in \text{SCrs}_2 \) so that for each \( n \geq 0 \), \( \Sigma_n = (G, C_n, \delta_n) \) is a \( G \)-crossed module. The simplicial groupoid \( \overline{W}_2(G, \Sigma) \) has the same object as \( G \). Its groupoid of 0-simplices is \( \overline{W}_2(G, \Sigma)_0 = G \); its groupoid of 1-simplices can be identified with

\[
\overline{W}_2(G, \Sigma)_1 = G \ltimes C_0
\]

with face and degeneration morphisms given by the following formulas:

\[
\begin{align*}
d_0(f, a_0) &= (\delta_0)_y(a_0) f, \\
d_1(f, a_0) &= d_1^f(f) = f, \\
s_0(f) &= (f, 0; C_0(y)).
\end{align*}
\]

In general, for \( n \geq 2 \), the set of arrows from \( x \) to \( y \) in the groupoid of \( n \)-simplices of \( \overline{W}_2(G, \Sigma) \) is given by

\[
\text{Hom}_{\overline{W}_2(G, \Sigma)}(x, y) = \text{Hom}_G(x, y) \times C_0(y) \times \cdots \times C_{n-1}(y).
\quad (36)
\]

We therefore denote

\[
\overline{W}_2(G, \Sigma)_n = G \ast C_0 \ast \cdots \ast C_{n-1}
\]

the groupoid of \( n \)-simplices of \( \overline{W}_2(G, \Sigma) \). The composition in this groupoid is given by the formula:

\[
(g, b_0, b_1, \ldots, b_{n-1})(f, a_0, a_1, \ldots, a_{n-1})
= (gf, b_0 + (\delta_1)_1; (b_1)_{(b_2)}(b_{n-1}) \mathbf{s} a_0, \ldots, b_{n-2} + (\delta_{n-1})_{(b_{n-1})} \mathbf{s} a_{n-2}, b_{n-1} + \mathbf{s} a_{n-1}).
\]
Note that again this groupoid is a kind of semidirect product. The face and degeneration operators
\[ \left( \overline{W}_2(\mathcal{G}, \Sigma) \right)_n \overset{s_j}{\leftarrow} \left( \overline{W}_2(\mathcal{G}, \Sigma) \right)_{n+1} \overset{d_i}{\rightarrow} \left( \overline{W}_2(\mathcal{G}, \Sigma) \right)_{n-1} \]
are given by the formulas:
\[
\begin{align*}
d_0(f, a_0, a_1, \ldots, a_{n-1}) &= \left( (\delta_{n-1})_y(a_{n-1})f, a_0, \ldots, a_{n-2} \right), \\
d_0(f, a_0, a_1, \ldots, a_{n-1}) &= (f, a_0, \ldots, a_{n-i-2}, a_{n-i-1} + d_0a_{n-i}, d_1a_{n-i+1}, \ldots, d_{i-1}a_{n-1}), \\
d_0(f, a_0, a_1, \ldots, a_{n-1}) &= (f, d_1a_1, \ldots, d_{n-1}a_{n-1}), \\
s_j(f, a_0, a_1, \ldots, a_{n-1}) &= (f, a_0, \ldots, a_{n-j-1}, 0_{C_{n-j}(y)}, s_0a_{n-j}, \ldots, s_{j-1}a_{n-1}).
\end{align*}
\] (37)

**Example.** If \( \Pi \) is a groupoid regarded as a discrete constant simplicial crossed module, then \( \overline{W}_2(\Pi) \) is equal to \( \Pi \) regarded as a constant simplicial groupoid.

If \( \Pi \) is a groupoid, as a consequence of \( \overline{W}_2(\Pi) = \Pi \), the functor \( \overline{W}_2 : Scrs_2 \rightarrow Scrs_1 \) induces a functor \( \overline{W}_2 : Scrs_2/\Pi \rightarrow Scrs_1/\Pi \). Let \( A : \Pi \rightarrow \text{Ab} \) be a \( \Pi \)-module. If, as before, for \( n \geq 1 \), \( \tilde{A}_n = \text{ins}_n(A) \) is regarded as an abelian group object in \( Crs_n/\Pi \), we have:

**Proposition 4.16.**
\[ \overline{W}_2(K(\tilde{A}_2, m)) = K(\tilde{A}_1, m + 1). \]

**Proof.** \( K(\tilde{A}_2, m) \) is the simplicial crossed module \( (\Pi, \Sigma) \) where
\[ \Sigma_n = (\Pi, C_n, 0) \]
with
\[ C_n(y) = \begin{cases} 
0, & \text{if } n < m, \\
A(y), & \text{if } n = m, \\
A(y)^{m+1}, & \text{if } n = m + 1.
\end{cases} \]

Thus, formula (36) yields in this case,
\[
\overline{W}_2(K(\tilde{A}_2, m))_n(x, y) = \begin{cases} 
\Pi(x, y), & \text{if } n \leq m, \\
\Pi(x, y) \times A(y), & \text{if } n = m + 1, \\
\Pi(x, y) \times A(y)^{m+2}, & \text{if } n = m + 2.
\end{cases}
\]
Thus, one obtains for \( K(\tilde{A}_1, m + 1) \). The details are in [13, Corollary 4.3.8]. □
We define now a left adjoint to $\overline{W}_2$, which is similar to the “loop groupoid” functor $F_1 = G$ defined in [11].

Let $\Sigma_n$ be the groupoid of $n$-simplices of the simplicial groupoid $\Sigma$. We have a simplicial diagram of pre-crossed modules which in dimension $n$ has the pre-crossed module $(\Sigma_0, K_{n-1}, \delta)$ which, furthermore, has a split augmentation by $(\Sigma_0, K_0, \delta)$ where for $n \geq 0$. $K_n$ denotes the $\Sigma_0$-group associated to the totally disconnected groupoid defined by $\ker(d_1d_2 \cdots d_{n+1} : \Sigma_{n+1} \rightarrow \Sigma_0)$ with action given by conjugation via $s_0s_{n-1} \cdots s_0 : \Sigma_0 \rightarrow \Sigma_{n+1}$, and $\delta$ is the natural transformation whose $x$-component, for $x \in \text{obj}(\Sigma_0)$, is

$$
\delta_x(u) = \begin{cases} 
  d_0(u), & \text{if } u \in K_0(x), \\
  d_0d_2 \cdots d_{n+1}(u), & \text{if } u \in K_n(x) \text{ with } n \geq 1.
\end{cases}
$$

Note that the face and degeneration operators

$$(\Sigma_0, K_{n+1}, \delta) \xleftarrow{\sigma_j} (\Sigma_0, K_n, \delta) \xrightarrow{\delta_i} (\Sigma_0, K_{n-1}, \delta)$$

for $1 \leq i \leq n$ and $0 \leq j \leq n$, and also the augmentation $\sigma_0 : (\Sigma_0, K_n, \delta) \rightarrow (\Sigma_0, K_{n+1}, \delta)$ for all $n \geq 0$, are given by restrictions of the faces $d_{i+1} : \Sigma_{n+1} \rightarrow \Sigma_n$ and of the degeneracies $s_{j+1} : \Sigma_{n+1} \rightarrow \Sigma_{n+2}$ of the simplicial groupoid $\Sigma$.

From this we build an augmented split simplicial crossed module by factoring out in each $\Sigma_0$-module, the Peiffer elements as well as those who are images by $s_0$. Such quotient determines, in each dimension, a crossed module $(\Sigma_0, \tilde{K}_n, \delta)$ and the face and degeneration operators go well with the quotients. In order to obtain the simplicial crossed module $F_2(\Sigma)$ we just need to add in each dimension a new morphism of crossed modules which will be given by the morphism of $\Sigma_0$-groups $\delta_0 : \tilde{K}_n \rightarrow \tilde{K}_{n-1}$, in turn determined by the natural transformation $[d_0, d_1] : K_n \rightarrow K_{n-1}$ whose component on an object $x \in \text{obj}(\Sigma_0)$ is given by

$$
[d_0, d_1]_x(u) = (d_1)_x(u)(d_0)_x(u)^{-1}(s_0d_1d_0)_x(u),
$$

for each $u \in K_n(x)$. The functor so defined is left adjoint to $\overline{W}_2$ (see [13, Proposition 4.3.9]).

If we regard now a groupoid $\Pi$ first as a constant simplicial groupoid and then as a constant simplicial crossed module, it is easy to check that $F_2(\Pi) = \Pi$. In fact, we still have an adjoint pair of functors

$$
F_2 \dashv \overline{W}_2, \quad \text{SCrs}_2/\Pi \xrightarrow{F_2} \text{SCrs}_1/\Pi \xrightarrow{\overline{W}_2} \text{SCrs}_2/\Pi.
$$

We next show that in certain cases the functors $F_2$ and $\overline{W}_2$ preserve homotopy classes. First we note that in order that two morphisms $(F, \alpha), (G, \beta) : (G, \Sigma) \rightarrow (G', \Sigma')$ in $\text{SCrs}_2$ be homotopic, both must have the same functor at the level of base groupoids, that is, $F = G$.
Lemma 4.18. Let $F, F': \Sigma \to \Sigma'$ be two morphisms in $\text{SCrs}_1$ such that $F_0 = F'_0$ and let $H$ be a homotopy between them, such that

$$H^0_0 = s_0 F_0 \quad \text{and} \quad H^j_i = s_i F_j, \quad \text{if } i < j$$
(note that $H^j$ is arbitrary for $j > 0$), then the morphisms of simplicial crossed modules $\mathcal{F}_2(F)$ and $\mathcal{F}_2(F')$ in $\text{SCrs}_2$ are homotopic.

The two previous lemmas can easily be proved for slice categories. We have:

**Lemma 4.19.** For each groupoid $\Pi$, the functor $\overline{W}_2 : \text{SCrs}_2/\Pi \to \text{SCrs}_1/\Pi$ preserves homotopy classes of simplicial morphisms in the corresponding slice categories.

**Lemma 4.20.** Let $\Pi$ be a groupoid, $F$ and $F'$ two morphisms in $\text{SCrs}_1/\Pi$

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{F} & \Sigma' \\
\downarrow & & \downarrow \\
\Pi & \xrightarrow{F'} & \Pi
\end{array}
\]

such that $F_0 = F'_0$ and let $H$ be a homotopy between them in $\text{SCrs}_1/\Pi$

\[
H^0_0 = s_0F_0 \quad \text{and} \quad H^j_0 = s_iF_j, \quad \text{if } i < j.
\]

Then the morphisms of simplicial crossed modules $\mathcal{F}_2(F)$ and $\mathcal{F}_2(F')$ in the slice category $\text{SCrs}_2/\Pi$ are homotopic.

Note that if in Lemma 4.20 we take $\Sigma' = \mathcal{K}(\tilde{A}_1, n) \in \text{SCrs}_1/\Pi$ for any $\Pi$-module $A$, then for any two homotopic morphisms in $\text{SCrs}_1/\Pi$ with codomain $\mathcal{K}(\tilde{A}_1, n)$ there is a homotopy $H$ in the hypothesis of the said lemma and therefore we can conclude that the functor $\mathcal{F}_2$ preserves homotopy classes of morphisms with codomain $\mathcal{K}(\tilde{A}_1, n)$.

From the preceding reasoning it follows immediately,

**Proposition 4.21.** Let $\Pi$ be a groupoid, $X$ a simplicial groupoid above $\Pi$, and let $A$ be $\Pi$-module. Then the adjunction $\mathcal{F}_2 \dashv \overline{W}_2$,

\[
\begin{array}{ccc}
\text{SCrs}_2/\Pi & \xrightarrow{\mathcal{F}_2} & \text{SCrs}_1/\Pi \\
\downarrow \overline{W}_2 & & \downarrow \overline{W}_2
\end{array}
\]

induces an isomorphism in homotopy classes:

\[
\left[\mathcal{F}_2(X), \mathcal{K}(\tilde{A}_2, n)\right]_{\text{SCrs}_2/\Pi} \cong \left[X, \overline{W}_2(\mathcal{K}(\tilde{A}_2, n))\right]_{\text{SCrs}_1/\Pi}. \quad (38)
\]
4.2.3. The lifting to $\text{SCrs}_n$ for $n \geq 3$

For $n \geq 3$ each object of $\text{SCrs}_n$ can be represented by a diagram of this form:

$$
\xymatrix{
\cdots & C_n^i \ar[r]^{d_i} \ar[dr]^{s_i} & C_{n-1}^i \ar[d]^{d_i} & \cdots & C_n^1 \ar[r]^{d_1} \ar[dr]^{s_1} & C_{n-1}^1 \ar[d]^{d_1} & \cdots & C_n^0 \ar[r]^{d_0} \ar[dr]^{s_0} & C_{n-1}^0 \ar[d]^{d_0} & \cdots \\
& C_{n-2} \ar[d] & \vdots & \vdots & \vdots & \vdots & C_2 \ar[d] & \vdots & \vdots & 1_G \\
& & & & & & & & & 
}
$$

(39)

Giving such an object is equivalent to giving its “head:”

$$
\xymatrix{
C_n : & \cdots & C_n^i \ar[r]^{d_i} & C_{n-1}^i & \cdots & C_n^1 \ar[r]^{d_1} & C_{n-1}^1 & \cdots & C_n^0 \ar[r]^{d_0} & C_{n-1}^0 & \cdots \\
& & & & & & & & & (40)
}
$$

(a simplicial complex of $G$-modules, where $G$ is the base groupoid of all the involved crossed complexes), its “tail”

$$
\mathcal{C} = (C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \rightarrow \cdots \rightarrow C_2 \xrightarrow{\partial_2} 1_G)
$$

(an $(n-1)$-truncated crossed complex), and an augmentation $C_n \xrightarrow{\partial_0} C_{n-1}^0$ of $C_n$ over $C_{n-1} = \text{cell}_{n-1} (\mathcal{C})$ such that the compositions

$$
C_n^0 \xrightarrow{\partial_0} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \xrightarrow{\partial_2} 1_G \\
\widehat{\mathcal{C}}_2 \xrightarrow{\alpha_2} G \xrightarrow{\partial_2} \text{Ab}
$$

are trivial.

We will represent the above $n$-truncated crossed complex by the pair $(\mathcal{C}, C_n)$ or the triple $(G, \mathcal{C}, C_n)$ in case we want to make explicit the base groupoid.

A map from $(\mathcal{C}, C_n)$ to $(\mathcal{C}', C_n')$ in $\text{SCrs}_n$ is a pair $(f, \alpha)$ where $f : \mathcal{C} \rightarrow \mathcal{C}'$ is a morphism of $(n-1)$-truncated crossed complexes with $F = \text{base}(f)$ and $\alpha : C_n \rightarrow F^* C_n'$ is a simplicial map of $G$-modules where $G = \text{base}(\mathcal{C})$, $G' = \text{base}(\mathcal{C}')$, and $F^* : \text{Ab}^G \rightarrow \text{Ab}^{G'}$ is the functor induced by $F = \text{base}(f)$.
It is possible to give a definition of $\overline{W}_n$ in terms of the Artin–Mazur diagonal as for the case $n = 2$. But for $n > 2$ it is also possible to give a direct description without going through double simplicial $n$-truncated crossed complexes:

$$\overline{W}_n(C, C_n) = (T_{n-2}(C), C_{n-1}),$$

where $C_{n-1}$ is the simplicial $G$-module (where $G = \text{base}(C)$) augmented over $C_{n-2} = \text{col}_n-2(C)$ given by

$$C_{n-1}^0 = C_{n-1} \quad \text{and} \quad C_{n-1}^i = C_{n-1}^{i-1} \oplus \cdots \oplus C_n^0 \oplus C_{n-1}, \quad \text{if } 1 \leq i,$$

where $C_n^i = (G, C_n^i, 0)$ for $i \geq 0$, with augmentation $\partial_{n-1}^0 = \partial_{n-1} : C_{n-1} \to C_{n-2}$ and with face and degeneration operators

$$(d_0)_x(u_{i-1}, \ldots, u_0, u) = (u_{i-2}, \ldots, u_0, \partial_{n-1}^{-1}(u_{i-1}) + u),$$
$$(d_i)_x(u_{i-1}, \ldots, u_0, u) = (d_{i-1}u_{i-1}, \ldots, d_iu_1, u),$$
$$(d_j)_x(u_{i-1}, \ldots, u_0, u) = (d_{i-1}u_{i-1}, \ldots, d_ju_{j-1}, d_0u_{j-1} + u_{i-j}, \ldots, u_0, u) \quad \text{if } 1 \leq j < i,$$
$$(s_j)_x(u_{i-1}, \ldots, u_0, u) = (s_{j-1}u_{i-1}, \ldots, s_0u_{i-j}, 0, u_{i-j-1}, \ldots, u_0, u) \quad \text{if } 0 \leq j \leq i.$$

It is obvious how $\overline{W}_n$ acts on morphisms.

If a groupoid $G$ is regarded as the simplicial $n$-truncated crossed complex $(G, 0)$, we have

$$\overline{W}_n(G) = \overline{W}_n(G, 0) = (G, 0) = G.$$

As a consequence, $\overline{W}_n$ induces a functor between slice categories,

$$\overline{W}_n : \text{SCrs}_n/G \to \text{SCrs}_{n-1}/G.$$

**Lemma 4.22.** If $(C, C_n) \in \text{SCrs}_n$ is such that $C_n$ has trivial Moore complex in dimensions $\geq m$, then upon applying $\overline{W}_n$, we get

$$\overline{W}_n(C, C_n) = (T_{n-2}(C), C_{n-1}),$$

where $C_{n-1}$ again has trivial Moore complex in dimensions $\geq m + 1$.

**Example.** Let $\Pi$ be a groupoid and $A : \Pi \to \text{Ab}$ a $\Pi$-module, for each $m \geq 1$ the triple $(\Pi, \Pi, K(A, m))$ determines a simplicial $n$-truncated crossed complex in $\text{SCrs}_n$. By the previous lemma,

$$\overline{W}_n(\Pi, \Pi, K(A, m)) \cong (\Pi, \Pi, K(A, m + 1)).$$
On the other hand,

\[ K(\bar{\Pi}_n, m) = (\Pi, \Pi, K(A, m)) \]

and therefore we have,

**Proposition 4.23.** For every \( n \geq 1 \),

\[ \bar{W}_n(K(\bar{\Pi}_n, m)) = K(\bar{\Pi}_{n-1}, m + 1) \].

We define now the left adjoint to \( \bar{W}_n \) on objects. Given \( (G, C, C_{n-1}) \in \text{SCrs}_{n-1} \), the \( G \)-modules

\[ K_0 = \ker(d_1: C_{n-1}^1 \to C_{n-1}^0), \]
\[ K_i = \ker(d_1 d_2 \cdots d_{i+1}: C_{n-1}^{i+1} \to C_{n-1}^0), \quad i \geq 1, \]

together with the operators \( d_j: K_i \to K_{i-1} \) and \( s_{j-1}: K_{i-1} \to K_i \) induced by the face operators \( d_j: C_{n-1}^{i+1} \to C_{n-1}^i \) and degeneracies \( s_{j-1}: C_{n-1}^i \to C_{n-1}^{i+1} \), for \( 2 \leq j \leq i + 1 \) define an augmented split simplicial complex of \( G \)-modules. Factoring out by the \( s_0: C_{n-1}^i \to C_{n-1}^{i+1} \)-image of \( K_{i-1} \), we get \( G \)-modules

\[ \tilde{K}_i = K_i / s_0(K_{i-1}), \]

which again determine an augmented split simplicial complex of \( G \)-modules with face operators \( \delta_j \) and degeneracies \( \sigma_j \) induced by the corresponding quotients by the operators \( d_{i+1} \) and \( s_{j+1} \). Finally, the natural transformation whose component in an object \( x \in \text{obj}(G) \) is given by

\[ (\delta_0)_x(u) = (d_1)_x(u)(d_0)_x(u)^{-1}(s_0d_1d_0)_x(u), \]

for each \( u \in K_i(x) \), determines another, also be denoted \( \delta_0: \tilde{K}_i \to \tilde{K}_{i-1} \), for \( i > 0 \), which together with the previous diagram provides us with a simplicial complex of \( G \)-modules which will be denoted \( \tilde{K}_n \). Furthermore, the restriction to \( K_0 \) of the face operator \( d_0: C_{n-1}^1 \to C_{n-1}^0 \) induces a morphism \( \tilde{\delta}_0: \tilde{K}_0 \to C_{n-1}^0 \) such that \( \tilde{K}_n / \tilde{\delta}_0 \to C_{n-1}^0 \) is an augmented simplicial \( G \)-module. Thus, we define

\[ \mathcal{F}_n(G, C, C_{n-1}) = (G, C_{n-1}^0, \tilde{K}_n), \]

where \( C_{n-1}^0 \) is the \((n - 1)\)-truncated crossed complex whose \((n - 1)\) truncation is \( C \) \((T_{n-1}(C_{n-1}^0) = C)\), and is such that \( \text{cehr}_{n-1}(C_{n-1}^0) = C_{n-1}^0 \).

It is now a routine exercise to define this functor on arrows and to verify that it is left adjoint to \( \bar{W}_n \).
Note 1. For every \( n > 3 \), the functor \( \overline{W}_n : SCrs_n \to SCrs_{n-1} \) is an equivalence of categories whose inverse is \( \mathcal{F}_n \), and as a consequence, for \( n > 3 \) the categories \( SCrs_n \) are equivalent to \( SCrs_3 \).

Let \( \Pi \) be a groupoid, regarded as a 1-truncated crossed complex. Evidently \( \Pi \) can be regarded as a \( k \)-truncated crossed complex for any \( k \), and we will do so as needed. Furthermore, we can consider the constant simplicial crossed complex \( \Pi \) as an object in \( SCrs_k \) for any \( k \). Such objects verify \( \overline{W}(\Pi^{c1}) = \Pi^{c1} \) so that the functors \( \overline{W}_k : SCrs_k \to SCrs_{k-1} \) induce functors \( \overline{W}_k : SCrs_k / \Pi^{c1} \to SCrs_{k-1} / \Pi^{c1} \). In fact, for every \( k \geq 1 \), we still have an adjoint pair of functors

\[
\mathcal{F}_k : SCrs_k / \Pi \rightleftarrows SCrs_{k-1} / \Pi : \overline{W}_k.
\]

We next show that in certain cases the functors \( \mathcal{F}_n \) and \( \overline{W}_n \) preserve homotopy classes. First, we note that the restriction to \( SCrs_n \) of the functor \((T_{n-1})_* : SCrs_n^{\Delta^{op}} \to SCrs_n^{\Delta^{op}} SCrs_{n-1} \) takes homotopies to homotopies. Thus, a homotopy \( \bar{h} : (F, f, \alpha) \to (F, f, \beta) \) is given as a pair

\[
\bar{h} = (f, h),
\]

with

\[
h = \left\{ h^i_j : C^i_n \to C^{i+1}_n ; 0 \leq j \leq i \right\} : \alpha \rightsquigarrow \beta : C_n \to F^* C'_n
\]

a homotopy in the category \( SCrs_n^{\Delta^{op}} (\text{Ab}^G) \). Furthermore, the homotopy identities for \( \bar{h} \) follow from the homotopy identities for \( h \).

We can now prove

Lemma 4.24. The functor \( \overline{W}_n \) preserves homotopy classes of simplicial morphisms.

Proof. Let \( \bar{h} = (f, h) \) be a homotopy as before, and let us put \( \overline{W}_n(F, f, \alpha) = (F, T_{n-2}(f), \alpha') \) and \( \overline{W}_n(F, f, \beta) = (F, T_{n-2}(f), \beta') \), where \( \alpha' \) and \( \beta' \) are simplicial morphisms of \( G \)-modules given in dimension \( i \) by

\[
\alpha^{i}_{n-1}(u_{i-1}, \ldots, u_0, u) = (\alpha^{i-1}_n(u_{i-1}), \ldots, \alpha^0_n(u_0), \alpha_{n-1}(u)),
\]

\[
\beta^{i}_{n-1}(u_{i-1}, \ldots, u_0, u) = (\beta^{i-1}_n(u_{i-1}), \ldots, \beta^0_n(u_0), \beta_{n-1}(u)),
\]

for each \((u_{i-1}, \ldots, u_0, u) \in C^{i-1}_n(x) \oplus \cdots \oplus C^0_n(x) \oplus C_{n-1}(x)\). Then the homotopy \( \bar{h}' = (T_{n-2}(f), \bar{h}) \) with

\[
\bar{h} = \left\{ h^i_j ; 0 \leq j \leq i \right\} : \alpha' \rightsquigarrow \beta',
\]

where \( h^i_j \) is the natural transformation whose \( x \)-component for \( x \in \text{obj}(G) \) acts thus.
\[
\tilde{h}_j'(u_{i-1}, \ldots, u_0, u) = (h_{j-i}'(u_{i-1}), \ldots, h_0'(u_{i-1}), 0, \alpha_{n-1}^{i-j-1}(u_{i-j-1}), \ldots,
\alpha_n(u_0), \alpha_{n-1}(u)),
\]
for each \((u_{i-1}, \ldots, u_0, u) \in C_{n-1}^i(x) \oplus \cdots \oplus C_0^i(x) \oplus C_{n-1}(x)\). It is immediate to check that the homotopy identities for \(\tilde{h}\) follow from the corresponding identities satisfied by \(h\) (see [13, Lemma 4.3.23] for the details). \(\square\)

The functor \(F_n\) does not behave the same way as \(\overline{W}_n\) with respect to homotopies. However, in certain cases \(F_n\) does take homotopic morphisms to homotopic morphisms. One of these cases is the following:

**Lemma 4.25.** Let \((F, f, \alpha), (F, f, \beta) : (G, C, C_{n-1}) \rightarrow (G', C', C'_{n-1})\) be morphisms in the category \(\text{SCrs}_{n-1}\) such that \(\alpha_0 = \beta_0 : C_{n-1}^0 \rightarrow C_{n-1}^0 F\) and let \(\tilde{h} = (f, h)\) be a homotopy between the above morphisms such that

\[
h_0^0 = s_0 \alpha_{n-1}^0 \quad \text{and} \quad h_i^j = s_i \alpha_{n-1}^j, \quad \text{if } i < j
\]

(note that \(h_i^j\) is arbitrary for \(j > 0\)). Then the morphisms of simplicial \(n\)-truncated crossed complexes \(F_n(F, f, \alpha)\) and \(F_n(F, f, \beta)\) in \(\text{SCrs}_n\) are homotopic.

The two previous lemmas can easily be proved for slice categories. We have:

**Lemma 4.26.** For each groupoid \(\Pi\), the functor

\[
\overline{W}_n : \text{SCrs}_n / \Pi \rightarrow \text{SCrs}_{n-1} / \Pi
\]

preserves homotopy classes of simplicial morphisms in the corresponding slice categories.

**Lemma 4.27.** Let \(\Pi\) be a groupoid, \((F, f, \alpha)\) and \((F, f, \beta)\) two morphisms in the slice category \(\text{SCrs}_{n-1} / \Pi\),

\[
\begin{tikzcd}
(G, C, C_{n-1}) & (G', C', C'_{n-1}) \\
\Pi \\
\end{tikzcd}
\]

such that \(\alpha_0 = \beta_0 : C_{n-1}^0 \rightarrow C_{n-1}^0 F\), and let \(\tilde{h} = (f, h) : (F, f, \alpha) \rightarrow (F, f, \beta)\), with

\[
h = \{h_j^i : C_{n-1}^i \rightarrow C_{n-1}^{i+1} F; \ 0 \leq j \leq i\}.
\]

be a homotopy in \(\text{SCrs}_{n-1} / \Pi\) from \((F, f, \alpha)\) to \((F, f, \beta)\) such that

\[
h_0^0 = s_0 \alpha_{n-1}^0 \quad \text{and} \quad h_i^j = s_i \alpha_{n-1}^j, \quad \text{if } i < j.
\]
Then the morphisms $F_n(F, f, \alpha)$ and $F_n(F, f, \beta)$ in the category $\text{SCrs}_n/\Pi$ are homotopic.

Note that if $A$ is any $\Pi$-module and in Lemma 4.27 we take 

$$(G', C', C_{n-1}') = K(\tilde{\Lambda}_{n-1}, m) = (\Pi, \Pi, K(A, m)) \in \text{SCrs}_{n-1}/\Pi$$

for $m > 0$, then for any two homotopic morphisms in $\text{SCrs}_{n-1}/\Pi$ with codomain $K(\tilde{\Lambda}_{n-1}, m)$ there is a homotopy $\bar{h}$ satisfying the hypothesis of the said lemma and therefore we can conclude that the functor $F_n$ preserves homotopy classes of morphisms with codomain $K(\tilde{\Lambda}_{n-1}, m)$ and we have

**Proposition 4.28.** Let $G$ be a groupoid, $A$ a $\Pi$-module and $(G, C, C_{n-1})$ a simplicial object in $\text{SCrs}_{n-1}/\Pi$. Then the adjunction $F_n \dashv \overline{W}_n$, 

$$\text{SCrs}_n/\Pi \xrightarrow{F_n} \text{SCrs}_{n-1}/\Pi$$

induces an isomorphism in homotopy

$$[F_n(G, C, C_{n-1}), K(\tilde{\Lambda}_{n}, m)]_{\text{SCrs}_n/\Pi} \cong [(G, C, C_{n-1}), \overline{W}_n(K(\tilde{\Lambda}_{n}, m))]_{\text{SCrs}_{n-1}/\Pi}. \tag{41}$$

**Proof.** The proof follows trivially after the previous observations, together with Lemma 4.26 and the fact that the adjunction isomorphisms for $\overline{W}_n \dashv F_n$ are obtained by applying the functors $\overline{W}_n$ and $F_n$, and composing with the unit and counit of the said adjunction. $\square$

**4.2.4. The representation of singular cohomology of $n$-truncated crossed complexes as homotopy classes of maps of simplicial $n$-truncated crossed complexes**

Let now

$${\mathfrak{B}}_n = F_n F_{n-1} \cdots F_1.$$ 

Then, by repeated application of (38), one gets

$$[[\mathfrak{B}_n \text{Ner}(\mathcal{C}), K(\tilde{\Lambda}_n, m)]_{\text{SCrs}_n/\Pi} \cong [\text{Ner}(\mathcal{C}), \overline{W}_1 \cdots \overline{W}_n(K(\tilde{\Lambda}_n, m))]]_{\mathcal{S}/\text{Ner}(\Pi)}.$$

Combining this isomorphism with Proposition 4.23, and the generalized Eilenberg–MacLane representation (32) of the singular cohomology, we get:

**Corollary 4.29.** Let $1 \leq m \leq n$. There is a natural isomorphism

$$H^{n+m}_{\text{sing}}(\mathcal{C}, A) \cong [[\mathfrak{B}_n \text{Ner}(\mathcal{C}), K(\tilde{\Lambda}_n, m)]_{\text{SCrs}_n/\Pi}.$$
4.3. Obtaining the topological invariants from the algebraic ones

The main tool we use in this section is Duskin’s representation theorem, which in our context can be particularized like this:

Theorem 4.30 (Duskin’s Representation Theorem). If \( \tilde{A}_n \) is any internal abelian group object in \( \text{SCrs}_n/\Pi \),

\[
H^m_{G_n}(\mathcal{C}, \tilde{A}_n) \cong [G^*_n(\mathcal{C}), K(\tilde{A}_n, m)]_{\text{SCrs}_n/\Pi}.
\]

We use this theorem together with Corollary 4.29 to prove the following:

Theorem 4.31. Let \( \mathcal{C} \in \text{Crs}_n \), \( \Pi = \pi_1(\mathcal{C}) \), and let \( A: \Pi \to \text{Ab} \), and \( \tilde{A}_n \) the abelian group object in \( \text{SCrs}_n/\Pi \) defined in (29). For every \( m \geq 0 \) there is a natural map

\[
H^m_{G_n}(\mathcal{C}, \tilde{A}_n) \rightarrow H^{n+m}_{\text{sing}}(\mathcal{C}, A).
\]

Proof. By Corollary 4.29 and Duskin’s representation theorem (Theorem 4.30) it is sufficient to define a map

\[
\alpha: [\mathcal{C}_n(\mathcal{C}), K(\tilde{A}_n, m)]_{\text{SCrs}_n/\Pi} \rightarrow [\mathcal{F}_n \text{Ner}(\mathcal{C}), K(\tilde{A}_n, m)]_{\text{SCrs}_n/\Pi}.
\]

For this, in turn, it is sufficient to give a morphism

\[
\eta: \mathcal{F}_n \text{Ner}(\mathcal{C}) \to \mathcal{C}_n(\mathcal{C})
\]

(42) in \( \text{SCrs}_n \) such that it defines a map in \( \text{SCrs}_n/\Pi \). Note that in that case the induced map of sets

\[
\eta_*: \text{SCrs}_n(\mathcal{C}_n(\mathcal{C}), K(\tilde{A}_n, m)) \to \text{SCrs}_n(\mathcal{F}_n \text{Ner}(\mathcal{C}), K(\tilde{A}_n, m))
\]

automatically preserves homotopy classes.

To define \( \eta \) we first observe that the simplicial object \( \mathcal{F}_n \text{Ner}(\mathcal{C}) \) is free with respect to the cotriple \( G_n \) [13] and therefore to give \( \eta \) it is sufficient to give a morphism \( \eta_{-1}: \pi_0(\mathcal{F}_n \text{Ner}(\mathcal{C})) \to \mathcal{C} \) where \( \pi_0: \text{SCrs}_n \to \text{Crs}_n \) is the connected components functor, defined as the left adjoint to the diagonal \( \Delta: \text{Crs}_n \to \text{SCrs}_n \). Then we consider the two adjunctions

\[
\pi_0 \vdash \Delta, \quad \text{SCrs}_n \xrightarrow{\pi_0} \text{Crs}_n \quad \text{and} \quad \mathcal{F}_n \xrightarrow{\pi_0(\mathcal{F}_n \text{Ner}(\mathcal{C}))} \mathcal{W}_n = W_1 \cdots W_n, \quad \text{SCrs}_n \xrightarrow{\mathcal{F}_n} \text{SCrs}_0.
\]

and observe that the composition \( \mathcal{W}_n \Delta \) is just the nerve functor Ner. Therefore \( \pi_0(\mathcal{F}_n \text{Ner}(\mathcal{C})) = \pi_0(\mathcal{F}_n \mathcal{W}_n \Delta(\mathcal{C})) \) and we can take \( \eta_{-1} \) as the \( \mathcal{C} \)-component of the counit of the adjunction \( \pi_0(\mathcal{F}_n) \to \mathcal{W}_n \Delta \). \( \square \)
By taking the case \( m = 2 \), we obtain the desired map
\[
H^2_{G_n}(\mathcal{C}, \tilde{\pi}_n) \rightarrow H^{n+2}_{\text{sing}}(\mathcal{C}, A).
\]
This implies that for an arbitrary crossed complex \( \mathcal{C} \), if we denote \( \tilde{\pi}^{(n)}_{n+1}(\mathcal{C}) \) the abelian group object \( \tilde{\pi}_n \) corresponding to the local coefficient system determined by \( A = \pi_{n+1}(\mathcal{C}) \), then we have a morphism
\[
H^2_{G_n}(P_n(\mathcal{C}), \tilde{\pi}^{(n)}_{n+1}(\mathcal{C})) \rightarrow H^{n+2}_{\text{sing}}(P_n(\mathcal{C}), \pi_{n+1}(\mathcal{C})).
\]

The topological Postnikov invariant of a crossed complex \( \mathcal{C} \) is the image by this map of the algebraic Postnikov invariant \( k_{n+1} \in H^2_{G_n}(P_n(\mathcal{C}), \tilde{\pi}^{(n)}_{n+1}(\mathcal{C})) \).

For any space \( X \) having the homotopy type of a crossed complex, we can obtain its Postnikov invariants by simply calculating the topological Postnikov invariants of the fundamental crossed complex of its singular complex \( \mathcal{C} = \Pi(X) \).

### Appendix A. 2-Torsors and cotriple cohomology

Torsors were developed by Duskin as the appropriate algebraic structure “representing” general cotriple cohomology. The main references for this subject are [10,14]. Our definitions differ slightly from those found in those references in the sense that we put special emphasis in the groupoid fibre of a torsor. This is closer to the way torsors are defined and used, for example, in [8].

#### A.1. 2-Torsors

Every arrow in a groupoid establishes a group isomorphism between the endomorphism group of its domain and that of its codomain. A **connected** 2-torsor with coefficients in a given abstract group \( G \) is a connected groupoid (meaning that it has one single connected component) together with a “coherent” system of isomorphisms from the different groups of endomorphisms to the abstract group \( G \). Thus, in order to specify a connected 2-torsor with coefficients in \( G \) we must give a connected groupoid \( \mathcal{G} \) (called “the fiber” of the 2-torsor) and a natural system \( \alpha = \{ \alpha_x \}_{x \in \text{obj}(\mathcal{G})} \) of group isomorphisms \( \alpha_x : \text{End}_{\mathcal{G}}(x) \rightarrow G \).

This definition can be easily generalized to 2-torsors with a non-necessarily connected fiber groupoid. As it turns out, the general definition can be obtained as a particular case of the above, provided it is expressed in such a way that it makes sense in more general categories.

Let \( \mathcal{E} \) be a Barr exact category. Associated to \( \mathcal{E} \) we have the category \( \text{Gpd}(\mathcal{E}) \) of internal groupoid objects in \( \mathcal{E} \) and internal functors between them. If \( s, t : M \rightarrow O \) are the “source” and “target” structural morphisms of an internal groupoid \( \mathcal{G} \), we say that \( \mathcal{G} \) is connected if the coequalizer of \( s \) and \( t \) is the terminal object, and we say that \( \mathcal{G} \) is totally disconnected if \( s = t \).

For a given object \( O \) in \( \mathcal{E} \), \( \text{TdGpd}_{\mathcal{G}}(\mathcal{E}) \) denotes the category whose objects are totally disconnected internal groupoids in \( \mathcal{E} \) having \( O \) as object of objects, and whose arrows are
internal functors whose component at the level of objects is the identity of \( O \). This category can be identified with the category of internal group objects in the slice category \( \mathcal{E}/O \).

If \( \mathcal{G} \) is an internal groupoid in \( \mathcal{E} \), having \( O \) as object of objects, the category \( \text{Gr}(\mathcal{E})^{\mathcal{G}} \) of internal \( \mathcal{G} \)-groups in \( \mathcal{E} \) is now defined in terms of \( \mathcal{G} \)-actions. Then an internal \( \mathcal{G} \)-group consists of a totally disconnected groupoid \( H \in Td\mathcal{G}pd_O \) together with a \( \mathcal{G} \)-action, that is a map in \( \mathcal{E} \):

\[
M \times_O H \to H; \quad (f, h) \mapsto fh,
\]

where \( M \) and \( H \) denote the objects of arrows of \( \mathcal{G} \) and \( \mathcal{H} \), respectively, and \( M \times O H \) is the pullback object of the diagram \( M \to O \leftarrow H \), such that it satisfies the usual axioms for an action of groups. A morphism of internal \( \mathcal{G} \)-groups \( \alpha: \mathcal{H} \to \mathcal{H}' \) is an equivariant functor in \( Td\mathcal{G}pd_O(\mathcal{E}) \).

One of the basic examples of an internal \( \mathcal{G} \)-group is the \( \mathcal{G} \)-group of endomorphisms of \( \mathcal{G} \), \( \text{End}(\mathcal{G}) \), defined by the totally disconnected groupoid \( \text{End}(\mathcal{G}) \) (the equalizer of \( s \) and \( t \) with the groupoid structure given by restriction of that in \( \mathcal{G} \)) and the action by conjugation in \( \mathcal{G} \).

If we write \( \mathbf{1} \) for the internal groupoid in \( \mathcal{E} \), having the terminal object as both object of arrows and object of objects, we can identify the category \( \text{Gr}(\mathcal{E})^{\mathbf{1}} \) with \( \text{Gr}(\mathcal{E})^{\mathcal{G}} \) and therefore the canonical \( \mathcal{G} \to \mathbf{1} \) induces a functor \( \text{Gr}(\mathcal{E})^{\mathcal{G}} \to \text{Gr}(\mathcal{E})^{\mathcal{G}'} \) which allows us to regard any internal group \( \mathcal{G} \) in \( \mathcal{E} \) as a \( \mathcal{G} \)-group (the trivial action of \( \mathcal{G} \) on \( \mathcal{G} \)).

Note that any internal functor \( f: \mathcal{G} \to \mathcal{G}' \) defines, in a standard way, a functor \( f^*: \text{Gr}(\mathcal{E})^{\mathcal{G}} \to \text{Gr}(\mathcal{E})^{\mathcal{G}'} \).

Then, a connected 2-torsor in \( \mathcal{E} \) consists of a triple \( (\mathcal{G}, G, \alpha) \) where \( G \) is an internal group in \( \mathcal{E} \), called the coefficients, \( \mathcal{G} \) is a connected groupoid in \( \mathcal{E} \), called the fiber, and \( \alpha \), the cocycle, is an isomorphism in the category \( \text{Gr}(\mathcal{E})^{\mathcal{G}} \) from \( \text{End}(\mathcal{G}) \) to \( G \) (the trivial action of \( \mathcal{G} \) on \( G \)). Note that to give the cocycle \( \alpha \) is equivalent to giving an arrow \( \alpha: \text{End}(\mathcal{G}) \to G \) that makes the following square a pullback in \( \mathcal{E} \):

\[
\text{End}(\mathcal{G}) \xrightarrow{\alpha} G \\
\uparrow s = t \downarrow \uparrow \phantom{\alpha} \\
\text{obj}(\mathcal{G}) \xrightarrow{\phantom{\alpha}} \mathbf{1}
\]

and satisfies:

- \( \alpha(ab) = \alpha(a)\alpha(b) \) for all composable endomorphisms \( a, b \) of \( \mathcal{G} \),
- \( \alpha(fa) = \alpha(faf^{-1}) = \alpha(a) \), for all arrows \( f \) and all endomorphisms \( a \) in \( \mathcal{G} \), such that \( s(f) = s(a) \).

The connected 2-torsors in \( \mathcal{E} \) whose group of coefficients is \( G \) are the objects of a category, denoted \( \text{Tor}^2(1, G) \), whose arrows form \( (\mathcal{G}, G, \alpha) \) to \( (\mathcal{G}', G, \alpha') \) are internal functors \( f: \mathcal{G} \to \mathcal{G}' \) compatible with the cocycles \( \alpha, \alpha' \) in the sense that

\[ \alpha = f^*(\alpha') = \alpha' \circ f. \]
If $T$ is an object in a Barr exact category $\mathcal{E}$, then the slice category $\mathcal{E}/T$ is again a Barr exact category and a 2-torsor above $T$ in $\mathcal{E}$ is defined as a connected 2-torsor in $\mathcal{E}/T$. In this definition it is understood that the coefficients are taken in an internal group object in $\mathcal{E}/T$. If $T$ is an object and $G$ an internal group in $\mathcal{E}$, then the canonical projection $G \times T \to T$ gives an object of $\mathcal{E}/T$ which has a canonical structure of internal group object in $\mathcal{E}/T$.

In this situation, a $(G, 2)$-torsor above $T$ in $\mathcal{E}$ is defined as a connected 2-torsor in $\mathcal{E}/T$ with coefficients in $G \times T$, and the category of such torsors is denoted $\text{Tor}^2(T, G)$. By $\text{Tor}^2[T, G]$ we denote the set of connected components of $\text{Tor}^2(T, G)$.

Let us suppose now that we have a functor $U : \mathcal{E} \to \mathcal{S}$ from a Barr exact category to a category with finite limits. Let $\mathcal{G}$ be an internal groupoid in $\mathcal{E}$ with source and target maps $s, t : M \to O$. If $q : O \to T$ is the coequalizer of $s$ and $t$ and $O \times_T O$ is the pullback of $q$ with itself, there is an induced map $(s, t) : M \to O \times_T O$. We say that the groupoid $\mathcal{G}$ is $U$-split if the maps $U(q)$ and $U(s, t)$ split in $\mathcal{S}$.

A $U$-split $(G, 2)$-torsor above $T$ is a $(G, 2)$-torsor above $T$ such that its fiber groupoid is $U$-split. The full subcategory of $\text{Tor}^2(T, G)$ determined by those $(G, 2)$-torsors which are $U$-split is denoted $\text{Tor}^2_U(T, G)$. Correspondingly, the set of connected components of $\text{Tor}^2_U(T, G)$ is denoted $\text{Tor}^2_U[T, G]$.

**Proposition A.1.** $\text{Tor}^2(T, G)$ is a category fibred over $\mathcal{E}$ via the composite functor

$$
\text{Tor}^2(T, G) \xrightarrow{\text{fib}} \text{Gpd}(\mathcal{E}) \xrightarrow{\text{obj}} \mathcal{E}.
$$

Furthermore, if $(\mathcal{G}, \alpha) \in \text{Tor}^2(T, G)$ is $U$-split for some left exact functor $U : \mathcal{E} \to \mathcal{S}$, then $(\mathcal{G'}, \alpha')$ is $U$-split if and only if the projection $q' : O' \to T$ corresponding to $(\mathcal{G'}, \alpha')$ is $U$-split.

**Proof.** For the first part we have to prove that if $(\mathcal{G}, \alpha)$ is a $(G, 2)$-torsor above $T$ and $f : O' \to O = \text{obj}(\mathcal{G})$ is a morphism in $\mathcal{E}$, then there is a $(G, 2)$-torsor $(\mathcal{G'}, \alpha')$ above $T$ and an $(\text{obj} \times \text{fib})$-cartesian morphism of torsors $f' : (\mathcal{G'}, \alpha') \to (\mathcal{G}, \alpha)$ above $f$.

The idea for the construction of $\mathcal{G'}$ with object of objects $O'$ is that its arrows from $x \in O'$ to $y \in O'$ are the arrows $f(x) \to f(y)$ in $\mathcal{G}$ and that the identity of $x \in O'$ is the identity of $f(x)$. The composition in $\mathcal{G'}$ will then be clearly induced by that of $\mathcal{G}$. Thus, the object of arrows of $\mathcal{G'}$, together with its source and target maps can internally be defined by the following pullback:

$$
\begin{array}{ccc}
M' & \xrightarrow{(g', f')} & O' \\
\downarrow{u} & & \downarrow{f \times f} \\
M & \xrightarrow{(s, t)} & O \times_T O',
\end{array}
$$

where $O \times_T O$ is the pullback of $q$ with itself and $O' \times_T O'$ that of $q' = qf$ with itself. This construction produces a functor $f' : \mathcal{G'} \to \mathcal{G}$ whose component on arrows is $u$ (and it
Theorem A.3. In the above conditions, for any object \( T \) in \( \mathcal{E} \), the cocycle map \( \alpha' \) is defined as the image of \( \alpha \) by the induced functor \( f^* : \text{Gr}(\mathcal{E}) \to \text{Gr}(\mathcal{E})^\mathcal{G} \), that is \( \alpha' = f^* \alpha \).

Let us now assume that \((\mathcal{G}, \alpha)\) is \( U \)-split. For \((\mathcal{G}', \alpha')\) to be \( U \)-split it is necessary that \( q' \) be \( U \)-split. From the exactness of \( U \) and a splitting of \( U(s, t) \) it follows that \((s', t')\) is \( U \)-split, therefore that \( q' \) be \( U \)-split is also sufficient for \((\mathcal{G}', \alpha')\) to be \( U \)-split.

It only remains to prove that \( f' \) is cartesian. Let \( g : \mathcal{G}' \to \mathcal{G} \) be a morphism of internal groupoids in \( \mathcal{E} \) such that \( g_0 = \text{obj}(g) : O'' \to O \) factors through \( f \) as \( g_0 = fh \). Then it is clear which is the only way to define a factorization \( g = f' h' \) such that \( \text{obj}(h') = h \). This condition determines \( h' \) on objects and \( g \) determines it on arrows. \( \square \)

By a reasoning similar to the one given in [14] to prove Theorem 5.7.5, it is easy to prove that from any diagram in \( \text{Tor}_2^U(T, G) \) of the form \((\mathcal{G}, \alpha) \to (\tilde{\mathcal{G}}, \tilde{\alpha}) \to (\mathcal{G}', \alpha') \) one can obtain another of the form \((\mathcal{G}, \alpha) \leftrightarrow (\mathcal{G}'', \alpha'') \to (\mathcal{G}', \alpha') \) (see [13, Lemma 4.2.6] for the details). As a consequence, we have the following useful necessary (and, obviously, also sufficient) condition satisfied by torsors in the same connected component of \( \text{Tor}_2^U(T, G) \).

**Proposition A.2.** If \((\mathcal{G}, \alpha)\) and \((\mathcal{G}', \alpha')\) are \( U \)-split 2-torsors which are in the same connected component of \( \text{Tor}_2^U(T, G) \), then there is a torsor \((\mathcal{G}'', \alpha'')\) and maps
\[
(\mathcal{G}, \alpha) \leftrightarrow (\mathcal{G}'', \alpha'') \to (\mathcal{G}', \alpha')
\]
in \( \text{Tor}_2^U(T, G) \).

### A.2. Cotriple cohomology

Let \( \mathcal{E} \) be a tripleable category over a category \( \mathcal{S} \) with cotriple \( \mathcal{G} \). For any object \( T \in \mathcal{E} \) and any abelian group object \( A \) in \( \mathcal{E}/T \), the cotriple cohomology groups \( H_2^\mathcal{G}(T, A) \) can be represented in terms of homotopy classed of simplicial maps from the cotriple simplicial resolution of \( T \) to the Eilenberg–Mac Lane complex \( K(A, n) \). On the other hand, Duskin’s interpretation theorem for cotriple cohomology [10] provides an interpretation of the elements of \( H_2^\mathcal{G}(T, A) \) in terms of \( U \)-split torsors, where \( U : \mathcal{E} \to \mathcal{S} \) is the monadic functor associated to the cotriple \( \mathcal{G} \). In the particular case \( n = 2 \), Duskin’s theorem implies the following:

**Theorem A.3.** In the above conditions, for any object \( T \in \mathcal{E} \) and any abelian group object \( A \) in \( \mathcal{E}/T \), there is a natural bijection
\[
H_2^\mathcal{G}(T, A) \cong \text{Tor}_2^U[T, A].
\]

In the presence of a cotriple, Proposition A.1 has the following consequence:

**Proposition A.4.** In the hypothesis of Theorem A.3, if \((\mathcal{G}, \alpha)\) is a \( U \)-split \((A, 2)\)-torsor above \( T \) with object of objects \( O \), there is a \( U \)-split \((A, 2)\)-torsor above \( T \), \((\mathcal{G}', \alpha')\), whose object of objects is \( \mathcal{G}(T) \) and whose projection is the counit \( \varepsilon_T : \mathcal{G}(T) \to T \). Furthermore, \((\mathcal{G}', \alpha')\) is connected to \((\mathcal{G}, \alpha)\) by a morphism \((\mathcal{G}', \alpha') \to (\mathcal{G}, \alpha)\) in \( \text{Tor}_2^U(T, A) \).
**Proof.** Let $s : U(T) \to U(O)$ be a section of the image $U(q)$ of the projection $q : O \to T$ of $(G, \alpha)$. Use Proposition A.1 with $O' = G(T)$ and $f$ equal to the composite $G(T) \xrightarrow{F(s)} G(O) \xrightarrow{\varepsilon} O$, where $F$ is the left adjoint to $U$ and $\varepsilon$ is the counit of $G (= FU)$. Then we obtain an $(A, 2)$-torsor above $T$, $(G', \alpha')$, whose projection is the composite

$$qf = q\varepsilon : OF(s) = \varepsilon_T FU(q) F(s) = \varepsilon_T,$$

and a map $f' : (G', \alpha') \to (G, \alpha)$ which is given by $f$ at the level of objects. Since $\varepsilon_T$ is a $U$-split map (with $U$-section given by $\eta_{U(O)}$), it follows that $(G', \alpha')$ is $U$-split. \qed

Let now $\text{Tor}_U^2(T, A)$ denote the full subcategory of $\text{Tor}_U^2(T, A)$ determined by those torsors whose object of objects is $G(T)$ and whose projection is the counit $\varepsilon_T$. Then Propositions A.4 and A.2 imply the following:

**Proposition A.5.** In the hypothesis of Theorem A.3, let $F : E \to \text{Tor}_U^2(T, A)$ be a full and faithful functor such that the inclusion $\text{Tor}_U^2(T, A) \hookrightarrow \text{Tor}_U^2(T, A)$ factors through $F$. Then, $F$ establishes a bijection between the set $[E]$ of connected components of $E$ and $\text{Tor}_U^2(T, A)$. Hence there is a natural bijection

$$H^2_G(T, A) \cong [E].$$

**Proof.** Let $A, B \in E$ such that $F(A)$ and $F(B)$ are in the same connected component of $\text{Tor}_U^2(T, A)$. We just need to show that $A$ and $B$ are in the same connected component in $E$. By Proposition A.4 there are 2-torsors $X, Y \in \text{Tor}_U^2(T, A)$ and morphisms $h : X \to F(A)$ and $k : Y \to F(B)$, such that $X$ and $Y$ are in the same connected component of $\text{Tor}_U^2(T, A)$. Hence, by Proposition A.2 we get a diagram $X \xrightarrow{f} Z \xrightarrow{g} Y$ in $\text{Tor}_U^2(T, A)$ where, by Proposition A.4 we can suppose that $Z$ is in $\text{Tor}_U^2(T, A)$. By the hypothesis that the inclusion of $\text{Tor}_U^2(T, A)$ factors through $F$, we get a diagram

$$A' \leftarrow C \rightarrow B'$$

(A.1)

in $E$ such that $F(A') = X$ and $F(B') = Y$. Thus, the maps $h, k$ and the fullness of $F$ allow us to extend diagram (A.1) to a diagram

$$A \leftrightarrow A' \leftrightarrow C \rightarrow B' \rightarrow B$$

proving that $A$ and $B$ are in the same connected component. \qed

**References**


Further reading


