Coherence measures are a tool to compare those fuzzy sets that are sensitive to their own similarity as well as to their fuzzy nature. Within this article we can find three generalizations made about the definition of coherence measures: a first one for any fuzzy set, a second one for any definition about strong negation, and a final one for an extension in those coherence measures that, as a result, do not cause a value in the unit interval, but a fuzzy set in that interval. Tools and properties are offered to create coherence measures. © 2005 Wiley Periodicals, Inc.

1. INTRODUCTION

It is well known that the difficulty of comparing measures, properties, classes, or, to put it briefly, “sets,” is a priority in any scientific area, but more relevant in engineering fields. This subject is of great relevance within the Intelligent Systems field. In this field the qualitative and subjective nature of comparisons is linked to the intelligence notion we could make use of. When dealing with those situations that take into consideration any kind of fuzziness, methodologies based on fuzzy sets need to be used. Consequently the way of evaluating similarities between fuzzy sets, which is a basic and fundamental theoretical aspect, is of a great importance when dealing with any practical application that might come from the real world.

The similarity measures of fuzzy sets start from the axiomatic definition of autosimilarity in a fuzzy set, which is to say the maximum similarity that arises between a fuzzy set and itself. Therefore this property does not include the fuzziness of that set. To prevent this misfunction, the aim of those coherence measures proposed by the authors in Refs. 1 and 2 was to offer a similarity tool, for comparing fuzzy sets, sensible to the fuzziness of the considered sets. In this way, from a very simple axiomatic, the definition of coherence measures was shown for both
finite and nonfinite sets \(X\). Despite the rigorous study taken in Ref. 1 and then in Ref. 2, the integration of these measures in a common shared theoretical framework is still pending.

From this point of view, the double objective of this article is, first, to generalize and unify, as much as possible, the definition of these measures, and, second, to supply several tools to build them. From now on, we consider \(X\) a referential set for a linguistic variable. \(X\) could be a finite set (although not void with cardinality) or a nonfinite set. For this second case, we will consider that \(X\) is at least a measurable Lebesgue set with \(\lambda(X) = n \neq 0\). All the cases of linguistic variables fulfilled this condition, because referential sets for this kind of fuzzy sets are always real intervals. So, when we write “\(X\) as a referential set,” we signify both cases: finite and nonfinite, with the former sense. In the above mentioned papers\(^1,2\) we considered only a type of usual operators for \(X\) and \([0,1]\), max, min, and \(1 - x\). Now, to consider a more general scope we have also considered the general operators \(T, S, N\) on \(P^f(X)\), where \(T\) is a \(t\)-norm, \(S\) is a \(t\)-conorm, and \(N\) a negation that form a dual triple. Here, we consider \(N\) a strong negation. These operators’ properties are usual and references can be found.\(^3,4\) Similarly, we use the same operators in unit intervals \([0,1]\). This makes sense as \(T, S,\) and \(N\) only act with those membership values of the \(X\) fuzzy subset found within \([0,1]\).

The following presentation integrates three generalizations: one about finite and nonfinite cases, a second about general class of operators on \(P\), and a third one about coherence measures and fuzzy coherence measures. Concretely, this generalization brings out a fuzzy set that stands for those coherence measures that are found between two fuzzy sets \(A, B\). As we have previously stated, these three generalizations make up and provide us with a framework common to the different previous contributions by the authors.\(^1,2\) These generalizations have also generalized and unified the approximations carried out in this same context by other authors.\(^5–7\) Particularly, some of the properties pointed out in Ref. 5 are studied, analyzed, and broadened when we consider here nonfinite referential sets.

Consequently, the article is developed as follows. In Section 2, the two first generalizations are carried out at the same time. Subsequently, in Section 3 the third generalization is developed. To distinguish them, the two first generalizations will be named Coherence Measures and the third one Fuzzy Coherence Measures. Some of the conclusions are useful for closing the topic we are dealing with.

## 2. COHERENCE MEASURES ON \(P^f(X)\)

**Definition 1.** Let \(X\) be a referential set and \(P^f(X)\) the set of fuzzy sets on \(X\), \(N\) a strong negation on \(P^f(X)\). One says that

\[
\text{cohe}: P^f(X) \times P^f(X) \to [0,1]
\]

is a coherence measure on \((P^f(X), N)\) if and only if the three following axioms hold:

- \(C1\). \(\text{cohe}(A, B) = \text{cohe}(B, A)\)
- \(C2\). \(\text{cohe}(A, N(B)) = N(\text{cohe}(A, B))\)
- \(C3\). \(\text{cohe}(\emptyset, X) = 0.\)
It is clear that C1 shows the map cohe as a symmetric measure. C2 presents the basic idea on the concept of coherence measure: If the term coherence is meant as the coexistence of two evaluations, then is clear that (in the [0,1] interval) such coexistence between A and N(B) is the opposite to another such between A and B. Finally C3 shows that the minimum coherence is to be attained when the sets ∅ and X are compared.

Note. From the continuity of function N we can assure that there is a unique g ∈ [0,1] such as N(g) = g. We hold the notation of g for this number throughout this article. Then [0,1] = [0,g] ∪ [g,1] and we need to point out that if x ∈ [0,g], then N(x) ∈ [g,1] and vice versa. For the sake of simplicity we shall call I = [0,g] and II = [g,1].

**Lemma 1.** Let cohe : \(P^f(X) \times P^f(X) \rightarrow [0,1]\) be a coherence measure. Then

(a) \(\text{cohe}(N(A), B) = \text{cohe}(A, N(B)) = N(\text{cohe}(A, B))\)

(b) \(\text{cohe}(N(A), N(B)) = \text{cohe}(A, B)\)

(c) \(\text{cohe}(\emptyset, \emptyset) = \text{cohe}(X, X) = 1\)

(d) If \(A^*(x) = g \forall x\), then \(\forall A \in P^f(X) \text{cohe}(A, A^*) = g\).

**Proof.**

(a) \(\text{cohe}(N(A), B) = \text{cohe}(B, N(A)) = N(\text{cohe}(B, A)) = N(\text{cohe}(A, B))\)

(b) \(\text{cohe}(N(A), N(B)) = N(\text{cohe}(N(A), B)) = N(\text{cohe}(A, B)) = \text{cohe}(A, B)\)

(c) \(\text{cohe}(\emptyset, \emptyset) = \text{cohe}(\emptyset, N(X)) = N(\text{cohe}(\emptyset, X)) = N(0) = 1; \text{cohe}(X, X) = [\text{by a}]\)

\(\text{cohe}(\emptyset, \emptyset) = 1\)

(d) If \(A^*(x) = g \forall x\), then \(A^* = N(A^*)\); hence \(\text{cohe}(A, A^*) = \text{cohe}(A, N(A^*)) = N(\text{cohe}(A, A^*))\), which implies \(\text{cohe}(A, A^*) = g\).

**Lemma 2.** Let \(f : [0,g]^2 \rightarrow [0,1]\) be a strictly monotonous function such that \(f(a, b) = N(f(b, a))\). Then the function \(h : [0,1]^2 \rightarrow [0,1]\) defined by

\[h(a, b) = \begin{cases} f(a, b) & \text{if } a, b \in I \\
N(f(a, N(b))) & \text{if } a \in I \text{ and } b \in II \\
N(f(N(a), b)) & \text{if } a \in II \text{ and } b \in I \\
f(N(a), N(b)) & \text{if } a, b \in II \end{cases}\]

for this number throughout this article.

Verifies that

(a) \(h\) is injective in any variable

(b) \(h(a, N(b)) = N(h(a, b))\)

(c) \(h(a, b) = N(h(b, a))\)

(d) \(h(a, g) = g\).

**Proof.** \(h\) is well defined because \(a \in II\) iff \(N(a) \in I\).
Notation. From now on, we will call $h$ “a coherence generator function.”

Notes. There are different ways to obtain a function such as $f$ in the previous lemma:

(a) A general way would be the following: If $N$ is a strong negation there exists\(^3\) a $k$ bijection verifying

$$N(x) = k^{-1}(k(1) - k(x))$$

Bearing this in mind, if $f(a,b) = N(f(b,a))$, then $f(a,b) = k^{-1}(k(1) - k(f(b,a)))$. Hence

$$k(f(a,b)) + k(f(b,a)) = k(1)$$

so the definition for a general strong negation $N$ would be $f(a,b) = k^{-1}((k(1)/2) - (b - a))$. By definition, $f$ is strictly monotonous.

(b) If we consider $N(x) = (1 - x^w)^{1/w}$ as the strong negation of Yager, where $w > 0$ is a parameter, we could easily obtain that $k(x) = x^w$ and $f(a,b) = (1/2 - (b - a))^{1/w}$ being a function of Lemma 2.

### Table I.

<table>
<thead>
<tr>
<th>Domain</th>
<th>$h(a,N(b))$</th>
<th>$N(h(a,b))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a,b \in I$</td>
<td>$N(f(a,b))$</td>
<td>$N(f(a,b))$</td>
</tr>
<tr>
<td>$a \in I$ and $b \in II$</td>
<td>$f(a,N(b))$</td>
<td>$f(a,N(b))$</td>
</tr>
<tr>
<td>$a \in II$ and $b \in I$</td>
<td>$f(N(a),b)$</td>
<td>$f(N(a),b)$</td>
</tr>
<tr>
<td>$a,b \in II$</td>
<td>$N(f(N(a),N(b)))$</td>
<td>$N(f(N(a),N(b)))$</td>
</tr>
</tbody>
</table>

- (a) If $h(a,b) = h(a',b)$. If we suppose $a < a'$ by definition of $h$ and by the monotony of $f$ and $N$ we could conclude that $h(a,b) < h(a',b)$ or $h(a,b) > h(a',b)$, but not the equality. Then our supposition on $a < a'$ is false. We can use a similar proof for the second variable.
- (b) Taking into account that $N(N(b) = b)$, we can see the results in Table I. This shows the equality.
- (c) Similarly, we have the results in Table II.
- (d) $h(a,g) = h(a,N(g)) = N(h(a,g)) \rightarrow h(a,g) = g$. $lacksquare$

### Table II.

<table>
<thead>
<tr>
<th>Domain</th>
<th>$h(a,b)$</th>
<th>$N(h(b,a))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a,b \in I$</td>
<td>$f(a,b)$</td>
<td>$N(f(b,a)) = f(a,b)$</td>
</tr>
<tr>
<td>$a \in I$ and $b \in II$</td>
<td>$N(f(a,N(b)))$</td>
<td>$f(N(b),a) = N(f(a,N(b)))$</td>
</tr>
<tr>
<td>$a \in II$ and $b \in I$</td>
<td>$N(f(N(a),b))$</td>
<td>$f(b,N(a)) = N(f(N(a),b))$</td>
</tr>
<tr>
<td>$a,b \in II$</td>
<td>$f(N(a),N(b))$</td>
<td>$N(f(N(b),N(a))) = f(N(a),N(b))$</td>
</tr>
</tbody>
</table>
Fuzzy Coherence Measures

(c) Another way to obtain \( f \) is by using any monotonous function \( \delta \) such as \( \delta(a, a) = g \). Then

\[
\begin{cases}
\delta(a, b) & \text{if } a < b \\
g & \text{if } a = b \\
N(\delta(a, b)) & \text{if } a > b
\end{cases}
\]

Theorem 1. Let \( X \) be a referential set with \( \lambda(X) = m \) and \( P^f(X) \) be the set of fuzzy sets on \( X \). Let \( d : P^f(X) \times P^f(X) \to [0, 1] \) be a bounded metric defined by

\[
d(A, b) = \left( \int_X p(A(x), B(x)) \lambda(X) \right)^{1/r}, \quad r \geq 1
\]

Let \( h \) be a coherence generator with \( f(0, 0) = 1 \). Then from \( d \), a coherence could be constructed such as

\[
\text{cohe}(A, B) = h(d(A, N(B)), N(d(A, B)))
\]

if and only if

\[
\begin{align*}
(a) \quad & p(0, 1) = 1/m \\
(b) \quad & p(a, N(b)) = p(N(a), b).
\end{align*}
\]

Proof. Let us suppose that \( \text{cohe} \) is a coherence measure.

(a) \( \text{cohe}(\emptyset, X) = 0 \); hence \( h(d(\emptyset, N(X)), N(d(\emptyset, X))) = 0 \). Then \( h(d(\emptyset, \emptyset)), N(d(\emptyset, X))) = 0 \), and so \( h(0, d(\emptyset, X)) = 1 \), but \( h(0, 1) = f(0, 0) = 1 \) and \( h \) injective, we have that \( d(\emptyset, X) = 1 \). But this means that

\[
d(\emptyset, X) = \left( \int_X p(0,1) \lambda(X) \right)^{1/r} = 1 \rightarrow p(0,1) \int_X \lambda(X) = 1 \rightarrow p(0,1) = \frac{1}{m}
\]

(b) \( \text{cohe}(A, B) = \text{cohe}(B, A) \). Let us consider \( A(x) = a \) and \( B(x) = b \ \forall x \in X \). Then we have that

\[
\begin{align*}
h(d(A, N(B)), N(d(A, B))) &= h(d(B, N(A)), N(d(B, A))) \\
N(h(d(A, N(B)), d(A, B))) &= N(h(d(B, N(A)), d(B, A))) \\
h(d(A, N(B)), d(A, B)) &= h(d(B, N(A)), d(A, B))
\end{align*}
\]

and considering the injectivity of \( h \), we could conclude that \( d(A, N(B)) = d(B, N(A)) \). This is formulated in

\[
\left( \int_X p(a, N(b)) \lambda(X) \right)^{1/r} = \left( \int_X p(b, N(a)) \lambda(X) \right)^{1/r} \rightarrow p(a, N(b)) = p(b, N(a))
\]

and from symmetry of \( d \) we obtain \( p(a, N(b)) = p(N(a), b) \).
We will start now with the fulfillment of a and b. We have to prove C1–C3.

C1. First, we can see that from b, we obtain easily \(d(A,N(B)) = d(N(A),B) = d(B,N(A))\); then \(\text{cohe}(A,B) = h(d(A,N(B)),N(d(A,B))) = N(h(d(B,N(A)),d(B,A))) = \text{cohe}(B,A)\)

C2. \(\text{cohe}(A,N(B)) = h(d(A,B),N(d(A,N(B)))) = N(h(d(A,B)),d(A,N(B))) = N(\text{cohe}(A,B))\)

C3. \(\text{cohe}(\emptyset,X) = h(0,N(d(\emptyset,X))) = h(0,0) = 1.\)

Lemma 3. For any strong negation \(N\), we have that \(N(x) = k^{-1}(k(1) - k(x))\), if we define

\[
d(A,B) = \left(\int_X p(A(x),B(x))\lambda(X)\right)^{1/r}, r \geq 1
\]

where \(p(A(x),B(x)) = (1/m\cdot k(1))\cdot |k(B(x)) - k(A(x))|\). \(d\) is a bounded distance on \(P^f(X) \times P^f(X)\) that holds the previous theorem.

Proof. We begin testing whether \(d\) is a metric: \(d(A,A) = 0\) obviously, and if \(d(A,B) = 0\), as \(p(A(x),B(x)) \geq 0\), this implies that \(k(B(x)) = k(A(x))\), and by the bijection of \(k\), \(A(x) = B(x)\). \(d(A(x),B(x)) = d(B(x) \cdot A(x))\) by the properties of an absolute value function. The triangular inequality holds by the same reason. We continue checking that \(p\) holds a and b:

\[
P(0,1) = (1/m\cdot k(1))\cdot |k(1) - k(0)| = (1/m\cdot k(1))\cdot |k(0)| = 1/m
\]

because \(k:[0,1] \rightarrow [0,1]\) and

\[
0 = N(1) = k^{-1}(k(1) - k(1)) = k^{-1}(0)
\]

and then \(k(0) = 0\),

\[
p(a,N(b)) = (1/m\cdot k(1))\cdot |N(b) - k(a)| = (1/m\cdot k(1))\cdot |k(1) - k(b) - k(a)|
\]

\[
= (1/m\cdot k(1))\cdot |k(b) - (k(1)) - k(a)| = p(N(a),b)
\]

Examples. We can consider the Yager negation \(N(x) = (1 - x^w)^{1/w}\). We will easily obtain from the definition that \(k(x) = x^w\) that in this case \(k(1) = 1\), and then we have that \(p(A(x),B(x)) = (1/m)\cdot |B^w(x) - A^w(x)|\), so a definition for \(d(A,B) = (1/m)(\int_X |B^w(x) - A^w(x)|\lambda(X))^{1/r}\) for \(r \geq 1\).

All the definitions presented above can meet together in the following algorithm for computing coherence measures based on Theorem 1 with Yager negation.
Step 1. Calculate $N(B)$

Step 2. Calculate $q_1 = d(A, N(B))$ and $d(A, B)$

Step 3. Calculate $q_2 = N(d(A, B))$

Step 4.

Case 1. $q_1, q_2 \leq (2)^{-1/w}$
   Calculate $f(q_1, q_2) = ((1 - (q_1 - q_2)))/2^{1/w}$

Case 2. $q_1 \leq (2)^{-1/w}, q_2 > (2)^{-1/w}$
   Calculate $N(f(q_1, N(q_2)))$

Case 3. $q_1 > (2)^{-1/w}, q_2 \leq (2)^{-1/w}$
   Calculate $N(f(N(q_1), q_2))$

Case 4. $q_1, q_2 > (2)^{-1/w}$
   Calculate $N(f(N(q_1), N(q_2)))$

3. FUZZY COHERENCE MEASURES ON $P^f(X)$

**Definition 2.** Let $X$ be a measurable set and $P^f(X)$ the set of fuzzy sets on $X$. Let $N$ be the negation fuzzy subset of $P^f(X)$, $N(x) = N(x)$. One says that $cohef : P^f(X) \times P^f(X) \to P^f([0,1])$ is a fuzzy coherence measure on $P^f(X)$ if and only if the three following axioms hold:

C1. $cohef(A, B) = cohef(B, A)$

C2. $cohef(A, N(B)) = N(cohef(A, B))$

C3. $cohef(\emptyset, X) = N$.

The only remarkable change in this definition is the image set of the measure, $P^f([0,1])$ of course, and the third axiom. If we want to associate a fuzzy set on $[0,1]$ that stands for a maximum incoherence between fuzzy subsets, $N$ will be the fuzzy opposite to the identity. In fact, the membership degree of the fuzzy coherence 0 is 1, and so forth.

**Lemma 4.** Let $cohef : P^f(X) \times P^f(X) \to P^f([0,1])$ be a fuzzy coherence measure; then

(a) $cohef(N(A), B) = cohef(A, N(B)) = N(cohef(A, B))$

(b) $cohef(N(A), N(B)) = cohef(A, B)$

(c) $cohef(\emptyset, \emptyset) = cohef(X, X) = Id$

(d) If $A^*(x) = g \; \forall x$, then $\forall A \in P^f(X)cohef(A, A^*) = A^*$.

**Proof.** Both a and b are equivalent to Lemma 2.
Proof. Let us check the three axioms:

(c) \( \text{cohef}(\emptyset, \emptyset) = \text{cohef}(\emptyset, N(X)) = N(\text{cohef}(\emptyset, X)) = N(N) = \text{Id} \); \( \text{cohef}(X, X) = [\text{by a}] \text{cohef}(\emptyset, \emptyset) = \text{Id} \)

(d) If \( A^*(x) = g \forall x \), then \( A^* = N(A^*) \); hence \( \text{cohef}(A, A^*) = \text{cohef}(A, N(A^*)) = N(\text{cohef}(A, A^*)) \), which implies \( \text{cohef}(A, A^*) = g \forall x \); then \( \text{cohef}(A, A^*) = A^* \). □

Lemma 5. Let \( f : [0, g] \to [0, 1] \) be a symmetric function with a boundary condition \( f(0, a) = N(a) \); then the function \( q : [0, 1]^2 \to [0, 1] \) defined by

\[
q(a, b) = \begin{cases} 
  f(a, b) & \text{if } a, b \in I \\
  N(f(a, N(b))) & \text{if } a \in I \text{ and } b \in II \\
  N(f(N(a), b)) & \text{if } a \in II \text{ and } b \in I \\
  f(N(a), N(b)) & \text{if } a, b \in II
\end{cases}
\]

verifies

(a) \( q \) is symmetric
(b) \( q(a, N(b)) = N(q(a, b)) \)
(c) \( q(0, a) = N(a) \)
(d) \( q(a, g) = q(g, a) = g \).

Proof. First, \( q \) is well defined because \( a \in II \) iff \( N(a) \in I \). Moreover,

(a) \( a \) is clear by construction of \( q \)
(b) \( b \) is direct, regarding construction of \( q \)
(c) \[
q(0, a) = \begin{cases} 
  f(0, a) & \text{if } a \in I \\
  N(f(0, N(a))) & \text{if } a \in II
\end{cases} = \begin{cases} 
  N(a) & \text{if } a \in I \\
  N(N(a)) & \text{if } a \in II
\end{cases} = N(a) \text{ for all } a \in [0, 1]
\]

(d) \( q(a, g) = q(a, N(g)) = N(q(a, g)) \); then \( q(a, g) = g \) for all \( a \). □

Note. A function \( f \) with the conditions of the previous lemma is, for example, \( f(a, b) = (aN(b) + bN(a))/|b - a| \).

Theorem 2. Let \( f \) and \( q \) be functions in the conditions of the lemma above, and let \( \text{cohe} \) be a coherence measure on \( P^f(X) \times P^f(X) \). Then, \( \text{cohe} : P^f(X) \times P^f(X) \to P^f[0, 1] \) defined by

\[
\text{cohe}(A, B)(x) = q(\text{cohe}(A, B), x)
\]
is a fuzzy coherence measure.

Proof. Let us check the three axioms:

C1. \( \text{cohe}(A, B)(x) = q(\text{cohe}(A, B), x) = q(\text{cohe}(B, A), x) = \text{cohe}(B, A)(x) \to \text{cohe}(A, B) = \text{cohe}(B, A) \)
C2. cohef\((A, N(B))(x) = q(cohe(A, N(B)), x) = N(q(cohe(A, B)), x) = N(q(cohe(A, B), x)) = N(cohe(A, B))\) and therefore cohef\((A, N(B)) = N(cohe(A, B))\)

C3. cohef\((\mathcal{Z}, X)(x) = q(cohe(\mathcal{Z}, X), x) = q(0, x) = N(x)\) and then cohef\((\mathcal{Z}, X) = N\).  

**Lemma 6.** Let \(f\) and \(q\) be functions in the conditions of the above lemma. Then, the fuzzy coherence measure defined in the previous theorem’s terms verifies the following properties:

(a) If cohe\((A, B) = g\) then cohe\((A, B) = A^*\)
(b) cohe\((A, B)(g) = g\)
(c) cohe\((A, B)(0) = N(cohe(A, B))\)
(d) cohe\((A, B)(1) = cohe(A, B)\).

**Proof.**

(a) cohe\((A, B)(x) = q(cohe(A, B), x) = q(g, x) = g\) for all \(x\); then cohe\((A, B) = A^*\)
(b) cohe\((A, B)(g) = q(cohe(A, B), g) = g\)
(c) cohe\((A, B)(0) = q(cohe(A, B), 0) = N(cohe(A, B))\)
(d) cohe\((A, B)(1) = q(cohe(A, B), 1) = N(q(cohe(A, B), 0)) = N(N(cohe(A, B))) = cohe(A, B)\).

**Lemma 7.** Let \(N\) be a strong negation on \([0,1]\). Let \(k\) be a function defined as \(N(x) = k^{-1}(k(1) - k(x))\). If we consider \(f(a, b) = N(k^{-1}(|k(b) - k(a)|))\), this \(f\) verifies the hypothesis of Lemma 5 and the coherence measure defined in Theorem 2 holds

(a) if cohe\((A, B) \neq g\), then cohe\((A, B)(cohe(A, B)) = 1\)
(b) if cohe\((A, B) \neq g\), then cohe\((A, B)(cohe(N(A), B)) = cohe(A, B)(cohe(A, N(B))) = 0\).

**Proof.** First we prove that \(f\) is well defined, that verifies symmetry, and \(f(0, a) = N(a)\). As \(N, k, k^{-1}\) are well-defined functions in \([0,1]\) to \([0,1]\), \(f\) is well defined as well. If absolute value is used, then the symmetry of \(f\) is guaranteed. \(f(0, a) = N(k^{-1}(|k(a) - k(0)|))\), but from \(N(1) = 0\) we obtain \(N(1) = k^{-1}(0)\), so \(k(0) = 0\). This reveals that \(f(0, a) = N(a)\) by the definition of \(f\), \(f(a, a) = N(k^{-1}(0)) = 1\), indeed.

Now we consider the definition of cohef\((A, B)\) in Theorem 2:

\[
\text{cohef}(A, B)(x) = q(\text{cohe}(A, B), x)
\]

We can affirm that cohef\((A, B)(\text{cohe}(A, B)) = q(\text{cohe}(A, B), \text{cohe}(A, B))\). From the definition of \(q\) and taking into account that cohe\((A, B) \neq g\), we can consider two cases: Either cohe\((A, B) < g\) or cohe\((A, B) > g\):
This result can be illustrated as in Figure 1. Of course, this picture is symbolic. Continuous lines do not have to be straight, and $g$ could not be in the central position. Furthermore, \(\text{cohe}(A, B)\) is not equal to \(1 - N(\text{cohe}(A, B))\) in general. Only those points marked with a circle are guaranteed by the results obtained here. We have already proved that if \(\text{cohe}(A, B) = g\) then \(\text{cohef}(A, B) = A^*\). That is to say that \(\text{cohef}(A, B)(x) = g\) is for all \(x \in X\).

### 3.1. Ambiguity Measures, Coherence Measures Based in Metrics, and Other Combinations

As is well known, in Fuzzy Sets Theory the ambiguity measures have been typically used as measures of fuzziness, but formerly they came from the classical Sets Theory.\(^6\) The extension of this definition to the fuzzy case, that is, to fuzzy sets, can be made straightforward by using the usual operations of union, intersection, and complementation.\(^7,8\) In previous articles, the relationship between the ambiguity and coherence measures has been proved by Extension Theorems in those cases of finite referentials,\(^1\) measurable Lebesgue,\(^2\) and in fact, recently in a generalization of any of the strong negations \(N.\(^5\) Also, in Refs. 1 and 2, methods were shown to construct coherence measures based on metrics, and in Ref. 5,  

![Figure 1](image-url)  
**Figure 1.** Example of fuzzy coherence measures.
general methods using a certain family of functions were shown. This allows us to make use of all of these constructions of coherence measures that arise from these theorems in this new generalized context. Table III presents a sketch of the possible combination of the elements that constitute the coherence measures in this last case.

4. CONCLUSIONS

A wide generalization of coherence measure (from \( P^f(X) \times P^f(X) \) to \([0,1]\)) for fuzzy sets has been presented. It is valid for a finite and nonfinite referential set \( X \), and for a general class of operators on \( P^f(X) \) and \([0,1]\): \( T, S, N \) where \( T \) is a \( t \)-norm, \( S \) is a \( t \)-conorm, and \( N \) is a strong negation that forms a dual triple with \( T \) and \( S \). Properties of these measures have been shown. Procedures to build a broad class of coherence measures have been also presented.

Finally, this definition has been extended in a natural way to fuzzy coherence measures (from \( P^f(X) \times P^f(X) \) to \( P^f([0,1]) \)) and methods to build a fuzzy coherence measure from the coherence measure have been shown.

References