THE REGION OF SOLVABILITY OF A PARAMETERIZED
BOUNDARY VALUE PROBLEM CAN BE DISCONNECTED

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A celebrated result by Amann, Ambrosetti and Mancini [1] implies the connectedness of
the region of existence for some parameter-depending boundary value problems which
are resonant at the first eigenvalue. The analogous thing does not hold for problems
which are resonant at the second eigenvalue.

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1. Introduction

Consider the following nonlinear Dirichlet boundary value problem

\[-u'' - \lambda_k u + r(t, u, u') = \mu \sin(kt/2), \quad t \in [0, 2\pi],
\]
\[u(0) = u(2\pi) = 0,\]

where \(\lambda_k = k^2/4\) is the \(k\)th eigenvalue of the operator \(u \mapsto -u''\) when acting
on \(H^1_0(0, 2\pi)\); \(r : [0, 2\pi] \times \mathbb{R}^2 \to \mathbb{R}\) is continuous and bounded, and \(\mu \in \mathbb{R}\) is a
parameter.

In the special case of \(k = 1\), Amann, Ambrosetti and Mancini [1], showed that, if
(1) has a lower and an upper solution (not necessarily ordered), then it is solvable.
This result does not extend to higher eigenvalues; indeed, for any \(k \geq 2\), the linear
problem

\[-u'' - \lambda_k u = h(t), \quad t \in [0, 2\pi], \quad u(0) = u(2\pi) = 0,
\]
is solvable for \(h(t) = \pm \sin(t/2)\), since \(\pm \int_0^{2\pi} \sin(t/2) \sin(kt/2) dt = 0\), but not for
any \(h \in C[0, 2\pi]\) with \(\int_0^{2\pi} h(t) \sin(kt/2) dt \neq 0\), and, of course, there are many
of such functions \(h\) lying between \(t \in [0, 2\pi] \mapsto -\sin(t/2)\) and \(t \in [0, 2\pi] \mapsto \sin(t/2)\).

Observe, however, that an immediate consequence of the Amann–Ambrosetti–
Mancini results is the following: if \(k = 1\), for any given \(r\), the set of those \(\mu \in \mathbb{R}\) such
that (1) is solvable is an interval. Now, it is easy to check that this fact continues to
hold for linear problems which are resonant at higher eigenvalues, and, on the other hand, for any $k \geq 2$ it follows from the results in [2] that it is also the case provided $r = r(t, u)$ does not depend on $u'$ and the partial derivative $\partial_u r$ is continuously defined on $[0, 2\pi] \times \mathbb{R}$ and verifies $-\lambda_{k+1} < -\lambda_k + \partial_u r(t, u) < -\lambda_{k-1}$ $\forall (t, u) \in [0, 2\pi] \times \mathbb{R}$. Thus, Ortega [4] asked whether the same result was valid without this, apparently artificial, assumption.

This paper is devoted to show that no direct extension of the mentioned consequence of the Amann–Ambrosetti–Mancini results to the second eigenvalue is possible:

**Theorem 1.1.** There exists a $C^\infty$, bounded mapping $r : [0, 2\pi] \times \mathbb{R}^2 \to \mathbb{R}$ and $\mu_- < 0 < \mu_+$ such that the boundary value problem

\[
-u'' - u + r(t, u, u') = \mu \sin t, \quad t \in [0, 2\pi],
\]

\[
u(0) = 0 = u(2\pi),
\]

is solvable for $\mu = \mu_\pm$, but not for $\mu = 0$.

Let remark that our construction is not analytic, neither it has a variational structure, so that it remains an open problem to give analytic/variational counterexamples to Ortega’s question or prove that they do not exist. On the other hand, Ortega originally raised the problem for nonlinearities of the special type $f(t, u) = g(u) - h(t)$, and, also in this framework, the question remains open.

Finally, the Amann–Ambrosetti–Mancini results were originally formulated in a PDE setting. A PDE counterexample for higher eigenvalues remains unknown up to the date.

### 2. A Sketch of the Proof

Theorem 1.1 easily follows from the result below:

**Proposition 2.1.** There exists a Lipschitz function $f : [0, 2\pi] \times \mathbb{R}^2 \to \mathbb{R}$ such that

1. For any $c \in \mathbb{R}$, the solution $x_c$ of the IVP

\[
-x'' + f(t, x, x') = 0, \quad t \in [0, 2\pi], \quad x(0) = 0, \quad x'(0) = c,
\]

verifies $x_c(2\pi) = 0$.

2. There exists a solution $x_*$ of the IVP

\[
-x_*'' + f(t, x_*, x_*') = \sin t, \quad t \in [0, 2\pi], \quad x_*(0) = 0,
\]

which verifies $x_*(2\pi) < 0$.

3. There exists some $B > 0$ such that

$$f(t, x, y) = -x \quad \forall (t, x, y) \in [0, 2\pi] \times \mathbb{R}^2 \text{ with } \| (x, y) \| > B.$$
Thus, we will first construct such a function \( f \). We will finish the paper by showing how to obtain Theorem 1.1 from this.

**Outline of the construction.** Consider the following linear, homogeneous Dirichlet boundary value problem:

\[
-u'' + a(t)u' + b(t)u = 0, \quad t \in [0, 2\pi], \quad u(0) = u(2\pi) = 0,
\]

(4)

where \( a, b \in L^1[0, 2\pi] \). Assume that it is resonant and let \( v : [0, 2\pi] \to \mathbb{R} \) be a nonzero solution of the adjoint problem:

\[
-v''(t) - a(t)v'(t) + b(t)v(t) = 0, \quad t \in [0, 2\pi], \quad v(0) = v(2\pi) = 0.
\]

The set

\[
\mathcal{E}(a, b) := \left\{ p \in L^1[0, 2\pi] : \int_0^{2\pi} p(s)v(s)ds = 0 \right\}
\]

is made up of those functions \( p \) such that every solution of the nonhomogeneous initial value problem

\[
-u'' + a(t)u' + b(t)u = p(t), \quad t \in [0, 2\pi], \quad u(0) = 0,
\]

(5)

verifies \( u(2\pi) = 0 \). If \( p \notin \mathcal{E}(a, b) \), no solutions of (5) can satisfy this equality, and then, either every solution \( u \) verifies \( u(2\pi) > 0 \) or any solution \( u \) verifies \( u(2\pi) < 0 \). In another words, \( (L^1[0, 2\pi]) \setminus \mathcal{E}(a, b) = \mathcal{N}_-(a, b) \cup \mathcal{N}_+(a, b) \), where

\[
\mathcal{N}_\pm(a, b) := \{ p \in L^1[0, 2\pi] \text{ such that any solution } u \text{ of (5) verifies } \pm u(0) > 0 \}.
\]

Observe, for instance, that the function \( t \mapsto \sin t \) belongs to \( \mathcal{N}_+(0, -1) \), since all solutions of the IVP

\[
-u'' - u = \sin t, \quad t \in [0, 2\pi], \quad u(0) = 0,
\]

(which can be easily shown to be parameterized in the family \( \{ t \mapsto (1/2)t \cos t + c \sin t \}_{c \in \mathbb{R}} \)) are positive at \( 2\pi \).

Our function \( f \) will be built in the following way: We will first find functions \( a, b \in \text{Lip}(0, 2\pi) \) such that (4) is resonant and the function \( t \in [0, 2\pi] \mapsto \sin t \) belongs to \( \mathcal{N}_-(a, b) \). We will choose a solution \( v_+ \) of (5) for \( p(t) = \sin t \) and take a cylinder of the type \([0, 2\pi] \times \{(x, y) \in \mathbb{R}^2 : \| (x, y) \| < A \} \) (which will be referred to as “the small cylinder”) containing this solution. In this set, we will let \( f(t, x, y) := a(t)x + b(t)y \). Outside a bigger cylinder of the type \([0, 2\pi] \times \{(x, y) \in \mathbb{R}^2 : \| (x, y) \| > B \} \) (here, \( B > A \)), our function will simply be defined by \( f(t, x, y) := -x \). The key point in the construction will be how to do it in such a way that, for any \( c \in \mathbb{R} \), the solution \( x_c \) of (3) verifies \( x_c(2\pi) = 0 \).
3. The Construction in the Small Cylinder

Lemma 3.1. There exists an increasing, $C^\infty$ diffeomorphism $\phi : [0, 2\pi] \to [0, 2\pi]$ such that

$$\int_0^{2\pi} (\sin t) \sin(\phi(t))\phi'(t)^2 dt < 0.$$

Proof. For any $0 < \epsilon < 1$, we consider the piecewise linear function $p_\epsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$p_\epsilon(t) := \begin{cases} \frac{\pi/4}{(3\pi/2) - \epsilon} t & \text{if } t \leq (3\pi/2) - \epsilon, \\ \frac{\pi/4 + \epsilon}{(3\pi/2) + \epsilon} t - \frac{(3\pi/2) + \epsilon}{4\epsilon} & \text{if } (3\pi/2) - \epsilon \leq t \leq (3\pi/2) + \epsilon, \\ 3\pi/4 + \frac{5\pi/4}{(\pi/2) - \epsilon} (t - (3\pi/2) - \epsilon) & \text{if } (3\pi/2) + \epsilon \leq t. \end{cases}$$

It is not a $C^\infty$ function, but it is strictly increasing and verifies $p_\epsilon(0) = 0$, $p_\epsilon(2\pi) = 2\pi$, and

$$\lim_{\epsilon \to 0} \int_0^{2\pi} (\sin t) \sin(p_\epsilon(t))p'_\epsilon(t)^2 dt = -\infty.$$

To check this, simply decompose this integral as the sum of three elements; the first one being the integral between $0$ and $(3\pi/2) - \epsilon$ of $t \mapsto (\sin t) \sin(\phi(t))\phi'(t)^2$, the second one being the integral between $(3\pi/2) - \epsilon$ and $(3\pi/2) + \epsilon$ and the third one the integral between $(3\pi/2) + \epsilon$ and $2\pi$. Then, both the first and the third summand are bounded, while the second one diverges to $-\infty$ as $\epsilon \to 0$.

Fix $\epsilon_* > 0$ be small enough so that $\int_0^{2\pi} (\sin t) \sin(p_\epsilon(t))p'_\epsilon(t)^2 dt < 0$. Take next an approximation of the unity made of even functions $\{h_n\}_n$ and define, for each $n \in \mathbb{N}$,

$$\phi_n := h_n \ast p_\epsilon : [0, 2\pi] \to \mathbb{R}.$$ 

Then, $\{\phi_n\}_n$ is a sequence of $C^\infty[0, 2\pi]$ functions and, $h_n$ being even, $\phi'_n(t) > 0 \forall t \in [0, 2\pi]$ and $\phi_n(0) = 0$. $\phi_n(2\pi) = 2\pi$ for $n$ big enough. Further, $\lim_{n \to \infty} \phi_n(t) = p_\epsilon(t)$ for any $t \in [0, 2\pi]$, $\lim_{n \to \infty} \phi'_n(t) = p'_\epsilon(t)$ for any $t \in [0, 2\pi\setminus((3\pi/2) - \epsilon_*, (3\pi/2) + \epsilon_*)$, and the sequence $\{\phi'_n\}_n$ is bounded in $L^\infty[0, 2\pi]$. The dominated convergence theorem implies that

$$\lim_{n \to \infty} \int_0^{2\pi} (\sin t) \sin(\phi_n(t))\phi'_n(t)^2 dt = \int_0^{2\pi} (\sin t) \sin(p_\epsilon(t))p'_\epsilon(t)^2 dt < 0,$$

so that it suffices to take $\phi := \phi_{n_*}$ for some $n_* \in \mathbb{N}$ big enough. \hfill $\Box$

Lemma 3.2. Let $\phi$ be the diffeomorphism given in Lemma 3.1. Then, the solution $w$ of the linear IVP
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\[ \begin{align*}
-u'' - u &= \sin(\phi(t))\phi'(t)^2, \quad 0 \leq t \leq 2\pi, \\
u(0) &= 0, \quad u'(0) = 1,
\end{align*} \]

(6)

verifies \( w(2\pi) < 0 \).

**Proof.** Write \( \sin(\phi(t))\phi'(t)^2 = \tilde{h}(t) + \tilde{h}\sin t \), where \( \tilde{h} := \frac{1}{\pi} \int_0^{2\pi} \sin(\phi(t))\phi'(t)^2\sin t \ dt < 0 \) and the \( C^1[0, 2\pi] \) mapping \( \tilde{h} \) verifies

\[ \int_0^{2\pi} \tilde{h}(t) \sin t \ dt = 0. \]

(7)

Then, \( u = \hat{u} + (\tilde{h}/2)(t \cos t - \sin t) \), \( \hat{u} : [0, 2\pi] \to \mathbb{R} \) being the solution of the IVP

\[ \begin{align*}
-u'' - u &= \tilde{h}(t), \quad 0 \leq t \leq 2\pi, \\
u(0) &= 0, \quad u'(0) = 1,
\end{align*} \]

Expression (7) implies that \( \hat{u}(2\pi) = 0 \). Consequently,

\[ u(2\pi) = 2\pi\tilde{h}/2 < 0. \]

\[ \square \]

**Lemma 3.3.** There exists a \( C^\infty \), increasing diffeomorphism \( \varphi : [0, 2\pi] \to [0, 2\pi] \) and a solution \( v_+ : [0, 2\pi] \to \mathbb{R} \) of the equation

\[ -v'' + \frac{\varphi''(t)}{\varphi'(t)} v' - \varphi'(t)^2 v = \sin t, \quad 0 \leq t \leq 2\pi, \]

(8)

which verifies \( v_+(0) = 0, v_+(2\pi) < 0 \).

**Proof.** Let \( \phi \) be as given in Lemma 3.1, and let \( w \) be given as in Lemma 3.2. We define \( \varphi := \phi^{-1} : [0, 2\pi] \to [0, 2\pi] \), and \( v_+ := w \circ \varphi : [0, 2\pi] \to \mathbb{R} \). Then, \( v_+(0) = w(0) = 0, v_+(2\pi) = w(2\pi) < 0 \), and

\[ v'_+ (t) = \varphi'(t)w'(\varphi(t)), \quad v''_+ (t) = \varphi''(t)w'(\varphi(t)) + \varphi'(t)^2 w''(\varphi(t)), \quad t \in [0, 2\pi]. \]

Since \( w \) is a solution of (6), a straightforward computation shows that \( v_+ \) solves (8).

\[ \square \]

4. From the Small Cylinder to the Whole Space

Let now \( \varphi : [0, 2\pi] \to [0, 2\pi] \) and \( v_+ : [0, 2\pi] \to \mathbb{R} \) as given by Lemma 3.3. Let the constants \( 0 < m < 1 < M \) verify \( m < \varphi'(t) < M \) for any \( t \in [0, 2\pi] \) and let \( A > 0 \) verify

\[ ||(v_+(t), v'_+(t))|| < A \ \forall \ t \in [0, 2\pi]. \]

(9)

Next, choose \( \rho \in C^1(\mathbb{R}) \) verifying, for some \( N > A/m \), some \( P > N \) and some \( Q > P + 1 \), the following facts:
(1) \( \rho(-c) = -\rho(c) \quad \forall c \in \mathbb{R} \),
(2) \( \rho'(c) > 0 \quad \forall c \in \mathbb{R} \),
(3) \( \rho'(c) > 2(M + 1)\rho(c)/m \quad \forall c \in [P, P + 1] \),
(4) \( \rho(c) = c \quad \forall c \in \mathbb{R} \) with \( |c| > Q \) and \( |c| < N \).

Let now \( s \in C^\infty(\mathbb{R}) \) verify

\[
s(t) = 0 \quad \forall t \leq 0, \quad s(t) = 1 \quad \forall t \geq 1, \quad 0 < s'(t) < 2 \quad \forall t \in [0, 1],
\]

and let \( \Phi : [0, 2\pi] \times \mathbb{R} \to [0, 2\pi] \) be the \( C^\infty \) homotopy between \( \varphi \) and the identity mapping on \([0, 2\pi] \), defined by

\[
\Phi(t, c) := (1 - s(|c| - P))\varphi(t) + s(|c| - P)t.
\]

Finally, consider the function \( H : [0, 2\pi] \times \mathbb{R} \to \mathbb{R} \) given below

\[
H(t, c) := \rho(c) \sin(\Phi(t, c)), \quad (t, c) \in [0, 2\pi] \times \mathbb{R},
\]

together with the \( C^1 \) mapping

\[
\Psi : [0, 2\pi] \times \mathbb{R} \to [0, 2\pi] \times \mathbb{R}^2, \quad \Psi(t, c) := (t, H(t, c), \partial_t H(t, c)).
\]

**Lemma 4.1.** \( \Psi \) is injective, and \( \Psi'(t_0, c_0) \) is injective for any \( (t_0, c_0) \in [0, 2\pi] \times \mathbb{R} \).

**Proof.** Let \( t \in [0, 2\pi] \) and \( c_1, c_2 \in \mathbb{R} \) verify

\[
(H(t, c_1), \partial_t H(t, c_1)) = (H(t, c_2), \partial_t H(t, c_2)),
\]
or, what is the same,

\[
\rho(c_1)\sin(\Phi(t, c_1)), \partial_t \Phi(t, c_1) \cos(\Phi(t, c_1)) = \rho(c_2)\sin(\Phi(t, c_2)), \partial_t \Phi(t, c_2) \cos(\Phi(t, c_2))
\]

The point in the left belongs to the ellipse \((x/\rho(c_1))^2 + (y/\rho(c_1)\partial_t \Phi(t, c_1))^2 = 1\), while the point in the right belongs to the ellipse \((x/\rho(c_2))^2 + (y/\rho(c_2)\partial_t \Phi(t, c_2))^2 = 1\). However, the two functions \( c \in \mathbb{R} \mapsto \rho(c) \) and \( c \in \mathbb{R} \mapsto \rho(c)\partial_t \Phi(t, c) \) are odd and strictly increasing, as it follows from items (1) and (2) above and the Claim below. It means that, either \( c_1 = c_2 \) or \( c_1 = -c_2 \neq 0 \). Assume that the second possibility holds. Then, \( \Phi(t, c_1) = \Phi(t, c_2) \) and \( \partial_t \Phi(t, c_1) = \partial_t \Phi(t, c_2) \), so that it follows from expression (11) above that \( \rho(c_1) = \rho(c_2) \) and then, \( c_1 = c_2 \), a contradiction.

Let us prove now the claimed strictly monotonicity of the function \( c \in \mathbb{R} \mapsto \rho(c)\partial_t \Phi(t, c) \) for any \( t \in [0, 2\pi] \).

**Claim.**

\[
\frac{d}{dc}[\rho(c)\partial_t \Phi(t, c)] > 0 \quad \forall (t, c) \in [0, 2\pi] \times \mathbb{R}.
\]

**Proof.** Since \( \rho(-c)\partial_t \Phi(t, -c) = -\rho(c)\partial_t \Phi(t, c) \) \( \forall (t, c) \in [0, 2\pi] \times \mathbb{R} \), we only have to show that the inequality above holds for \( (t, c) \in [0, 2\pi] \times [0, +\infty] \). Thus, let
Lemma 4.2. Let $t \in [0, 2\pi]$ and $c \in [0, +\infty]$ be given. The following hold
\[
\frac{d}{dc} [\rho(c) \partial_t \Phi(t, c)] = \rho'(c) \partial_t \Phi(t, c) + \rho(c) \partial_c \partial_t \Phi(t, c)
\]
\[
= \rho'(c) [(1 - s(c - P))\varphi'(t) + s(c - P)] + \rho(c)[s'(c - P)(1 - \varphi'(t))]
\]
\[
\geq \begin{cases} 
\rho'(c)\varphi'(t) > 0, & \text{if } 0 \leq c \leq P, \\
mp\rho'(c) - 2(M + 1)\rho(c) > 0, & \text{if } P \leq c \leq P + 1, \\
\rho'(c) > 0, & \text{if } c \geq P + 1,
\end{cases}
\]
the last inequality following from item (3) in the choice of $\rho$. This proves (12).

Finally, in view of the definition of $\Psi$ in (10), the fact of $\Psi'(t_0, c_0)$ being injective for any $(t_0, c_0) \in [0, 2\pi] \times \mathbb{R}$ is equivalent to the fact that $\partial_t H$ and $\partial_c \partial_t H$ do not vanish simultaneously. Assume, otherwise, that $\partial_c \Psi(t_0, c_0) = \partial_t \partial_c \Psi(t_0, c_0) = 0$ for some $(t_0, c_0) \in [0, 2\pi] \times \mathbb{R}$. Then,
\[
\left. \frac{d}{dt} \right|_{t=c_0} [\rho(c) \partial_t \Phi(t_0, c)] = \rho'(c_0) \partial_t \Phi(t_0, c_0) + \rho(c_0) \partial_c \partial_t \Phi(t_0, c_0) = 0,
\]
contradicting (12).

Next result shows that $\Psi^{-1} : \Psi([0, 2\pi] \times \mathbb{R}^2) \rightarrow \mathbb{R}^2$ is a locally Lipschitz mapping of a uniform Lipschitz constant. From now on, we will denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^p$, $p = 2, 3$.

Lemma 4.2. There exist constants $L, \delta > 0$ such that, for any $t_1, t_2 \in [0, 2\pi]$ and $c_1, c_2 \in \mathbb{R}$ with $\|\Psi(t_1, c_1) - \Psi(t_2, c_2)\| < \delta$, we have:
\[
\|(t_1 - t_2, c_1 - c_2)\| \leq L\|\Psi(t_1, c_1) - \Psi(t_2, c_2)\|.
\]

Proof. Assume the lemma is not true. Then, there exist sequences $\{(t_1^n, c_1^n)\}_n, \{(t_2^n, c_2^n)\}_n \subset [0, 2\pi] \times \mathbb{R}$ with $\Psi(t_1^n, c_1^n) - \Psi(t_2^n, c_2^n) \rightarrow 0$ and
\[
\|\Psi(t_1^n, c_1^n) - \Psi(t_2^n, c_2^n)\| < \frac{1}{n}\|(t_1^n - t_2^n, c_1^n - c_2^n)\| \quad \forall n \in \mathbb{N}.
\]

Observe, however, that, for any $(t, c) \in [0, 2\pi] \times \mathbb{R}$ with $|c| > Q$, $H(t, c) = c \sin t$, and consequently, $\Psi(t, c) = (t, c \sin t, c \cos t)$, so that, if $(t_1, c_1), (t_2, c_2) \in [0, 2\pi] \times (\mathbb{R} \setminus [-Q, Q])$,
\[
\|\Psi(t_1, c_1) - \Psi(t_2, c_2)\|
\]
\[
= \|(t_1 - t_2, c_1 \sin t_1 - c_2 \sin t_2, c_1 \cos t_1 - c_2 \cos t_2)\|
\]
\[
\geq \frac{1}{2}\|t_1 - t_2\| + \frac{1}{2}\|(c_1 \sin t_1 - c_2 \sin t_2, c_1 \cos t_1 - c_2 \cos t_2)\|
\]
\[
\geq \frac{1}{2}\|t_1 - t_2\| + \frac{1}{2}\|c_1| - |c_2|\| \geq \frac{1}{2}\|(t_1 - t_2, |c_1| - |c_2|)\|.
\]

Let us show next that either $\{(t_1^n, c_1^n)\}_n$ or $\{(t_2^n, c_2^n)\}_n$ has a bounded subsequence. Since $\|\Psi(t, c)\| \rightarrow +\infty$ as $|c| \rightarrow +\infty$ uniformly with respect to
$t \in [0, 2\pi)$, we will deduce that, after possibly passing to a subsequence, we can assume that both $\{c_1^n\}_n$ and $\{c_2^n\}_n$ are bounded. Otherwise, we should have $\{(|t_1^n, c_1^n|), \{(|t_2^n, c_2^n|)\}_n \to +\infty$, or, what is the same, $\{|c_1^n|, \{|c_2^n|\}_n \to +\infty$. On the other hand, since $\{\Psi(t_1^n, c_1^n) - \Psi(t_2^n, c_2^n)\}_n \to 0$, (15) implies that both $\{|t_1^n - t_2^n|\}_n$ and $\{|c_1^n - c_2^n|\}_n$ converge to zero. Now, if the set of $n \in \mathbb{N}$ such that $c_1^n c_2^n > 0$ is infinite, (15) implies that, for $n$ big enough inside this set, $\|\Psi(t_1^n, c_1^n) - \Psi(t_2^n, c_2^n)\| \geq \frac{1}{2} \|(t_1^n - t_2^n, c_1^n - c_2^n)\|$, which contradicts (14). If not, $c_1^n c_2^n < 0$ for $n$ big enough, and we deduce from (15) that $\{c_1^n + c_2^n\}_n \to 0$ and, thus, $\{c_1^n/c_2^n\}_n \to 1$. It follows that

$$\frac{1}{|c_1^n|}\|(|c_1^n| \sin t_1^n - c_2^n \sin t_2^n, c_1^n \cos t_1^n - c_2^n \cos t_2^n)\|$$

$$= \left\| \left( \frac{c_1^n}{c_1^n} \sin t_1^n, \cos t_1^n - \frac{c_1^n}{c_2^n} \cos t_2^n \right) \right\| \to 2,$$

and, since $|c_1^n| \to +\infty$, it implies that $\|\Psi(t_1^n, c_1^n) - \Psi(t_2^n, c_2^n)\| \to +\infty$, in contradiction with the choice of these sequences.

Thus, passing again to a subsequence, we may assume that $\{(t_1^n, c_1^n)\}_n \to (t_1^1, c_1^1) \quad \text{and} \quad \{(t_2^n, c_2^n)\}_n \to (t_2^1, c_2^1)$ for given points $(t_1^1, c_1^1), (t_2^1, c_2^1) \in [0, 2\pi) \times \mathbb{R}$. We deduce from (14) that $\Psi(t_1^1, c_1^1) = \Psi(t_2^1, c_2^1)$, and then, since, as shown in Lemma 4.1 above, $\Psi$ is injective, $(t_1^1, c_1^1) = (t_2^1, c_2^1)$. Together with inequality (14), this contradicts the fact that, as shown in Lemma 4.1 above, $\Psi'(t_1^1, c_1^1)$ is injective. Indeed, any accumulation point of the sequence $\{\frac{\Psi(t_1^n, c_1^n) - \Psi(t_2^n, c_2^n)}{\|\Psi(t_1^n, c_1^n) - \Psi(t_2^n, c_2^n)\|}\}_n$ must belong to the kernel of $\Psi'(t_1^1, c_1^1)$.

Define

$$S := \Psi([0, 2\pi] \times \mathbb{R}) \subset [0, 2\pi] \times \mathbb{R}^2, \quad g := (\partial_t^2 H) \circ \Psi^{-1} : S \to \mathbb{R}.$$

**Lemma 4.3.** The following hold:

1. $g(t, x, y) = -\phi'(t)^2 x + \phi''(t) y$ for any $(t, x, y) \in S$ with $\|(x, y)\| < mn$,
2. $g(t, x, y) = -x$ for any $(t, x, y) \in S$ with $\|(x, y)\| > MQ$,
3. There are constants $L_\epsilon, \delta > 0$ such that

$$|g(t_1, x_1, y_1) - g(t_2, x_2, y_2)| \leq L_\epsilon \|(t_1 - t_2, x_1 - x_2, y_1 - y_2)\|,$$

for any $(t_1, x_1, y_1), (t_2, x_2, y_2) \in S$ with $\|(t_1 - t_2, x_1 - x_2, y_1 - y_2)\| < \delta$.

**Proof.** Let $(t, x, y) \in S$ be given. There exists some point $c \in \mathbb{R}$ such that $x = \rho(c) \sin (\Phi(t, c))$ and $y = \rho(c) \partial_t \Phi(t, c) \cos (\Phi(t, c))$. Then,

$$\|(x, y)\| = \sqrt{(\rho(c))^2 \sin (\Phi(t, c))^2 + (\rho(c))^2 \partial_t \Phi(t, c) \cos (\Phi(t, c))^2}$$

$$= |\rho(c)| \sqrt{\sin (\Phi(t, c))^2 + \partial_t \Phi(t, c) \cos (\Phi(t, c))^2} \in [m|\rho(c)|, M|\rho(c)|]. \quad (16)$$

Consequently, if $\|(x, y)\| < mn$, $m|\rho(c)| \leq \|(x, y)\| < mn$, and we deduce that $|\rho(c)| < N$. Since $\rho$ is strictly increasing and $\rho(N) = N, \rho(-N) = -N,$
on the other hand, since proving (1).

for any \( y \)
In general, the question of whether, a function obtained by pasting Lipschitzian sequences must be bounded, and then, (17) implies that

\[
g(t, x, y) = \partial_{tt}^2 H(t, c) = -c \sin(\varphi(t))\varphi'(t)^2 + c \cos(\varphi(t))\varphi''(t) = -\varphi'(t)^2 x + \frac{\varphi''(t)}{\varphi'(t)} y,
\]

proving (1).

On the other hand, in case \( \|x, y\| > MQ \), (16) implies that \( Q < |\rho(c)| \), and, since \( \rho \) is strictly increasing and \( \rho(Q) = Q \), \( \rho(-Q) = -Q \), we deduce that \( Q < |c| \).

Thus, \( x = \rho(c) \sin(\Psi(t, c)) = c \sin(\varphi(t)), y = \rho(c)\partial_t \Psi(t, c) \cos(\Psi(t, c)) = c\varphi'(t) \cos(\varphi(t)) \), and

\[
g(t, x, y) = \partial_{tt}^2 H(t, c) = -c \sin(\varphi(t))\varphi'(t)^2 + c \cos(\varphi(t))\varphi''(t) = -\varphi'(t)^2 x + \frac{\varphi''(t)}{\varphi'(t)} y,
\]

proving (2).

Finally, observe that \( \partial_{tt}^2 H \) is a \( C^1 \) mapping on \([0, 2\pi] \times \mathbb{R}\) which equals to \( (t, c) \to -c \sin t \) outside a compact set. Thus, it is Lipschitzian, and (3) becomes a direct consequence of Lemma 4.2 above.

Recall that we started our argument by choosing a constant \( A > 0 \) big enough to satisfy (9), which has not yet been employed. It is time to do so. Choose another constant \( B > MQ \) (observe that, in particular, \( B > A \)) and denote

\[
D = S \cup \{(t, x, y) \in [0, 2\pi] \times \mathbb{R}^2 : \|x, y\| \leq A\} \cup \{(t, x, y) \in [0, 2\pi] \times \mathbb{R}^2 : \|x, y\| \geq B\}.
\]

Consider, next, the mapping \( \bar{g} : D \to \mathbb{R} \) defined by

\[
\bar{g}(t, x, y) := \begin{cases} g(t, x, y), & \text{if } (t, x, y) \in S, \\ -\varphi'(t)^2 x + \frac{\varphi''(t)}{\varphi'(t)} y, & \text{if } \|t, x, y\| \leq A, \\ -x, & \text{if } \|t, x, y\| \geq B. \end{cases}
\]

In general, the question of whether, a function obtained by pasting Lipschitzian mappings defined in smaller closed sets, is Lipschitzian, is a delicate one which requires a careful analysis of the sets making up the final domain. However, in our case, things will be easier.

**Lemma 4.4.** \( \bar{g} \) is Lipschitzian.

**Proof.** Assume instead that it is not. Then, we can find sequences \( \{(t_1^n, x_1^n, y_1^n)\}_n \) and \( \{(t_2^n, x_2^n, y_2^n)\}_n \subset [0, 2\pi] \times \mathbb{R}^2 \) with

\[
|g(t_1^n, x_1^n, y_1^n) - g(t_2^n, x_2^n, y_2^n)| > n \|(t_1^n - t_2^n, x_1^n - x_2^n, y_1^n - y_2^n)\|
\]

for any \( n \in \mathbb{N} \). Since, as a consequence of Lemma 4.3 above, \( g \) is continuous, and on the other hand, \( \bar{g}(t, x, y) = -x \) outside a compact set, we deduce that both sequences must be bounded, and then, (17) implies that \( \{(t_1^n - t_2^n, x_1^n - x_2^n, y_1^n - y_2^n)\}_n \to 0.\)
Thus, after possibly passing to a subsequence, we may assume that there exists some element $(t_*, x_*, y_*) \in \mathcal{D}$ such that

$$\{(t_1^n, x_1^n, y_1^n)\}_n, \{(t_2^n, x_2^n, y_2^n)\}_n \rightarrow (t_*, x_*, y_*)$$

Now, three possibilities appear:

1. $\|(x_*, y_*)\| < mN$. Then, there exists some $r > 0$ such that $\tilde{g}(t, x, y) = -\varphi'(t)^2 x + \frac{\varphi''(t)}{\varphi(t)} y$ for all $(t, x, y) \in \mathcal{D}$ with $\|(t - t_*, x - x_*, y - y_*)\| < r$, and we deduce that $\tilde{g}$ is locally Lipschitz at $(t_*, x_*, y_*)$. This contradicts (17) and (18).

2. $\|(x_*, y_*)\| > MQ$. Then, there exists some $r > 0$ such that $\tilde{g}(t, x, y) = -x$ for all $(t, x, y) \in \mathcal{D}$ with $\|(t - t_*, x - x_*, y - y_*)\| < r$, and we deduce that $\tilde{g}$ is locally Lipschitz at $(t_*, x_*, y_*)$. This contradicts also (17) and (18).

3. $mN \leq \|(x_*, y_*)\| \leq MQ$. Then, there exists some $r > 0$ such that $(t, x, y) \in \mathcal{D}$ with $\|(t - t_*, x - x_*, y - y_*)\| < r$, and therefore, $\tilde{g}(t, x, y) = g(t, x, y)$ for all $(t, x, y) \in \mathcal{D}$ with $\|(t - t_*, x - x_*, y - y_*)\| < r$. Thus, Lemma 4.3 implies that $\tilde{g}$ is locally Lipschitz at $(t_*, x_*, y_*)$, giving again a contradiction with (17) and (18).

**Corollary 4.5.** If $B \in ]A, +\infty[\right)$ is big enough, there exists a Lipschitz mapping $f : [0, 2\pi] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

1. $f(t, x, y) = -\varphi'(t)^2 x + \frac{\varphi''(t)}{\varphi(t)} y$ for any $(t, x, y) \in [0, 2\pi] \times \mathbb{R}^2$ with $\|(x, y)\| < A$,
2. $f(t, x, y) = -x$ for any $(t, x, y) \in [0, 2\pi] \times \mathbb{R}^2$ with $\|(x, y)\| > B$,
3. For any $c \in \mathbb{R}$, the solution $x_c$ of (3) verifies $x_c(2\pi) = 0$.

**Proof.** Let $B$ and $\tilde{g} : \mathcal{D} \rightarrow \mathbb{R}$ be defined as above. Since it is a Lipschitz mapping, it admits a Lipschitz extension $\tilde{f} : [0, 2\pi] \times \mathbb{R}^2 \rightarrow \mathbb{R}$, see, for instance, [3, Theorem 1, p. 80]. Items (1) and (2) follow easily now from the definitions and Lemma 4.4. To check (3) simply observe that, for any $c \in \mathbb{R}$, the mapping $t \in [0, 2\pi] \rightarrow H(t, c)$ is a solution of the IVP

$$-x'' + f(t, x, x') = 0, \quad t \in [0, 2\pi], \quad x(0) = 0, \quad x'(0) = \partial_t H(0, c).$$

Since, further, $\partial_t H$ is a continuous mapping with $\lim_{c \rightarrow \pm \infty} \partial_t H(0, c) = \pm \infty$, we deduce that every solution of (19) can be written as $H(\cdot, c)$ for some $c \in \mathbb{R}$. It implies (3).

Observe now that this Corollary together with Lemma 3.3 and the choice of $A$ immediately imply Proposition 2.1. Let us finally show how to deduce Theorem 1.1 from this.

**Proof of Theorem 1.1.** A standard convolution argument with a approximation of the unity made of even functions which vanish outside the unit ball in $\mathbb{R}^3$ shows the existence of a sequence $\{f_n\}_n \subset C^\infty([0, 2\pi] \times \mathbb{R}^2)$ such that $f_n \rightarrow f$ uniformly and $f_n(t, x, y) = f(t, x, y) = -x$ for all $(t, x, y) \in [0, 2\pi] \times \mathbb{R}^2$ with $\|(x, y)\| > B + 1$. Let
us define, for each \( n \in \mathbb{N} \) and each \( c \in \mathbb{R} \), \( z_n(c) \) as the value at \( 2\pi \) of the solution of the IVP

\[
-x'' + f_n(t, x, x') = 0, \quad t \in [0, 2\pi], \quad x(0) = 0, \quad x'(0) = c.
\]

The theorem of continuous dependence with respect to parameters of the solutions of a differential equation implies that the functional sequence \( z_n \) converges to the constant zero function uniformly on compact subsets of \( \mathbb{R} \). Since \( z_n|_{[-B-1, B+1]} \equiv 0 \) for any \( n \in \mathbb{N} \), we have in fact that \( z_n \to 0 \) uniformly on \( \mathbb{R} \). On the other hand, if we define, for each \( n \in \mathbb{N} \), \( a_n \) as the value at \( 2\pi \) of the solution of the IVP

\[
-x'' + f_n(t, x, x') = \sin t, \quad t \in [0, 2\pi], \quad x(0) = 0, \quad x'(0) = x''(0),
\]

then \( \{a_n\}_n \to x_*(2\pi) = 0 \). Thus, it is possible to find \( n_0 \in \mathbb{N} \) big enough such that \( z_{n_0}(c) > a_n \) \( \forall c \in \mathbb{R} \).

By construction, the boundary value problem

\[
-x'' - x + (x + f_{n_0}(t, x, x')) = \lambda \sin t, \quad t \in [0, 2\pi], \quad x(0) = 0, \quad x(2\pi) = a_{n_0},
\]

(20)

is solvable for \( \lambda = 1 \), but not for \( \lambda = 0 \). On the other hand, the function \( \tilde{r} : [0, 2\pi] \times \mathbb{R}^2 \to \mathbb{R}, (t, x, y) \mapsto x + f_{n_0}(t, x, y) \) is \( C^\infty \) and has a compact support, so that it is bounded.

**Claim.** There exists some \( \lambda_- < 0 \) such that (20) is solvable for \( \lambda = \lambda_- \).

**Proof.** Consider the continuous function \( \psi : \mathbb{R} \to \mathbb{R} \) defined as follows: \( \psi(\lambda) \) is, for any \( \lambda \in \mathbb{R} \) the value at \( 2\pi \) of the solution \( x_\lambda \) of the IVP

\[
-x_\lambda'' - x_\lambda + \tilde{r}(t, x_\lambda, x'_\lambda) = \lambda \sin t, \quad t \in [0, 2\pi], \quad x_\lambda(0) = 0, \quad x'_\lambda(0) = \lambda.
\]

Observe that, for any \( \lambda \neq 0 \), \( y_\lambda := x_\lambda/\lambda \) is a solution of the IVP

\[
-y_\lambda'' - y_\lambda + \tilde{r}(t, x_\lambda, x'_\lambda)/\lambda = \sin t, \quad t \in [0, 2\pi], \quad y_\lambda(0) = 0, \quad y'_\lambda(0) = 1.
\]

and the theorem of continuous dependence of the solutions of differential equations with respect to parameters implies that \( \lim_{\lambda \to \pm \infty} y_\lambda(t) = (1/2)t \cos t \) uniformly with respect to \( t \in [0, 2\pi] \). In particular, \( \lim_{\lambda \to \pm \infty} y_\lambda(2\pi) = \pi \), and we deduce that

\[
\lim_{\lambda \to -\infty} \psi(\lambda) = -\infty.
\]

Since \( \psi(0) = z_{n_0}(0) > a_{n_0} \), there must exist some \( \lambda_- < 0 \) with \( \psi(\lambda_-) = a_{n_0} \). Consequently, (20) is solvable for \( \lambda = \lambda_- \).

To complete the proof of Theorem 1.1, simply use the change of variables \( u(t) := x(t) - (a_{n_0}/2\pi)t \), which transforms problem (20) into

\[
-u'' - u + r(t, u, u') = \lambda \sin t, \quad t \in [0, 2\pi], \quad u(0) = u(2\pi) = 0,
\]

being \( r(t, u, v) := \tilde{r}(t, u+(a_{n_0}/2\pi)t, v+(a_{n_0}/2\pi)) - (a_{n_0}/2\pi)t \), \( (t, u, v) \in [0, 2\pi] \times \mathbb{R}^2 \).

}\qed
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References