SEMICLASSICAL STATES FOR COUPLED SCHRÖDINGER–MAXWELL EQUATIONS:
CONCENTRATION AROUND A SPHERE

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In this paper we study a coupled nonlinear Schrödinger–Maxwell system of equations. In this framework, we are concerned with the existence of semiclassical states. We use a perturbation scheme in a variational setting in order to study the concentration of the solutions when the Planck constant is supposed to be small enough.

Keywords: Semiclassical states; Schrödinger equation; Maxwell equation; variational method; perturbation techniques.

1. Introduction

The nonlinear Schrödinger equation for standing waves:

\[-\hbar^2 \Delta u(x) + u + V(x)u(x) = u^p(x), \quad x \in \mathbb{R}^n\]  (1.1)

has attracted much attention, and is still a stimulating field of research either for mathematicians as for physicists. In (1.1) the constant \(\hbar\) is the Planck constant, \(V\) is a bounded and positive potential, and the nonlinear term \(u^p\) is introduced to describe the electromagnetic interaction among many different particles.

A number of papers (see Refs. 4, 15–17, 23 and 25) are concerned with the question of semiclassical states, i.e. the behavior of solutions of problem (1.1) when \(\hbar\) tends to zero. In a sense, this describes the transition between quantum mechanics and classical mechanics. Roughly speaking, it has been shown that there exist spikes (solutions concentrating at a single point) on the nondegenerate critical points of the potential \(V\). Also, multibump solutions have been found (i.e. solutions which concentrate around several different points). Very recently the existence of solutions concentrating around a whole sphere in \(\mathbb{R}^n\) (for radial potentials) has been shown in Refs. 5 and 6.

Other way to try to describe the effect of interaction is through Maxwell equations, see Refs. 7 and 18. In this paper we will work under both types of interaction.
In doing so, we are led with a system of equations of the form (see for instance Ref. 12):

\[
\begin{align*}
-\hbar^2 \Delta w + w + \psi(x)w &= \gamma w^p, \quad x \in \mathbb{R}^3 \\
-\Delta \psi &= w^2(x), \quad x \in \mathbb{R}^3
\end{align*}
\]

where \( \gamma \) is a positive parameter and \( p > 1 \). We point out that, in a similar way, this problem also arises in the study of the Klein–Gordon–Maxwell equation.

This type of equations has been studied in recent years; some of the papers most closely related to (1.2) are Refs. 8, 9 and 11, for the case \( \gamma = 0 \), and Refs. 13 and 14 for the question of existence with \( \gamma \neq 0 \).

In this work we are concerned with the problem of semiclassical states for (1.2). Specifically, we study the existence of positive radial solutions concentrating around a sphere, in the spirit of Ref. 5. For physical reasons, we search solutions \( w \) verifying

\[
\int_{\mathbb{R}^3} w^2(x) \, dx = 1;
\]

so, \( w^2 \) tend to \( \delta_{S_r} \) (the Dirac delta around the sphere \( S_r \)) as \( \varepsilon = h \rightarrow 0 \). Then, it is quite natural to look for solutions \( w \) verifying

\[
-\varepsilon \Delta w + \varepsilon w + \psi(x)w = \varepsilon \gamma \varepsilon^{\frac{p-2}{2}} y^p, \quad x \in \mathbb{R}^3
\]

where \( y = \varepsilon \varepsilon^{\frac{1}{2}} \) tends to \( \delta_{S_r} \) as \( \varepsilon \rightarrow 0 \). Then, it is quite natural to look for solutions \( v(x) = \varepsilon^{1/2} w(x) \) would then be uniformly bounded (independently of \( \varepsilon \)) in \( L^\infty \), and our problem becomes:

\[
\begin{align*}
-\varepsilon^2 \Delta v + v + \psi(x)v &= \gamma \varepsilon^{\frac{p-2}{2}} y^p, \quad x \in \mathbb{R}^3 \\
-\Delta \psi &= \frac{1}{\varepsilon} v^2(x), \quad x \in \mathbb{R}^3
\end{align*}
\]

In order to study the effect of the nonlinear term \( v^p \), we will consider the case in which \( \gamma \varepsilon^{\frac{p-2}{2}} \) converges to a certain positive number as \( \varepsilon \rightarrow 0 \). Up to a convenient change of variable, we can consider the problem:

\[
\begin{align*}
-\varepsilon^2 \Delta v + v + \psi(x)v &= v^p, \quad x \in \mathbb{R}^3 \\
-\Delta \psi &= \frac{1}{\varepsilon} v^2(x), \quad x \in \mathbb{R}^3
\end{align*}
\]

As far as we know, the only work regarding semiclassical states of (1.4) is Ref. 12. In that paper D’Aprile proves the concentration of solutions around a sphere for \( 1 < p < 3/2 \). The techniques used in the proof are quite involved and make use of a constrained minimization procedure.

Our results improve those of Ref. 12 in several aspects. To start with, the range of \( p \) for which the concentration result holds is enlarged to \( 1 < p < 11/7 \). Moreover, we will also show how these functions look like, i.e. the limit profile of the rescaled functions \( u(x) = v(\varepsilon x) \). Furthermore, we will show that our result is optimal in the following sense: if there exists a sequence of solutions that concentrate around a sphere (in a sense to be defined, see Theorem 3.1), then \( 1 < p < 11/7 \). Also, the radius of convergence and the asymptotic profile of the functions are those we prove in the existence result. In other words, the only concentration phenomena that may occur on (1.4) are those we have found.

The necessary conditions for concentration are exposed in Theorem 3.1. The main hypothesis is the uniform \( L^\infty \) bound on the sequence of functions (which is quite a natural assumption, as we explained above).
The existence result is stated in Theorem 4.1. The proof is very different from that of Ref. 12, and follows the ideas of Ref. 5. The main tool of the proof is a perturbative technique which was first introduced in Refs. 2–4.

Basically, we define a manifold \( Z \) of “approximate solutions”, and try to find a solution of (1.4) close to it. Such manifold will be defined taking into account the necessary conditions we have found previously.

In order to find such a solution, a Lyapunov–Schmidt procedure is used. The infinite-dimensional equation is solved near any \( z \in Z \), as in Ref. 5, thanks to the Banach Contraction Theorem. Afterwards, the finite dimensional part is solved by evaluating the functional on the previous solutions and finding the critical points.

We point out that our problem turns out to be qualitatively different from Ref. 5. In that paper the authors work on the problem (1.1) with some potential \( V \) fixed, while our potential \( \phi \) is not fixed but depends on \( v \). In particular we shall see (see Theorem 4.1) that the solutions we find are a local minima of the energy functional. This is a remarkable difference with respect to Ref. 5, and could have important physical consequences, for instance, the orbital stability of the evolution problem (see Refs. 21 and 24 for more information about this question).

Once we have studied the concentration around spheres for (1.4), one could ask about the simpler problem of single spikes. Clearly, once we obtain a spike at some point, we have spikes at any other point just by translation. For the sake of completeness, we also prove the existence of such spikes by using the Implicit Function Theorem.

The paper is organized as follows. Section 2 is devoted to some notations and to the variational setting of the problem. Moreover, the existence of single spikes is proved. In Sec. 3 we state and prove the necessary conditions for concentration on spheres. Finally, in Sec. 4 we prove that actually the necessary conditions are also sufficient.

2. Preliminaries. Single Spikes

Let us fix some notations. We will write \( H^1(\mathbb{R}^3) \), \( D^{1,2}(\mathbb{R}^3) \) as the usual Sobolev spaces, and \( H^1_r \), \( D^{1,2}_r \) the corresponding subspaces of radial functions. The norm of \( H^1(\mathbb{R}^3) \) will be simply denoted by \( \| \cdot \| \), while \( \| \cdot \|_D \) is the norm of the space \( D^{1,2}(\mathbb{R}^3) \). Other norms used in the paper will be clear from the notation.

When working on estimates, we will write \( C \) to denote a positive constant, independent of \( \varepsilon \), which may change from expression to expression. Moreover, we will use the symbol \( \sim \) to denote two expressions of the same order (as \( \varepsilon \to 0 \)), i.e. \( a(\varepsilon) \sim b(\varepsilon) \) if and only if

\[
0 < \liminf_{\varepsilon \to 0} \frac{a(\varepsilon)}{b(\varepsilon)} \leq \limsup_{\varepsilon \to 0} \frac{a(\varepsilon)}{b(\varepsilon)} < +\infty.
\]

For the sake of clarity, we will omit the coefficient \( \omega_2 = 4\pi \) when we write integrals in polar coordinates.
Throughout the paper we will be interested in weak positive solutions \((v, \psi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) for the problem:

\[
\begin{cases}
-\varepsilon^2 \Delta v + v + \psi(x)v = v^p, & x \in \mathbb{R}^3 \\
- \Delta \psi = \frac{1}{\varepsilon} v^2, & x \in \mathbb{R}^3
\end{cases}
\]  

(2.1)

with \(p \in (1, 5)\). By making the change of variable \(u(x) = v(\varepsilon x)\), we are led with the problem:

\[
\begin{cases}
- \Delta u + u + \phi(x)u = u^p, & x \in \mathbb{R}^3 \\
- \Delta \phi = \varepsilon u^2, & x \in \mathbb{R}^3
\end{cases}
\]  

(2.2)

Clearly, we can write an integral expression for \(\phi\) in the form:

\[
\phi(x) = \varepsilon \int_{\mathbb{R}^3} u^2(y) G(x, y) \, dy,
\]

where \(G(x, y)\) is the Green function of the Laplacian in \(\mathbb{R}^3\). If \(u\) is a radial function, then \(\phi\) is also radial and has the expression:

\[
\phi(r) = \frac{\varepsilon}{r} \int_0^{+\infty} u^2(s) s \min\{r, s\} \, ds.
\]

First of all, let us study the variational setting of the problem. The linear operator \(T : D^{1,2}(\mathbb{R}^3) \rightarrow \mathbb{R}\) defined as

\[
T(\nu) = \int_{\mathbb{R}^3} u^2 \nu \leq \|u^2\|_{L^{5/4}} \|\nu\|_{L^6} \leq C\|u\|^2 \|\nu\|_D
\]

is continuous. Then, by the Lax–Milgram theorem there exists \(\phi \in D^{1,2}(\mathbb{R}^3)\) such that for any \(\nu \in D^{1,2}(\mathbb{R}^3)\),

\[
\int_{\mathbb{R}^3} \nabla \phi \nabla \nu = \varepsilon \int_{\mathbb{R}^3} u^2 \nu.
\]

Moreover, \(\|\phi\|_D \leq C\varepsilon\|u\|^2\). Hence, we obtain that

\[
\int_{\mathbb{R}^3} \phi u^2 \leq \varepsilon \|u\|_{L^{5/4}}^2 \|\phi\|_{L^6} \leq C\varepsilon\|u\|^4.
\]

In particular, the functional

\[
I_\varepsilon : H^1 \rightarrow \mathbb{R}
\]

\[
I_\varepsilon(u) = \int_{\mathbb{R}^3} \left( \frac{1}{2} (|\nabla u(x)|^2 + u(x)^2) - \frac{1}{p+1} |u(x)|^{p+1} + \frac{1}{4} \phi(x) u^2(x) \right) \, dy \, dx
\]

is well defined. Furthermore, \(I_\varepsilon\) is a \(C^2\) functional, with derivatives given by:

\[
I'_\varepsilon(u)[v] = \int_{\mathbb{R}^3} \nabla u \nabla v + uv - |u|^{p-1} uv + \phi uv,
\]

\[
I''_\varepsilon(u, v, w) = \int_{\mathbb{R}^3} \nabla v \nabla w + vw - p |u|^{p-1} vw + \phi vw + 2\varepsilon \phi_{1uv}.
\]
where \(-\Delta \phi = \varepsilon u^2, \quad -\Delta \phi_1 = \varepsilon uw\) (for more details, see Ref. 12). In particular, the critical points of \(I_\varepsilon\) correspond to the solutions of (2.2). The above expressions are well defined since:

\[
\int_{\mathbb{R}^3} \phi uv \leq C\|u\|^3\|v\|,
\]

\[
\int_{\mathbb{R}^3} \phi vw + 2\phi_1 uv \leq C\|u\|^2\|v\||w|.
\]

At this point, let us just say a few words about the regularity of solutions. Due to the definition, \(\phi \in D^{1,2} \subset L^6(\mathbb{R}^3)\). Thanks to Ref. 1 and Sobolev inclusions, \(u\) is a \(C^{0,\alpha}_{\text{loc}}\) function. Also, if \(u\) is radial, we have that \(u \in L^\infty(\mathbb{R}^3)\) (see Ref. 26). From Schauder estimates it follows that \(\phi \in C^{2,\alpha}_{\text{loc}}\) (again, if \(\phi\) is radial, \(\phi \in L^\infty(\mathbb{R}^3)\), see Ref. 10). We easily conclude that \(u\) is also in \(C^{2,\alpha}_{\text{loc}}\). So, in the end, the weak solutions of (2.1) will be solutions in the classical sense.

Now we study the existence of single spikes for (2.1). Since our problem is translation invariant, once we have a spike around a certain point, we also have the existence of a spike elsewhere. Therefore, let us focus on proving the existence of spikes around zero; to do so, we can restrict ourselves to the radial case.

**Proposition 2.1.** Let \(p \in (1,5)\) and \(\hat{U}\) be the unique positive radial solution in \(\mathbb{R}^3\) for the problem (2.2) (see Ref. 22):

\[-\Delta u + u = u^p.\]

Then, for \(\varepsilon\) small enough there exists a solution of the problem (2.2) such that \(u_\varepsilon \rightarrow \hat{U}\) in \(H^1\) as \(\varepsilon \rightarrow 0\).

**Proof.** Define \(\mathcal{L} : \mathbb{R} \times H^1 \rightarrow H^1, \quad \mathcal{L}(\varepsilon, u) = I'_\varepsilon(u)\). We will find the existence of solutions for the equation \(\mathcal{L}(\varepsilon, u) = 0\) through the Implicit Function Theorem. Actually, \(L(0, u) = 0\) has a unique radial positive solution \(\hat{U}\). Moreover,

\[
\frac{\partial}{\partial u}\mathcal{L}(0, \hat{U})|_{u=v} = I''_0(\hat{U})[v]
\]

which is clearly a Fredholm operator. Moreover, \(I''_0(\hat{U})\) is nondegenerate in \(H^1_\varepsilon\); take into account that the kernel of \(I''_0(\hat{U})\) in the space \(H^1(\mathbb{R}^3)\) is spanned by the functions \(\frac{\partial}{\partial \varepsilon}\hat{U}\), which are orthogonal to the space \(H^1_\varepsilon\) because of symmetry reasons.

Then, for \(\varepsilon\) small, there is a continuous map \(\varepsilon \mapsto u_\varepsilon\) such that \(\mathcal{L}(\varepsilon, u_\varepsilon) = 0\).

**3. Necessary Conditions**

We now focus on the main scope of the paper, i.e. to provide results of concentration of radial solutions around a sphere. In this section we shall obtain necessary conditions. Specifically, we will show that if there are solutions \(u_\varepsilon\) concentrating around a certain sphere (is a sense to be defined later on), then \(p < 11/7\). Moreover, the radius of the sphere is explicitly given.
First, let us introduce a family of functions which will be very important in the rest of the paper. For each \( a \geq 0 \) define \( U_a \) as the only positive even solution in \( \mathbb{R} \), decaying at zero, of the ODE:

\[
-U'' + U + aU = U^p_a.
\]

A simple computation gives that \( U_a \) can be written in the form:

\[
U_a(x) = \alpha U_0(\beta x), \quad \text{where } \alpha = (a + 1)^{1/(p-1)}, \quad \beta = (a + 1)^{1/2}.
\]  

(3.1)

Moreover, \( U_0 \) has the explicit expression:

\[
U_0(x) = \left( \frac{p + 1}{2} \right)^{-\frac{1}{p-1}} \frac{1}{\cosh \left( \frac{p - 1}{2} x \right)}.
\]

Thanks to Pohozaev inequality, the functions \( U_a \) can be shown to satisfy:

\[
(a + 1) \int_{\mathbb{R}} U_a^2 = \left( \frac{1}{2} + \frac{1}{p+1} \right) \int_{\mathbb{R}} U_{a+1}^{p+1},
\]

(3.2)

\[
\int_{\mathbb{R}} (U'_a)^2 = \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}} U_{a+1}^{p+1}.
\]

(3.3)

We now state and prove the main result of the section.

**Theorem 3.1.** Suppose that there exists \( \varepsilon_n \to 0 \) and a sequence of positive functions \( v_n \in H^1_{r_n} \) uniformly bounded in \( L^\infty \) so that

\[
\begin{cases}
-\varepsilon_n^2 \Delta v_n + v_n + \psi(x)v_n = v_n^p, & x \in \mathbb{R}^3 \\
-\Delta \psi = \frac{1}{\varepsilon_n} v_n^2, & x \in \mathbb{R}^3
\end{cases}
\]

(3.4)

for \( p \in (1, 5) \). Moreover, suppose that they concentrate around a sphere of radius \( \bar{r} \), in the sense that

\[
\forall \delta > 0, \ \exists \varepsilon_0 > 0, \ R > 0 : |v_n(x)| < \delta \ \forall \varepsilon_n < \varepsilon_0, \ |x - \bar{r}| > R \varepsilon_n.
\]

(3.5)

Define \( \xi_n > 0 \) so that \( v_n(\bar{r}) \) attains its maximum at \( \xi_n \) (because of the above condition, \( \xi_n \to \bar{r} \), and \( r_0 = \xi_n \)). Finally, define \( u_n(x) = v_n(\varepsilon_n) \).

Then,

\[
\begin{align*}
&u_n(\bar{r}) - U_a(\bar{r} - \rho_n) \to 0 \quad \text{in } H^1_{\bar{r}} \\
u_n(\bar{r} + \rho_n) - U_{\bar{a}}(\bar{r}) \to 0 \quad \text{in } C^2_{\text{loc}}(\mathbb{R})
\end{align*}
\]

where \( \bar{a} = \frac{8(p-1)}{11 - 7p} \) and \( p < 11/7 \). Moreover, \( \bar{r} \) is given by:

\[
\bar{r} = \frac{1}{M_0 (\bar{a} + 1)^{\frac{1}{1 - p}}}.
\]

(3.6)

with \( M_0 = \int_{\mathbb{R}} U_0^2 \).
Proof. Clearly, the functions $u_n$ verify the equation:

$$
\begin{align*}
-\Delta u_n + u_n + \phi_n(x)u_n &= u'_n, \quad x \in \mathbb{R}^3 \\
-\Delta \phi_n &= \varepsilon_n u_n^2(x), \quad x \in \mathbb{R}^3.
\end{align*}
$$

(3.7)

We divide the proof into four steps:

**Step 1.** There exists $C, \lambda > 0$ so that $u_n(r) < Ce^{-\lambda|r-\rho_n|}, |u'_n(r)| < Ce^{-\lambda|r-\rho_n|}$ for any $|r-\rho_n| > R$ and $n \in \mathbb{N}$.

Take $\delta > 0$ small enough, but fixed, and choose $\varepsilon_0$ and $R$ satisfying (3.5). So, for $r > \rho_n + R$, we can argue as in Ref. 26 (see also Ref. 10) to obtain that there exists $C > 0, \lambda > 0$ such that $u_n(r) \leq Ce^{-\lambda(r-\rho_n)}$. Observe that the ODE analysis used in Ref. 26 to prove this estimate implies that $C$ is independent of $n$.

For $r < \rho_n - R$, we just need to compare $u_n$ with the function $w_n$ defined as:

$$
-\Delta w_n + \frac{1}{2}w_n = 0, \quad x \in B(0, \rho_n - R),
$$

$$
w_n(x) = u_n(x), \quad x \in \partial B(0, \rho_n - R).
$$

(3.8)

We can choose $\delta$ small enough so that $t - t' \geq t/2$ for any $t \in (0, \delta)$. Then, the maximum principle implies that $u_n \leq w_n$. Actually, for any $r < \rho_n - R$, we have that $u_n + \phi_n u_n - u'_n > \frac{2}{\varepsilon_0}$, so it suffices now to compare (3.7) and (3.8).

From the properties of Bessel functions (see Ref. 19), we have that $w_n(x) \leq Ce^{-\lambda(r-\rho_n)}$ for certain $C, \lambda$ positive.

Before dealing with the derivative, we estimate the potentials $\phi_n$:

$$
\phi_n(r) = \frac{\varepsilon_n}{r} \int_0^{+\infty} u_n^2(s) \min\{r, s\} \, ds
$$

$$
\leq \varepsilon_n \left( \int_{\rho_n+R}^{\rho_n-R} u_n^2(s) \, ds + C \int_{|s-\rho_n|>R} e^{-2\lambda|s-\rho_n|} \, ds \right)
$$

$$
\leq 2MR + C,
$$

where $M$ is a uniform $L^\infty$ bound on the sequence $u_n$.

Then, $\phi_n$ is uniformly bounded. We can now show the exponential decay of $u'_n$ by using a regularity result. For an equation $-\Delta u = h$ in a ball $B_2$ of radius 2, the following holds: $||u||_{C^1(B_1)} \leq C(||h||_{L^\infty(B_2)} + ||u||_{L^\infty(\partial B_2)})$ (see, for instance, Ref. 20). By applying this inequality and the exponential decay of $u_n$, we obtain that $u'_n(r) \leq Ce^{-\lambda r} |r-\rho_n|$ for all $|r-\rho_n| > R + 2$.

**Step 2.** There holds:

$$
|\phi_n(r) - \phi_n(t)| < C\varepsilon_n |r-t| \quad \text{for all} \quad r, t \in (\rho_n - \sqrt{\rho_n}, \rho_n + \sqrt{\rho_n}).
$$
We compute
\[
|\phi_n(r) - \phi_n(t)| = \left| \frac{\varepsilon_n}{r} \int_0^{+\infty} u_n^2(s) \min\{r, s\} \, ds - \frac{\varepsilon_n}{t} \int_0^{+\infty} u_n^2(s) \min\{t, s\} \, ds \right|
\leq \frac{\varepsilon_n}{r} \int_0^{+\infty} u_n^2(s) \min\{r, s\} \, ds - \min\{t, s\} \, ds
\]
\[
+ \frac{\varepsilon_n}{r} \frac{\varepsilon_n}{t} \int_0^{+\infty} u_n^2(s) \min\{t, s\} \, ds
\leq \frac{\varepsilon_n}{r} \int_0^{+\infty} u_n^2(s) |r - t| \, ds
\leq \frac{\varepsilon_n}{r} \rho_n |r - t| + C \frac{\varepsilon_n |r - t|}{\rho_n^2} \rho_n^2.
\]

In the above inequalities, the uniform $L^\infty$ bound of $u_n$ and their exponential decay have been used.

**Step 3.** The convergence of $u_n$.

Consider the sequence $z_n(r) = u_n(r + \rho_n)$ restricted to the interval $(-\sqrt{\rho_n}, \sqrt{\rho_n})$. Clearly, $z_n$ satisfies the equation:
\[
-z_n'' - 2 \frac{z_n'}{\rho_n + r} + (1 + \phi_n(\rho_n + r)) z_n = z_n^p.
\]

Observe that $z_n$ is uniformly bounded in $L^\infty$ and attains its maximum at zero. By evaluating Eq. (3.9) at $r = 0$, we deduce that $z_n(0) \leq z_n^p(0)$, which implies that $z_n(0) \geq c_0 > 0$. Moreover, the function
\[
\tilde{\phi}_n : (-\sqrt{\rho_n}, \sqrt{\rho_n}) \to \mathbb{R}
\]
\[
\tilde{\phi}_n(r) = \phi_n(\rho_n + r)
\]
is bounded in $L^\infty$ (see Step 1) and verifies that $|\tilde{\phi}_n(r) - \tilde{\phi}_n(t)| \leq C \varepsilon_n$ (see Step 2). Therefore, there exist $\bar{a} \geq 0$ and $z \in C^1(\mathbb{R})$ so that, up to a subsequence,
\[
\tilde{\phi}_n \to \bar{a} \text{ in } C_{\text{loc}}, \quad z_n \to z \text{ in } C^2_{\text{loc}}.
\]

Moreover, $z$ satisfies the equation $-z'' + (1 + \bar{a})z = z^p$ and attains its maximum (which is strictly positive) at zero. Hence, $z = U_{\bar{a}}$. Finally, because of the exponential decay of $u_n$, $u'_n$, we conclude that $u_n(r) - U_{\bar{a}}(r - \rho_n) \to 0$ in $H^1_r$. 
Step 4. Conclusion.

We now intend to find the concrete values of \( \bar{a} \) and \( \bar{r} \). First, we compute again \( \phi_n \) in value \( r \in (\rho_n - \sqrt{\rho_n}, \rho_n + \sqrt{\rho_n}) \) to obtain:

\[
\bar{a} \leftarrow \phi_n(r) \sim \frac{\varepsilon_n}{r} \int_{-\rho_n}^{+\infty} U_n^2(s - \rho_n)\min\{r, s\} \, ds
\]

\[
= \frac{\varepsilon_n}{r} \int_{-\rho_n}^{+\infty} U_n a^2(s + \rho_n)\min\{r, s + \rho_n\} \, ds
\]

\[
\sim \frac{\rho_n \varepsilon_n}{r} \int_{\mathbb{R}} U_n^2(s) \min\{r, s + \rho_n\} \, ds
\]

\[
\sim \frac{\rho_n}{r} \int_{\mathbb{R}} U_n^2(s) \, ds
\]

So, it holds

\[
\bar{a} = \bar{r} \int_{\mathbb{R}} U_n^2(s) \, ds = \bar{r}(\bar{a} + 1) \frac{\gamma - 2}{\gamma - 1} \int_{\mathbb{R}} U_n^2(s) \, ds ,
\]

(3.10)

where we have used formula (3.1). Therefore, the radius of concentration \( \bar{r} \) and the coefficient \( \bar{a} \) are linked via Eq. (3.6).

We now turn our attention to the functions \( u_n \), which are solutions of the problem:

\[
(-r^2 u'' + r^2 (1 + \phi)) u = r^2 u^p .
\]

In the next computations, we write simply by \( u, u', u'' \) instead of \( u(r), u'(r), u''(r) \), i.e. we understand that those functions are evaluated on the variable \( r \). If another variable is used, it will be written explicitly.

By multiplying by \( u'_n \) and integrating (taking into account, again, the exponential decay of \( u_n, u'_n \)), we obtain:

\[
\int_0^{+\infty} r^2 u'_n u'' + r_2 u_n u'_r - r^2 e_n u'_n + u_n u'_r \varepsilon \int_0^{+\infty} r s \min\{r, s\} u_n^2(s) \, ds \, dr = 0 .
\]

From \( u'u'' = [(u')^2]/2, uu' = (u^2)/2, u^p u' = (u^{p+1})'/p + 1 \), and integrating by parts, we obtain:

\[
0 = \int_0^{+\infty} -r(u'_n)^2 - ru_n^2 + \frac{2}{p + 1} ru_{n+1}^p + \varepsilon \left( \frac{u_n^2}{2} \int_0^{+\infty} s^2 u_n^2(s) \, ds + ru_n^2 \int_r^{+\infty} s u_n^2(s) \, ds \right) dr
\]

\[
\sim \rho_n \int_{\mathbb{R}} -(U_{\bar{a}'}^2 - \bar{U}_{\bar{a}}^2 + \frac{2}{p + 1} \bar{U}_{\bar{a}}^p + 1 .
\]
Taking into account Eq. (3.10) and using equalities (3.2), (3.3), we conclude:

\[
\int_{\mathbb{R}} U_a^2 \left[ \frac{\bar{a}}{4} + \left( \frac{2}{p+1} - 1 \right) \frac{\bar{a} + 1}{2 + \frac{1}{p+1}} \right] = 0
\]

which is verified if and only if \( \bar{a} = \frac{8(p-1)}{11 - 7p} \). Since \( \bar{a} \geq 0 \), this implies that \( p \) must be necessary smaller than \( \frac{11}{7} \).

**Remark 3.1.** One could try to remove the hypothesis of the uniform \( L^\infty \) boundedness of the sequence \( v_n \) in Theorem 3.1, and try to prove \emph{a priori} bounds. Observe that we cannot argue as in Ref. 25 since \( \phi_n \) may also go to infinity. Formal computations make us suspect that such \emph{a priori} bounds hold at least for \( p \neq \frac{7}{3} \).

### 4. The Concentration Result

In Sec. 3 we found necessary conditions for the concentration phenomena on spheres. The following theorem basically asserts that those conditions are indeed sufficient.

**Theorem 4.1.** For any \( p \in (1, 11/7) \) there exists solutions \( v_\varepsilon \) of (2.1) concentrating (as \( \varepsilon \to 0 \)) around a sphere of radius \( \bar{r} \), where

\[
\bar{r} = \frac{1}{M_0} \frac{\bar{a}}{(\bar{a} + 1) \frac{8 - p}{3(p-1)}}
\]

(4.1)

with \( M_0 = \int_{\mathbb{R}} U_0^2 \) and \( \bar{a} = \frac{8(p-1)}{11 - 7p} \). Moreover, for \( \varepsilon \) small enough, the functions \( u_\varepsilon(x) = v_\varepsilon(\varepsilon x) \) are local minima of \( I_\varepsilon \) and their energy tend to \( -\infty \) as \( \varepsilon \to 0 \).

Throughout this section we will prove Theorem 4.1. Roughly speaking, our approach is as follows. First we consider a manifold \( Z \) of approximate solutions in the form \( U_a(\cdot - \frac{z}{\varepsilon}) \), where \( r \) is variable and \( a \) is chosen so that (4.1) holds with \( \bar{r} \), \( \bar{a} \) replaced with \( r, a \). The choice of condition (4.1) is made in view of Theorem 3.1. Afterwards, we use a Lyapunov–Schmidt procedure to decompose our problem in an infinite dimensional problem, which will be solved near any \( z \in Z \), and a one-dimensional equation, which will be verified only if \( a = \frac{8(p-1)}{11 - 7p} \).

So, we are interested in writing \( a \) as a function of the variable \( r \). Taking into account (4.1) and the preceding remarks, we are then interested in the inverse of the function

\[
g(a) = \frac{1}{M_0} \frac{a}{(1 + a)^{\frac{8 - p}{3(p-1)}}}
\]
However, it is easy to check that \( g \) is increasing from zero to \( a_0 = \frac{2(p-1)}{1-3p} \) and is decreasing from \( a_0 \) onwards, tending to 0 at infinity. Then, there are two possible inverses of \( g \) to be chosen. Taking into account that \( a = \frac{8(p-1)}{11-7p} > a_0 \) for any \( p \in (1, 11/7) \), we will consider the inverse of \( g \) between \( [a_0, +\infty) \), which will be denoted by \( f \).

Clearly, \( f : (0, g(a_0)] \to [a_0, +\infty) \) is decreasing and is \( C^1 \) in \( (0, g(a_0)] \).

Let us fix two numbers \( r_1 \) and \( r_2 \) so that

\[
0 < r_1 < \bar{r} = g(\bar{a}) < r_2 < g(a_0).
\]

Define the family of “approximate solutions”:

\[
z_{\varepsilon, \rho}(r) = \xi_{\varepsilon}(r)U_\rho(r - \rho),
\]

where \( \rho = f(\rho \varepsilon) \) and \( \rho \) is any number in the interval \( T_\varepsilon = \left( r_1/\varepsilon, r_2/\varepsilon \right) \). Here \( \xi_{\varepsilon} = \hat{\xi}(\varepsilon \rho) \) is a \( C^\infty \) function defined as:

\[
\hat{\xi}(r) = \begin{cases} 
0 & \text{if } r \leq \frac{r_1}{4}, \\
1 & \text{if } r \geq \frac{r_1}{2}.
\end{cases}
\]

We are interested in finding critical points for this functional near the following manifold:

\[
Z_\varepsilon = \{z_{\varepsilon, \rho}, \rho \in T_\varepsilon\}.
\]

For the sake of clarity, we sometimes write \( z \) instead of \( z_{\varepsilon, \rho} \) and \( Z \) instead of \( Z_\varepsilon \). For any \( z \in Z \), consider \( T_zZ \) the tangent space, and \( W = W_z = T_zZ^\perp \). Let \( P \), \( Q \) denote the orthogonal projections onto the spaces \( W \) and \( T_zZ \), respectively. We then decompose the equation:

\[
I'_\varepsilon(z + w) = 0 \iff \begin{cases} 
PI'(z + w) = 0 \quad \text{(auxiliary equation)}, \\
QI'(z + w) = 0 \quad \text{(bifurcation equation)}.
\end{cases}
\]

Here \( z \in Z \) and \( w \in C_\varepsilon \), which is a convex set defined as:

\[
C_\varepsilon = \{w \in H^1_T : \|w\| \leq C_1, \ |w(x)| \leq C_2 \varepsilon \ a.e. \ in \mathbb{R}^3 \}.
\]

In the above definition \( C_1 \) and \( C_2 \) are fixed positive constants to be defined later.

**Remark 4.1.** The preceding definitions are made in the spirit of Ref. 5. However, our definition of the convex set \( C_\varepsilon \) is different, and makes later proofs technically much easier than those of Ref. 5.

We first focus on solving the auxiliary equation on \( w \) for any \( z \in Z \) fixed. To do so, some preliminary estimates are in order:
Lemma 4.1. The following estimate hold:

\begin{align*}
\text{(E0)} & \quad \|z\| \sim \varepsilon^{-1}, \\
\text{(E1)} & \quad \|I'_{\varepsilon}(z)\| < C, \\
\text{(E2)} & \quad \|I''_{\varepsilon}(z + w)\| \leq C, \\
\text{(E3)} & \quad \|I'_{\varepsilon}(z + w)\| \leq C, \\
\text{(E4)} & \quad \|I''_{\varepsilon}(z + w) - I''_{\varepsilon}(z)\| \leq C\varepsilon^{1/(p-1)}
\end{align*}

for any \( z = z_{\varepsilon,\rho} \in Z_{\varepsilon}, w \in C_{\varepsilon} \).

\textbf{Proof.} (E0) \( \|z\| \sim \varepsilon^{-1}. \)

By using the exponential decay of the functions \( U_a, U'_a \), it follows:
\[
\|z\|^2 = \int_0^{+\infty} r^2(z^2 + z'^2) \, dr \\
\sim \int_{-\rho}^{+\infty} (s + \rho)^2(U'_a(s) + U'_a(s)^2) \, ds \sim \rho^2.
\]

(E1) \( \|I'_{\varepsilon}(z_{\varepsilon,\rho})\| < C. \)

For all \( v \in H^1_r \) we have:
\[
I'_{\varepsilon}(z)[v] = \int_0^{+\infty} r^2 (z'v' + zv - z^p v) \, dr \\
+ \varepsilon \int_0^{+\infty} \int_0^{+\infty} v(r)rs \min\{r, s\} z(r)z(s)^2 \, ds \, dr.
\]

The marked part becomes:
\[
\left| \int_0^{+\infty} r^2 (z'v') \, dr \right| = \left| - \int_0^{+\infty} v(r)(2rz' + r^2z'') \, dr \right| \\
\leq 2 \left( \int_0^{+\infty} r^2v^2 \, dr \right)^{1/2} \left( \int_0^{+\infty} z'^2 \, dr \right)^{1/2} \\
+ \left| \int_0^{+\infty} v(r)r^2z'' \, dr \right| \\
\leq C\|v\| + \int_0^{+\infty} r^2v(r)(\xi''(r)U_a(r - \rho) \\
+ 2\xi'(r)U'_a(r - \rho) + \xi(r)U''_a(r - \rho)v^2) \, dr.
\]

Since \( \xi', \xi'' \) are zero for \( r \geq r_1/(2\varepsilon) \), the first two terms of the sum above can be controlled as on p. 438 of Ref. 5. Then, we have:
A change of variable gives:

\[ I'_c(v) \sim \int_{0}^{+\infty} r^2 \xi(r) v(r)(-a + 1)U_a(r) + U_a^p(r - \rho) \]

\[ + r^2 \xi(r) v(r)U_a(r - \rho) - r^2 \xi^p(r) U_a^p(r - \rho) v(r) \, dr \]

\[ + \varepsilon \int_{0}^{+\infty} \int_{0}^{+\infty} v(r)rs \min\{r, s\} z(r)z(s)^2 \, ds \, dr \]

\[ = \int_{0}^{+\infty} r^2 \xi(r) v(r)((1 - \xi(r)^{p-1})U_a(r - \rho) - aU_a(r - \rho)) \, dr \]

\[ + \varepsilon \int_{0}^{+\infty} \int_{0}^{+\infty} v(r)rs \min\{r, s\} z(r)z(s)^2 \, ds \, dr . \]

The marked part can be proved to tend to zero taking into account that \( 1 - \xi^{p-1} \) vanishes when \( U_a(r - \rho) \) takes its more significant values. In what follows, we denote by \( M_a = \int_{\mathbb{R}} U_a^2 \). As was shown in the previous section, \( M_a \) has the expression \( M_a = (a + 1)^{\frac{1}{p-1}} M_0 \) [it suffices to use the expression (3.1)]. Observe that both \( a \) and \( M_a \) are bounded and not close to zero for any \( \varepsilon > 0, \rho \in T \). Then, we can write:

\[ I'_c(v) \sim \int_{0}^{+\infty} \int_{0}^{+\infty} -ar^2 v(r)\xi(r)U_a(r - \rho)U_a(s - \rho)^2 M_a^{-1} \]

\[ + \varepsilon v(r)rs \min\{r, s\} \xi(r)U_a(r - \rho)\xi(s)^2 U_a^2(s - \rho) \, ds \, dr \]

\[ = - \int_{0}^{+\infty} \int_{0}^{+\infty} v(r)r\xi(r)U_a(r - \rho)U_a^2(s - \rho)M_a^{-1} \]

\[ \cdot \min\{r, s\} \xi^2(s) \] .

A change of variable gives:

\[ I'_c(v) \sim \int_{\mathbb{R}} \int_{\mathbb{R}} (r + \rho)(r + \rho)\xi(r + \rho)U_a(r)U_a^2(s)M_a^{-1} \]

\[ \cdot (a(r + \rho) + M_a\varepsilon(s + \rho)\min\{r, s\} + \rho) \xi^2(s + \rho) \, dr \, ds . \]

As before, the term \( \xi^2(s + \rho) \) can be replaced with 1. We claim that the marked part is bounded by a polynomial on \( r, s \) independently of \( \varepsilon \). Actually, we have that:

\[ a(r + \rho) + M_a\varepsilon(s + \rho)\min\{r, s\} + \rho \]

\[ = [a - M_a\varepsilon\rho] + a - M_a\varepsilon(s\min\{r, s\}) - C\varepsilon\min\{r, s\} . \]

Recall that we have chosen \( a = f(\varepsilon\rho) \Leftrightarrow g(a) = \varepsilon\rho \). Using the explicit definition of \( g \), we have:

\[ a = \varepsilon\rho M_0(1 + a)^{\frac{1}{p-1}} = \varepsilon\rho M_a . \] (4.2)
Then, the claim follows for certain polynomial \( P(r, s) \). So, we finally conclude:

\[
|I_e''(z)[v]| \leq \int_R \int_R |v(r + \rho)(r + \rho)\xi(r + \rho)U_a(r)U_a^2(s)M_a^{-1}P(r, s)| \, dr \, ds
\]

\[
\leq \left( \int_R (r + \rho)^2 v(r + \rho)^2 \, dr \right)^{1/2} \left( \int_R \int_R |U_a(r)U_a^2(s)M_a^{-1}P(r, s)|^2 \, dr \, ds \right)^{1/2}
\]

\[
\leq C\|v\|.
\]

(E2) \( \|I_e''(z + w)\| \leq C \) for any \( z \in Z_\varepsilon, w \in C_\varepsilon \).

First of all, take into account that if \( w \in C_\varepsilon \), then

\[
\int_R w^2(r)r^k \, dr < C \quad \forall k \in [0, 2].
\]  

For the sake of brevity, we write \( \tilde{z} = z + w \). Recall the expression of the second derivative of \( I_e \) given in Sec. 2, to get:

\[
|I_e''(\tilde{z})[v, v]| = \|v\|^2 + p \int_0^{+\infty} r^2|\tilde{z}(r)|^{p-1}v(r)^2 \, dr
\]

\[
+ 2\varepsilon \int_0^{+\infty} \int_0^{+\infty} rs\tilde{z}(r)\tilde{z}(s)v(r)v(s) \min\{r, s\} \, dr \, ds
\]

\[
+ \varepsilon \int_0^{+\infty} \int_0^{+\infty} rs\tilde{z}^2(r)v^2(s) \min\{r, s\} \, dr \, ds.
\]

We estimate each integral expression separately, taking into account that \( \tilde{z} \) is uniformly bounded in \( L^\infty \) for any \( w \):

\[
\int_0^{+\infty} r^2|\tilde{z}(r)|^{p-1}v(r)^2 \, dr
\]

\[
\leq C\|v\|^2;
\]

\[
2\varepsilon \int_0^{+\infty} \int_0^{+\infty} rs\tilde{z}(r)\tilde{z}(s)v(r)v(s) \min\{r, s\} \, dr \, ds
\]

\[
\leq 2\varepsilon \left( \int_0^{+\infty} \int_0^{+\infty} r^2s^2v^2(r)v^2(s) \, dr \, ds \right)^{1/2}
\]

\[
\cdot \left( \int_0^{+\infty} \int_0^{+\infty} \tilde{z}^2(r)\tilde{z}^2(s)s^2 \, dr \, ds \right)^{1/2}
\]

\[
\leq C\|v\|^2 \cdot \left( \int_0^{+\infty} \tilde{z}^2(r) \, dr \right)^{1/2} \left( \int_0^{+\infty} \tilde{z}^2(s)s^2 \, ds \right)^{1/2}
\]

\[
\leq C\|v\|^2\varepsilon\|\tilde{z}\|_{L^2}.
\]
The above inequalities have been possible since
\[ \int_0^{+\infty} \tilde{z}^2(r) \, dr \leq 2 \int_0^{+\infty} \tilde{z}^2(r) \, dr + 2 \int_0^{+\infty} w^2(r) \, dr \leq C. \]
Thus, we are done since \( \|\tilde{z}\|_{L^2} \leq \|z\| + \|w\| \sim \varepsilon^{-1}. \)
We now estimate the last term:
\[ \varepsilon \int_0^{+\infty} \int_0^{+\infty} rs\tilde{z}^2(r)v^2(s) \min\{r, s\} \, dr \, ds \]
\[ \leq \varepsilon \int_0^{+\infty} s^2 v^2(s) \, ds \int_0^{+\infty} r\tilde{z}^2(r) \, dr \]
\[ \leq \|v\|^2 \varepsilon \int_0^{+\infty} r\tilde{z}^2(r) + rw^2(r) \, dr \]
\[ \sim \|v\|^2 \varepsilon \int (r + \rho)U_0^2(r) \, dr \leq C\|v\|^2. \]
(E3) \( \|I'_\varepsilon(z + w)\| \leq C \forall z \in Z, w \in C_z. \)
This inequality follows obviously from (E1) and (E2).

(E4) \( \|I''_\varepsilon(z + w) - I''_\varepsilon(z)\| \leq C\varepsilon^{1/(p-1)}. \)
As before, we write \( \tilde{z} = z + w. \) Then,
\[ |I''_\varepsilon(\tilde{z})[v, v] - I''_\varepsilon(z)[v, v]| \]
\[ = \left| \int_0^{+\infty} r^2(p|\tilde{z}|^{p-1} - pz^{p-1})v^2(r) \, dr \right| \]
\[ + 2\varepsilon \int_0^{+\infty} \int_0^{+\infty} rs(v(r)s) \min\{r, s\}(\tilde{z}(r)\tilde{z}(s) - z(r)z(s)) \, dr \, ds \]
\[ + \varepsilon \int_0^{+\infty} \int_0^{+\infty} rsv^2(s) \min\{r, s\}(\tilde{z}(r) - z^2(r)) \, dr \, ds \]
\[ \leq C \int_0^{+\infty} (|w| + |w|^{p-1})r^2v^2 \, dr \]
\[ + 2\varepsilon \|v\|^2 \left( \int_0^{+\infty} \int_0^{+\infty} \min\{r, s\}^2(\tilde{z}(r)\tilde{z}(s) - z(r)z(s))^2 \, dr \, ds \right)^{1/2} \]
\[ + \varepsilon \|v\|^2 \left( \int_0^{+\infty} r^2(\tilde{z}^2(r) - z^2(r))^2 \, dr \right)^{1/2}. \]
From the definition of \( C_\varepsilon, \) we have that
\[ \int_0^{+\infty} (|w| + |w|^{p-1})r^2v^2 \, dr \leq C(\varepsilon + \varepsilon^{p-1})\|v\|^2. \]
We now focus on the estimate of the term:

\[
2\varepsilon \left( \int_0^{+\infty} \int_0^{+\infty} \min\{r, s\}^2(w(r)z(s) + w(s)z(r) + w(r)w(s))^2 \, dr \, ds \right)^{1/2}
\]

\[
\leq C\varepsilon \left( \int_0^{+\infty} \int_0^{+\infty} \min\{r, s\}^2(w^2(r)z^2(s) + w^2(s)z^2(r) + w^2(r)w^2(s)) \, dr \, ds \right)^{1/2}
\]

\[
\leq C\varepsilon \left( \int_0^{+\infty} z^2(s) \, ds + \int_0^{+\infty} r^2 w^2(r) \, dr + \int_0^{+\infty} z^2(r) \, dr \int_0^{+\infty} s^2 w^2(s) \, ds 
\right)
\]

\[
+ \int_0^{+\infty} w^2(r) \, dr \int_0^{+\infty} s^2 w^2(s) \, ds \right)^{2} \leq C\varepsilon.
\]

In the same way we can estimate the third summand:

\[
\varepsilon \left( \int_0^{+\infty} r^2(2z(r)w(r) + w^2(r))^2 \, dr \right)^{1/2}
\]

\[
\leq C\varepsilon \left( \int_0^{+\infty} r^2 z^2 w^2 + r^2 w^4 \, dr \right)^{1/2}
\]

\[
\leq C\varepsilon \left( \|z\|_{L^\infty}^2 \int_0^{+\infty} r^2 w^2 \, dr + \|w\|_{L^\infty}^2 \int_0^{+\infty} r^2 w^2 \, dr \right)^{1/2} \leq C\varepsilon.
\]

The following result will be used quite frequently in the rest of the paper. Observe the analogy with the necessary results given in Theorem 3.1, Step 4.

**Lemma 4.2.** Define \( \phi_\varepsilon \) the solution of the problem \(-\Delta \phi_\varepsilon = z^2_{r, \rho} \) for any \( \rho = \rho(\varepsilon) \in T_\varepsilon \). Take \( \gamma = \gamma(\varepsilon) > 0 \) a function, possibly diverging at zero, but satisfying that \( \gamma(\varepsilon)\varepsilon \to 0 \) (with \( \varepsilon \to 0 \)). Then, \( \phi_\varepsilon \to f(\varepsilon \rho) = a \) uniformly in \((\rho - \gamma, \rho + \gamma)\), i.e.

\[
\forall \delta > 0, \exists \varepsilon_0 > 0 : \varepsilon < \varepsilon_0 \longrightarrow |\phi_\varepsilon(r) - a| < \delta \quad \forall r \in (\rho - \gamma, \rho + \gamma).
\]

**Proof.** By using the equality (4.2), we can write:

\[
|\phi(r) - a| = \varepsilon \left| \frac{1}{r} \int_0^{+\infty} z^2(s - \rho)s \min\{r, s\} ds - \rho \int_\mathbb{R} U^2_a(s) \, ds \right|.
\]

Then, we just need to estimate the expression:

\[
\left| \frac{1}{r} \int_0^{+\infty} z^2(s) s \min\{r, s\} ds - \rho \int_\mathbb{R} U^2_a(s) \, ds \right|
\]

\[
\sim \left| \frac{1}{r} \int_{-\rho}^{+\infty} U^2_a(s)(s + \rho) \min\{r, s + \rho\} ds - \rho \int_\mathbb{R} U^2_a(s) \, ds \right|
\]

\[
\sim \int_\mathbb{R} U^2_a(s) \left| \frac{(s + \rho)}{r} \min\{r, s + \rho\} - \rho \right| ds.
\]
\[= \int_{s \geq r - \rho} U_a^2(s) |s| ds + \int_{s < r - \rho} U_a^2(s) \left( \frac{s + \rho}{r} - (s + \rho) - \rho \right) ds \]
\[\leq C + \int_{s < r - \rho} U_a^2(s) \left( \frac{\rho}{r} (\rho - r) + 2 \rho s + \frac{s^2}{r} \right) ds \leq C + C\gamma. \]

With previous estimates in hand, we are in the position to solve the auxiliary equation. First, we study the second derivative of \(I_\varepsilon\) on \(z\). To do so, we will make use of the function:
\[
\frac{\partial^2 z}{\partial \rho} = \xi(r) \left( \varepsilon f'(\varepsilon \rho) \frac{\partial U_a}{\partial \rho} (r - \rho) - \frac{\partial U_a}{\partial r} (r - \rho) \right).
\]

From now on, we write \(\dot{z} = \frac{\partial z}{\partial \rho}\), which clearly verifies \(\|\dot{z}\| \sim \varepsilon^{-1}\). Recall that \(W = (T_2 Z)^\perp = (\dot{z})^\perp\). Next two lemmas are devoted to the study of \(I''_\varepsilon(z)\) in \(W\).

**Lemma 4.3.** There exists \(\alpha > 0\) such that for small \(\varepsilon\):
\[I''_\varepsilon(z)[v, v] \geq \alpha \|v\|^2 \]
for all \(v \perp z\), \(v \perp \dot{z}\).

**Proof.** We recall that the second derivative of \(I_\varepsilon\) is:
\[I''_\varepsilon(z)[v, v] = \int_{\mathbb{R}^N} [\nabla v]^2 + v^2 - p_2 z^{-1} v^2 + \phi v^2 + 2 \phi_1 z v,
\]
where \(\phi\) and \(\phi_1\) are solutions of the problems:
\[-\Delta \phi = \varepsilon z^2, \quad -\Delta \phi_1 = \varepsilon z v.
\]

Observe that the last term is always positive, since:
\[\int_{\mathbb{R}^N} \phi_1 z v = \varepsilon^{-1} \int_{\mathbb{R}^N} \phi_1 (-\Delta \phi_1)
\[= \varepsilon^{-1} \int_{\mathbb{R}^N} |\nabla \phi_1|^2 \geq 0.
\]

Take \(\mu > 0\) large enough, but fixed; we study the function \(\phi(r)\), where \(r \in (\rho - \mu, \rho + \mu)\). As in the proof on Theorem 3.1 (Step 2), we obtain that \(|\phi(r) - \phi(t)| \leq C \varepsilon \mu\) for any \(r, t \in (\rho - 2\mu, \rho + 2\mu)\).

Thanks to previous estimates and Lemma 4.2, the rest of the proof can be developed by just following the arguments used to prove Lemma 4.1 of Ref. 5. We only need to substitute the function \(V(\varepsilon r)\) considered in Ref. 5 with \(1 + \phi(r)\). \( \square \)

**Lemma 4.4.** There exists \(\alpha > 0\) such that, for \(\varepsilon\) small, we have:
\[I''_\varepsilon(\dot{z})[z, z] \geq \alpha \|z\|^2.\]
Proof.

\[ I''_\varepsilon(z)[z, z] = \int_{\mathbb{R}^N} |\nabla z|^2 + z^2 - pz^{p+1} + 3\phi z^2 \]
\[ = \int_{\mathbb{R}^N} |\nabla z|^2 + z^2 - z^{p+1} + \phi z^2 \, dx + \int_{\mathbb{R}^N} 2\phi z^2 - (p-1)z^{p+1} \, dx \]
\[ = I'_\varepsilon(z)[z] + \int_{\mathbb{R}^N} 2\phi z^2 - (p-1)z^{p+1}. \]

Because of estimate (E1), \( I'_\varepsilon(z)[z] \sim \varepsilon^{-1} \). We now estimate the remaining term, taking into account Lemma 4.2 and the exponential decay of \( z \):

\[ \int_{\mathbb{R}^N} 2\phi z^2 - (p-1)z^{p+1} \sim 2ap^2 \int_{\mathbb{R}} U_a^2(r) \, dr - (p-1)p^2 \int_{\mathbb{R}} U_a^p+1(r) \, dr. \]

By using equality (3.2), we obtain:

\[ I'_\varepsilon(z)[z, z] \sim \rho^2 \int_{\mathbb{R}} U_a^{p+1}(r) \, dr \left( \left( \frac{1}{2} + \frac{1}{p+1} \right) 2a(a+1)^{-1} + 1 - p \right). \]

We now show that the expression between the brackets is strictly positive, which concludes the proof.

Since \( a = f(\varepsilon \rho) \), we only need to prove the inequality:

\[ 2 - \frac{f(\varepsilon \rho)}{1 + f(\varepsilon \rho)} \left( \frac{1}{2} + \frac{1}{p+1} \right) + 1 - p > 0. \]

This holds if and only if:

\[ \frac{f(\varepsilon \rho)}{1 + f(\varepsilon \rho)} > \frac{(p-1)(p+1)}{p+3}. \]

Remember now that the image of \( f \) is \((2/(p-1) \cdot \varepsilon \rho, +\infty)\). Because of the increasing character of the function \( t \mapsto \frac{1}{1+t} \), we have that

\[ \frac{f(\varepsilon \rho)}{1 + f(\varepsilon \rho)} \in \left( \frac{2(p-1)}{5-p}, 1 \right). \]

So, it suffices to show that

\[ \frac{(p-1)(p+1)}{p+3} < \frac{2(p-1)}{5-p}. \]

But this inequality holds since, for \( p > 1 \), it is equivalent to:

\[ (p+1)(5-p) < 2(p+3) \Leftrightarrow -p^2 + 2p - 1 < 0. \]

We denote \( A_\varepsilon = -P(I''_\varepsilon(z)|W)^{-1} \), (recall that \( P \) is the orthogonal projection onto \( W \)). Because of previous lemmas, \( \|A_\varepsilon\| \leq C \) for any \( \rho \in T_\varepsilon \).

In the next proposition we find a solution of the problem \( PI'_\varepsilon(z + w) = 0 \) near \( z \), for any \( z \in Z \).
Proposition 4.1. The constants $C_1$ and $C_2$ in the definition of $C_\varepsilon$ can be chosen such that for any $\varepsilon > 0$, there exists $w = w_{\varepsilon, \rho} \in W \cap C_\varepsilon$ such that $PI'_\varepsilon(z + w) = 0$.

Proof. We look for $w \in W$ verifying $PI'_\varepsilon(z + w) = 0$, i.e.

$$w = S_\varepsilon(w) := A_\varepsilon P(I'_\varepsilon(z + w) - I''_\varepsilon(z)[w]).$$ (4.4)

So, we look for such a $w$ as a fixed point of $S_\varepsilon$ in $C_\varepsilon \cap W$. The idea is to use Banach contraction principle.

First, we need to show that $S_\varepsilon$ maps $C_\varepsilon \cap W$ into itself. First, we show that $\|S_\varepsilon(w)\|$ is bounded independently of $\varepsilon$. Actually, we have:

$$\|S_\varepsilon(w)\| \leq C \|I'_\varepsilon(z + w) - I''_\varepsilon(z)[w]\|$$

$$\leq C \left\| \int_0^1 (I''_\varepsilon(z + sw) - I''_\varepsilon(z))[w] \, ds + I'_\varepsilon(z) \right\|$$

$$\leq C\varepsilon^{1\lambda(p-1)} + C' < C_1.$$

Recall that we have already fixed the value $C_1$ of the definition of $C_\varepsilon$. At this point, we claim that:

Claim. There exists $C_2 > 0$ such that if $w \in C_\varepsilon$ and $\bar{w} = S_\varepsilon(w)$, then $|w(r)| \leq C_2 \varepsilon$.

If this is true, by choosing such a $C_2$ in the definition of $C_\varepsilon$, we have that $S_\varepsilon(C_\varepsilon \cap W) \subset C_\varepsilon \cap W$.

Equation (4.4) implies that, for certain $\lambda \in \mathbb{R}$,

$$I''_\varepsilon(z)[\bar{w}] = -I'_\varepsilon(z + w) + I''_\varepsilon(z)[w] + \lambda(-\Delta \bar{z} + \bar{z}).$$

In general, there holds:

$$I''_\varepsilon(z)[u] = -\Delta u + u + \phi u + 2\phi_1 z - p z^{p-1} u,$$

where $-\Delta \phi = \varepsilon z^2$, $-\Delta \phi_1 = \varepsilon u$. Moreover,

$$I'_\varepsilon(z + u) = -\Delta(z + u) + (z + u) + \phi_2(z + u) - |z + u|^{p-1}(z + u),$$

where $-\Delta \phi_2 = \varepsilon(z + u)^2$.

Then, $\bar{w} = S_\varepsilon(w)$ verifies the equation:

$$\begin{cases}
-\Delta \bar{w} + \bar{w} + \phi \bar{w} + 2\phi_1 z - p z^{p-1} \bar{w} \\
= \Delta z - z - p z^{p-1} w + |z + w|^{p-1}(z + w) \quad (4.5) \\
- \phi_2(z + w) + \phi w + 2\phi_3 z + \lambda(-\Delta \bar{z} + \bar{z}),
\end{cases}$$

where $-\Delta \phi = \varepsilon z^2$, $-\Delta \phi_1 = \varepsilon z \bar{w}$, $-\Delta \phi_2 = \varepsilon(z + w)^2$, $-\Delta \phi_3 = \varepsilon z w$.

We now recall the following general result due to Strauss\cite{26}:

$$|u(r)| \leq c_0 r^{-1} ||u|| \quad \forall u \in H^1_r, \ r \geq 1,$$
where $c_0$ is a fixed constant.

By applying this result, we can choose $C_2 > 0$ so that for any $u \in H^1_1$, $\|u\| \leq C_1$, there holds $|u(r)| < C_2 \varepsilon$ whenever $r \geq \frac{r_1}{4\varepsilon}$. Taking into account that $z(r) = \dot{z}(r) = 0$ for any $r < \frac{r_1}{4}$, we have that $\tilde{w}$ solves:

$$
\begin{cases}
-\Delta \tilde{w} + \dot{\theta} \tilde{w} + \phi \tilde{w} = |w|^{p-1}w - \phi_2 w + \phi w, & \text{if } |x| < \frac{r_1}{4\varepsilon}, \\
|\tilde{w}(x)| < C_2 \varepsilon, & \text{if } |x| = \frac{r_1}{4\varepsilon}.
\end{cases}
$$

(4.6)

We now estimate $|\phi_2 - \phi|$:

$$
|\phi_2(r) - \phi(r)| \leq \frac{\varepsilon}{r} \int_0^{+\infty} s \min\{r, s\}(w^2(s) + 2z(s)|w(s)|) \, ds
\leq \varepsilon \int_0^{+\infty} s w^2(s) + \varepsilon \left( \int_0^{+\infty} s w^2(s) \, ds \right)^{1/2} \left( \int_0^{+\infty} z^2(s) \, ds \right)^{1/2}
\leq C \varepsilon.
$$

(4.7)

Since $\phi \geq 0$, the inequality $|u(r)| < C_2 \varepsilon$ for $r < \frac{r_1}{4\varepsilon}$ follows from the maximum principle, and the claim is proved.

To complete the proof, we now show that $S_\varepsilon$ is a contraction with constant sufficiently small. Actually, we have:

$$
\|S_\varepsilon(w_1) - S_\varepsilon(w_2)\|
\leq C \|I''_\varepsilon(z + w_1) - I''_\varepsilon(z)\| |w_1 - w_2|
\leq C \left| \int_0^1 I''_\varepsilon(z + w_2 + s(w_1 - w_2)) |w_1 - w_2| \, ds - I''_\varepsilon(z) |w_1 - w_2| \right|
\leq C \left| \int_0^1 (I''_\varepsilon(z + w_2 + s(w_1 - w_2)) - I''_\varepsilon(z)) |w_1 - w_2| \, ds \right|
\leq C \varepsilon^{1/(p-1)} \|w_1 - w_2\|.
$$

Remark 4.2. We point out that, thanks to Lemmas 4.3, 4.4, and estimate (E4), we deduce that $PI''_\varepsilon(z + w)$ is positive definite. Hence, the critical point $z + w$ is actually a nondegenerate relative minimum of $I''_\varepsilon|(z+w)$.

Remark 4.3. Let us say a few words about the dependence of $w$ on $\rho$. For $\varepsilon$ fixed as small as necessary, and $\rho_0$ fixed, we have a solution of the problem $P_{\rho_0}I''_\varepsilon(z_{\rho_0} + w_{\rho_0})$.

The idea is just to apply the Implicit Function Theorem to this function. In order to do this, we need to compute:

$$
\frac{\partial}{\partial w} PI''_\varepsilon(z + w) = PI''_\varepsilon(z + w)
$$

and study its invertibility on $W$. But this operator is clearly invertible in $W$ because of (E4) and Lemmas 4.3 and 4.4.
Therefore, there exists a solution \( w_\rho \) for \( \rho \) close to \( \rho_0 \). Due to the uniqueness of \( w_\rho \) in a neighborhood of \( w \), \( w_\rho \) must be the solution we found before, and therefore belongs to \( C_\varepsilon \). The \( C^1 \) dependence on \( \rho \) then follows from the standard Implicit Function Theorem, since the curve \( \rho \mapsto z_{\varepsilon, \rho} \) is \( C^2 \) in \( H^1 \), and therefore the projection \( P(u) = u - \|\hat{\varepsilon}\|^{-2} \langle u, \hat{\varepsilon} \rangle \hat{\varepsilon} \) is \( C^1 \) with respect to \( \rho \).

In order to also solve the bifurcation equation, it suffices to study the critical points of the finite-dimensional function \( T_\varepsilon \equiv \rho \mapsto I_\varepsilon(z_{\varepsilon, \rho} + w_{\varepsilon, \rho}) \) (see Refs. 2 and 3). To do so, we will use the function \( M : (0, g(a_0)] \to \mathbb{R} \) defined as

\[
M(r) = r f(r) \left[ \frac{3p - 7}{4} f(r) + p - 1 \right].
\]

**Lemma 4.5.** For \( \varepsilon \) small, we have:

\[
\varepsilon^2 I_\varepsilon(z_{\varepsilon, \rho} + w_{\varepsilon, \rho}) = \varepsilon^2 I_\varepsilon(z_{\varepsilon, \rho}) + o(1) = M(\varepsilon \rho) + o(1).
\]

**Proof.** Thanks to estimate (E3), we easily get that

\[
|I_\varepsilon(z + w) - I_\varepsilon(z)| \leq C
\]

for certain constant \( C \). Then,

\[
I_\varepsilon(z + w) \sim \int_{\mathbb{R}^N} \frac{1}{2} \left( |\nabla z|^2 + z^2 \right) + \frac{\phi}{4} z^2 - \frac{1}{p + 1} z^{p+1}
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^N} \left[ |\nabla z|^2 + z^2 + \phi z^2 - z^{p+1} \right] dx
\]

\[
+ \int_{\mathbb{R}^N} \left[ -\frac{\phi}{4} z^2 + \left( \frac{1}{2} - \frac{1}{p + 1} \right) z^{p+1} \right] dx
\]

\[
= \frac{1}{2} I_\varepsilon^\prime(z)(z) + \int_{\mathbb{R}^N} \left[ -\frac{\phi}{4} z^2 + \left( \frac{1}{2} - \frac{1}{p + 1} \right) z^{p+1} \right] dx.
\]

The first term above is of order \( \varepsilon^{-1} \) and can be omitted. Thanks to Lemma 4.2 and the exponential decay of \( z \), we deduce:

\[
I_\varepsilon(z + w) \sim -\rho^2 \int R \frac{f(\varepsilon \rho)}{4} \int_{\mathbb{R}^N} U^a + \rho^2 \left( \frac{1}{2} - \frac{1}{p + 1} \right) \int_{\mathbb{R}^N} U^{p+1}.
\]

By using equalities (3.2), (4.2), we obtain:

\[
I_\varepsilon(z + w) \sim \rho^2 \int_{\mathbb{R}} U_a^2 \left[ -\frac{f(\varepsilon \rho)}{4} + (1 + f(\varepsilon \rho)) \left( \frac{p - 1}{p + 1} \right) \right]
\]

\[
= \varepsilon^{-2}(\varepsilon \rho)^2 M_a \left[ -\frac{f(\varepsilon \rho)}{4} + (1 + f(\varepsilon \rho)) \frac{p - 1}{p + 3} \right]
\]

\[
= \varepsilon^{-2}(\varepsilon \rho)^2 \frac{3p - 7}{4} f(\varepsilon \rho) + p - 1.
\]
We are now in a position to complete the proof of Theorem 4.1. In order to study the possible existence of critical points of $M$, we compose it with the function $g$, inverse of $f$. So, we obtain:

$$\tilde{M}(s) := M(g(s)) = \frac{1}{c(p+3)} \frac{s^2}{(1+s)^{\frac{p+3}{p-1}}} \left( \frac{3p-7}{4} s + p - 1 \right).$$

It is easy to verify that $a = \frac{4(p-1)}{11p}$ is a nondegenerate minimum of $\tilde{M}$, and $\tilde{M}(a) < 0$. Then, according to Lemma 4.5, there exists $r_\varepsilon$ in $T_\varepsilon$ such that $r_\varepsilon = \rho_\varepsilon \varepsilon \to g(\tilde{a}) = \tilde{r}$ and $u_\varepsilon = z_{\varepsilon, \rho_\varepsilon} + w_{\varepsilon, \rho_\varepsilon}$ is a critical point of $I_\varepsilon$. Thus, the functions $u_\varepsilon$ give rise to a solution of (2.2), and the functions $v_\varepsilon(x) = u_\varepsilon(\frac{x}{r_\varepsilon})$ are solutions of (2.1) which concentrate around the sphere of radius $r_\varepsilon$.

Furthermore, $u_\varepsilon$ is a local minimum of $I_\varepsilon$, since it is a minimum of $I_\varepsilon|_{(Z + W)}$ (see Remark 4.2) as well as along the curve $Z_\varepsilon$. In a similar way we can deduce that it is a nondegenerate minimum, that is, $I''_\varepsilon$ is positive definite. Finally, the fact that $I_\varepsilon(u_\varepsilon) \to -\infty$ follows from Lemma 4.5 and the fact that $M(\tilde{r}) = \tilde{M}(\tilde{a}) < 0$.

**Remark 4.4.** If we study the explicit expression of $\tilde{M}$, we see that it exhibits another critical point, which is $2 \frac{p-1}{p} = a_0$. However, this point is not in the rank we are working with. Even more, this point shows up because $\tilde{M} = M \circ g$ and $g'(a_0) = 0$. Actually, one can easily verify that

$$\lim_{r \to g(a_0)} M'(r) \neq 0.$$

**Remark 4.5.** We point out that the function $M$ defined above plays the role of “auxiliary potential”, which in Ref. 5 was played by the function $r^{n-1}V^\theta(r)$, where $V$ is a fixed potential and $\theta = \frac{n+1}{p-1} - \frac{1}{2}$. We recall that in Ref. 5 the critical points found do not correspond to local minima. Moreover, in Ref. 5 there are always pairs of radius of concentration, while in our case the concentration occurs only at a certain radius.

**Remark 4.6.** Let us say a few words about $I_\varepsilon$ for $\varepsilon$ sufficiently small. If $p \in (1, \frac{1}{2})$, we have found a local minimum of this functional in $H^1$. Observe that another local minimum is attained at the function 0. Then, our functional has the convenient geometric properties to apply a mountain-pass argument and, provided that (PS) holds, we would get another critical point. But we have also found a critical point of $I_\varepsilon$, namely, the single spike shown in Sec. 2. Moreover, it is quite easy to prove that the spike has Morse index equal to one. Hence, it is very reasonable to think that this spike coincides with the previous mountain-pass critical point.

**Note Added in Proof**

After the completion of the paper, A. Ambrosetti brought to our attention a preprint by T. D’Aprile and J. Wei, On bound states concentrating on spheres
for the Maxwell–Schrödinger equation. In the paper the authors prove the existence of solutions concentrating around a sphere for (2.1) with $1 < p < \frac{11}{7}$, like our Theorem 4.1.

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