Tangential residual as error estimator in the boundary element method

Alejandro E. Martínez-Castro, Rafael Gallego *

Department of Structural Mechanics, University of Granada, Ed. Politécnico Fuentenueva, Avda. Fuentenueva s/n, 18002 Granada, Spain

Accepted 10 September 2004
Available online 21 December 2004

Abstract

In this paper a new error estimator based on tangential derivative Boundary Integral Equation residuals for 2D Laplace and Helmholtz equations is shown. The direct problem for general mixed boundary conditions is solved using standard and hypersingular boundary integral equations. The exact solution is broken down into two parts: the approximated solution and the error function. Based on theoretical considerations, it is shown that tangential derivative Boundary Integral Equation residuals closely correlate to the errors in the tangential derivative of the solution. A similar relationship is shown for nodal sensitivities and tangential derivative errors. Numerical examples show that the tangential Boundary Integral Equation residual is a better error estimator than nodal sensitivity, because of the accuracy of the predictions and the lesser computational effort.

Keywords: Boundary element method; Error estimation; Boundary Integral Equation residual; Nodal sensitivity; Adaptivity; Mesh adaptation; Mesh refinement

1. Introduction

The computation of accurate and effective error estimators is a key step in the development of automated procedures for mesh adaptation and refinement within the family of Boundary Element Methods (see [1] for a recent review). The adaptive process consists of three steps: error estimation, adaptive tactics, and mesh refinement. The error estimation is the most important of them since the whole process depends on its accuracy.

Error estimation schemes can be classified in five types: residual, interpolation error, boundary integral error, node-sensitivity and solution difference. In this classification, the procedure described in the present paper is a residual-type error estimator.

Error estimators based on the approximation of the error function for potential or flux can lead to wrong error indicators. Two different displacement fields (potential) can be considered truly different if their tangential derivatives are different [2]. In this aspect, two types of error can be considered:

- Error defined by the difference of two values at a point: the exact solution and the approximated one.
- Error defined by the local interpolation space, through the shape functions.
The aim of the error estimation techniques is to locate the sources of error due to wrong interpolation functions. In general, these error sources are not necessarily located at the elements with the worst computed values. Error estimators based on the error predicted at the variable values require special norms to obtain an adequate error indicator. An easy example can illustrate this fact. For a fracture problem, Fig. 1 shows a schematic displacement field for the exact solution and the approximated solution. An error estimator based on the values of the displacements predicts the highest error in elements far to the crack tip. However, for this problem it is known that the behaviour of the displacement field is $O(\sqrt{r})$ close to the crack tip ($r$ = distance to the crack tip). The source of errors far from the tip is an incorrect shape function set close to the tip, but no the lack of refinement far from it.

In order to obtain an error estimator for the approximated fields, sensitivity analysis has been proposed for different formulations of the Boundary Element Method (BEM): elastostatics [2,3], fracture problems [4,5], standard and hypersingular integral equations for potential and acoustic problems [6,7]. Based on interpolation ideas, nodal sensitivity can be related to tangential derivative error [4,5], but for a BEM procedure, a similar relationship is not obvious, because all the nodal values change with respect to nodal perturbations.

Tangential derivatives of potential values have been considered in order to obtain an error estimator: one approach is based on two solution difference, with Hermitian and Lagrangian interpolations. For the Hermitian elements a boundary integral equation for tangential derivatives of potential (displacements) is used. This error estimator is applied to potential problems, 2D and 3D elasticity, and thermo-elastic problems [8–12].

Recently, it has been presented an error estimation approach based on tangential derivatives [13,14]. This work also presents the use of local error estimator in an adaptive mesh refinement procedure, based on a $r$-refinement approach.

The mathematical analysis in the error estimation for the collocation BEM has been introduced in the last years. As cited above the mathematical relationship between sensitivity and tangential error has been established, but for interpolation theory. More recently, various mathematical relations between error estimator for the residuals of the Boundary Integral Equations (BIEs) have been published [15,16].

The present paper describes a new error estimator based on tangential BIE residuals. A mathematical analysis is performed in order to show that the error function of the tangential derivative (flux or potential) is related to the residual of an appropriate BIE. A similar analysis is carried out for nodal sensitivities. Numerical tests validate the use of the proposed BIE residual as error estimator, and show that it is more accurate than nodal sensitivity, and requires less computational effort.

This paper is focused on the critical study for the new error estimator proposed. This analysis is the most important of the process, because based on the information provided by the error estimator, several mesh adaptation techniques can be implemented ($h$-, $p$-, $hp$-, $r$-). The implementation of different adaptive techniques based on this error estimator is being carried out by the authors.

2. Basic equations

Consider a 2D domain $\Omega$ whose boundary is $\Gamma$. Both the potential and acoustic problem, governed by the Laplace equation

$$\Delta u(x) = 0$$

and Helmholtz equation

$$\Delta u(x) + k^2 u(x) = 0$$

respectively, are considered in parallel. In Eq. (2) $k$ is the wave number.

The flux is given by $q(x) = \frac{\partial u}{\partial n} = \sum n_i$, where $n(x)$ is the outward normal to the boundary at $x$.

In both problems, boundary conditions are provided, $u(x) = \pi$ on $\Gamma_{\mu}$ and $q(x) = \eta$ on $\Gamma_{\eta}$ such that $\Gamma_{\mu} \cup \Gamma_{\eta} = \Gamma$ and $\Gamma_{\mu} \cap \Gamma_{\eta} = \emptyset$.

For the Helmholtz problem $u(x)$ and $q(x)$ are complex numbers while they are real for the Laplace problem. For the sake of simplicity $u(x)$ will be termed potential and $q(x)$ flux, regardless of the problem considered, and no distinction is made between both problems, save when explicitly stated.

For these problems, the potential Boundary Integral Equation, or $u$-BIE, is
where \( y \in \Gamma \) is a smooth boundary point; \( q^*(x, y) \) and \( u^*(x, y) \) are the fundamental solution flux and potential, respectively, for the problem under consideration (see Section 5).

Likewise, the flux Boundary Integral Equation, or \( q \)-BIE, at a smooth boundary point is

\[
\frac{1}{2} q(y) + \int_{\Gamma} q^*(x, y) u(x) \, d\Gamma(x) = \int_{\Gamma} s^*(x, y) u(x) \, d\Gamma(x) \tag{4}
\]

where \( d^*(x, y) \) and \( s^*(x, y) \) are the normal derivatives of the kernels in the \( u \)-BIE (see section 5). In the rest of the paper, the variables \( x \) and \( y \) inside the integral sign are dropped out, for the sake of brevity.

The right-hand side integral in (4) is understood in the sense of the Hadamard Finite Part.

### 3. Error estimators based on BIE residuals

An error estimator based on BIE residuals was proposed for the collocation BEM almost two decades ago [17]. The authors use a residual type error estimator for 3D potential problems. The residual error estimator is latter extended to elastostatics by the same authors [18,19] and much latter the idea is further developed using the hypersingular or stress BIE for linear elasticity [4,20–22]. The error indicator is obtained recomputing the stresses with the hypersingular or stress BIE at the collocation points and comparing their value with the approximated solution. Based on this work, the hypersingular error estimator is extended to other related numerical method based on integral equations: Symmetrized-Galerkin BEM [23], Meshless Boundary Node Method [24], Boundary Contour Method [25]. A recent review paper presents a thorough discussion of these ideas [26].

A brief explanation of the idea of hypersingular BIE residual as error estimator is presented here, since it is closely related to the estimator proposed in this paper.

Firstly, the exact solutions \( u(x) \) and \( q(x) \) can be decomposed in two additive terms: approximated potential or flux \( (\hat{u}(x), \hat{q}(x)) \) plus error for potential or flux \( (e_u(x), e_q(x)) \). Thus

\[
u(x) = u(x) + e_u(x) \tag{5}\]
\[
q(x) = q(x) + e_q(x) \tag{6}\]

Using standard BEM an approximate solution \( \hat{u}(x), \hat{q}(x) \) is obtained such that the \( u \)-BIE is fulfilled, within numerical accuracy, at a given set of collocation points \( y \). Then, at any collocation point where Dirichlet boundary conditions are prescribed, the \( q \)-BIE for the exact solution can be stated (4), and, taking into account (5) and (6) the \( q \)-BIE can be rearranged as

\[
\frac{1}{2} e_u(y) + \int_{\Gamma} (d^* e_q - s^* e_u) \, d\Gamma = \int_{\Gamma} (d^* \hat{q} - s^* \hat{u}) \, d\Gamma \tag{7}\]

where \( y \) is one of the collocation points.

Next, the residual of the \( q \)-BIE, \( e_q(y) \) is defined by the equation,

\[
\frac{1}{2} e_q(y) = \frac{1}{2} \hat{q}(y) - \frac{1}{2} d^* \hat{q} - s^* \hat{u} \quad d\Gamma \tag{8}\]

If the integral in the left-hand side were negligible, then

\[
\frac{1}{2} e_q(y) \approx \frac{1}{2} e_q(y) \]

Thus, the residual in the \( q \)-BIE can be used to approximate the error for the flux. This residual has a “symmetric” property [20]. In fact, at boundary points where Neumann conditions are prescribed, the \( q \)-BIE may be collocated to obtain the approximate solution and a new residual \( e_u \) based on the \( u \)-BIE could be used to approximate \( e_u \). This scheme is used to estimate the error at interior points, as well.

These BIE residuals, however, overestimate the exact errors, although they can be used as an error indicators [20]. The regression between this error estimators and the real errors is not 1:1, and their correlation factor is not close to 1.0.

In the present work, using a similar approach, the residuals in the \( u \)-BIE and \( q \)-BIE are replaced by the residuals in the BIEs for the tangential derivative of potential or flux. These residuals are error estimators of the error in tangential derivative of potential and flux, respectively, as it is demonstrated in the following section.

### 4. Tangential boundary integral equation residuals as error estimator

A stable second kind integral equation set is obtained by using the \( q \)-BIE at collocation points where the unknown is the flux, and the \( u \)-BIE at points where the unknown is the potential [28]. Thus, the approximate
solution is computed using both BIEs in the same problem. For general mixed boundary conditions:

- The \( \nu \)-BIE is collocated at points \( y \in \Gamma_q \) (flux known, potential unknown).
- The \( q \)-BIE at points \( y \in \Gamma_u \) (potential known, flux unknown).

With this information, the proposed error estimator for both the Neumann and Dirichlet parts of \( \Gamma \) can be described in a parallel scheme.

4.1. Sub-boundary with Neumann boundary conditions

The boundary integral equation for the tangential derivative of the potential, or \( u_t \)-BIE, at a smooth boundary point is [7]

\[
\frac{1}{2} u_t(y) + \int_{\Gamma} (q_t^*(x;y)u(x) - u_t^*(x;y)q(x)) \, d\Gamma(x) = 0 \quad (9)
\]

where \( u_t(y) \) is the tangential derivative of the potential at the boundary point \( y \),

\[
\frac{\partial u}{\partial T} = \frac{\partial u}{\partial y} T_i = u_t T_i 
\]

The kernels \( u_t^*(x;y) \) and \( q_t^*(x;y) \) are obtained from those of the \( u \)-BIE by differentiation, as shown in Section 5. The integral is understood in the sense of Cauchy Principal Value.

The exact tangential derivative of the solution, \( u_e \), can be decomposed as

\[
u(x,y) = \hat{u}(y) + \epsilon_u(y) \quad (10)
\]

where \( \hat{u} \) is the tangential derivative of the approximate solution \( u \), differentiating the shape functions, \( \frac{\partial}{\partial y} \), and \( \epsilon_u(y) \) is the true error in the potential tangential derivative.

Considering Eqs. (10), (5) and (6), the \( u_t \)-BIE (9) can be written as

\[
\frac{1}{2} \epsilon_u(y) + \int_{\Gamma} (q_t^* e_q - u_t^* e_q) \, d\Gamma = - \frac{1}{2} \hat{u}(y) \quad (11)
\]

Defining the residual for the \( u_t \)-BIE, \( \epsilon_u \), by the equation

\[
\frac{1}{2} \epsilon_u(y) = - \frac{1}{2} \hat{u}(y) \quad (12)
\]

Eq. (11) can be re-written as

\[
\frac{1}{2} \epsilon_u(y) + \int_{\Gamma} (q_t^* e_u - u_t^* e_q) \, d\Gamma = \frac{1}{2} \epsilon_u(y) \quad (13)
\]

If the integral terms in the left-hand side were negligible, Eq. (13) leads finally to

\[
\epsilon_u(y) \simeq \epsilon_u(y) \quad (14)
\]

With this equation, the error in the tangential derivative \( \epsilon_u \) can be approximated by a residual of a BIE \( \epsilon_u \), which can be readily computed once the approximate solution is obtained.

A different approach could be devised from (13): taking into account that, \( \epsilon_u \) and that \( \epsilon_u(y) = 0 \) on \( \Gamma_u \) and \( \epsilon_q(y) = 0 \) on \( \Gamma_q \). Eq. (13) can be discretized and solved using standard Boundary Element techniques, for the unknowns \( \epsilon_u \) and \( \epsilon_q \). But this approach is not promising since, even if the values obtained for \( \epsilon_u \) and \( \epsilon_q \) were accurate, the errors in the potential and fluxes do not detect directly errors due to the interpolation functions, as mentioned in the introduction, which is the final aim of any error estimator for mesh adaptation.

Eq. (14) implies that the error in the tangential derivative of the error on the boundary \( \Gamma_q \) is estimated by the amount by which the solution computed using \( u \)-BIE and \( q \)-BIE, fails to satisfy the \( u_t \)-BIE. This approach has several advantages worth stressing:

- The estimator can be computed at the nodal points or any other boundary point.
- It does not involve any adjustable parameters.
- It is defined by an analytical formula.
- It is intrinsic to the Boundary Element Method.
- Its evaluation can be performed at chosen points independently.
- No new system of equations has to be solved.
- It predicts errors due to the interpolation space.

A similar approach with all these characteristics, except the last one, has been proposed [20]. This last property, however, is very important to rightly prioritize the locations where the mesh refinement should be carried out.

A theoretical argument can be provided to show that the integral term in (13) is actually small. To do so, substituting in

\[
I = \int_{\Gamma} (q_t^* e_u - u_t^* e_q) \, d\Gamma
\]

the values of \( e_u = u - \hat{u} \) and \( e_q = q - \hat{q} \), and rearranging terms, one obtains

\[
I = \int_{\Gamma} (q_t^* u - u_t^* q) \, d\Gamma - \int_{\Gamma} (q_t^* \hat{u} - u_t^* \hat{q}) \, d\Gamma
\]

The first integral is simply \(-\frac{1}{2} u_t \), from (9), i.e., \(-\frac{1}{2} \) times the exact value of the tangential derivative of the potential, while the second one is \(-\frac{1}{2} u_{t|BIE} \) where \( u_{t|BIE} \) is the recomputed value of the potential tangential derivative using the corresponding \( u_t \)-BIE. Note that \( u_{t|BIE} \neq \hat{u} \), since this last value is the tangential derivative of the approximate solution \( \hat{u} \), and \( u_{t|BIE} \neq u_t \) as well, since the value in the left-hand side is computed using the approximate solution of the potential and flux, not the exact one.
Collecting these expressions one obtains
\[ \frac{1}{2} (q^p e_y - u^p e_y) d\Gamma = -\frac{1}{2} (u_t - u_t|_{\text{BIE}}) \]

It is well established that the approximation of the tangential derivative using the \( u_t \)-BIE is much better than the one obtained by direct differentiation of the shape functions [16], and therefore
\[ \epsilon^t_q = u_t - u_t \gg u_t - u_t|_{\text{BIE}} \]

inequality that justifies the approximation in (14).

4.2. Sub-boundary with Dirichlet boundary condition

On the part of the boundary where the potential \( u(y) \) is known, the \( q \)-BIE is collocated to compute the approximate solution. The boundary integral equation for the tangential derivative of flux, or \( q \)-BIE, at a smooth boundary point, is [7]
\[ \frac{1}{2} \hat{q}(y) + \int_f (d^i e_q - s^i e_u) d\Gamma(x) = 0 \]  

(15)

The kernels in this equation are obtained by differentiation from the kernels in the \( q \)-BIE (see Section 5).

The exact value of the flux tangential derivative, \( q_t(y) \), can be written as
\[ q_t(y) = \hat{q}(y) + \epsilon^t_q(y) \]  

(16)

where, again, \( \hat{q} \) is the tangential derivative of the approximate solution flux, and \( \epsilon^t_q \) is the error in the flux tangential derivative.

The integral equation (15) leads to a Hadamard Finite Part in the limit to the boundary [7]. To compute these Finite Parts is necessary to resort to regularization procedures to weaken the highly-singular kernels (up to \( O(r^{-3}) \) for 2D). In [7] these kernels have been employed to develop sensitivity analysis for the approximate solutions of the hypersingular boundary integral equation.

Substituting now Eqs. (16), (5) and (6), in Eq. (15), and rearranging terms the following equation is obtained:
\[ \frac{1}{2} \epsilon^t_q(y) + \int_f (d^i e_q - s^i e_u) d\Gamma = -\frac{1}{2} \hat{q}(y) - \int_f (d^i \hat{q} - s^i \hat{u}) d\Gamma \]  

(17)

The residual of the \( q \)-BIE is defined by the equation
\[ \frac{1}{2} \epsilon^t_q(y) = -\frac{1}{2} \hat{q}(y) - \int_f (d^i \hat{q} - s^i \hat{u}) d\Gamma \]  

(18)

and therefore Eq. (15) can be written as
\[ \frac{1}{2} \epsilon^t_q(y) + \int_f (d^i e_q - s^i e_u) d\Gamma = \frac{1}{2} \epsilon^t_q(y) \]  

(19)

Again, if the integral term in the left-hand side were small,
\[ \epsilon^t_q(y) \approx \epsilon^t_q(y) \]  

(20)

In conclusion, the residual for the \( q \)-BIE can be used to approximate the value of the error in the tangential derivatives of the approximated solution for the flux.

The reasoning to show that the integral term is actually negligible is similar to the one in the previous case, leading to
\[ \int_f (d^i e_q - s^i e_u) d\Gamma = -\frac{1}{2} q_t - q_t|_{\text{BIE}} \]

where \( q_t|_{\text{BIE}} \) is the value of the flux tangential derivative recomputed using the approximate solution in the \( q \)-BIE.

Again, the inequality
\[ \epsilon^t_q = q_t - \hat{q} \gg q_t - q_t|_{\text{BIE}} \]

justifies the approximation in (20).

4.3. Tangential derivative BIE residuals, nodal sensitivities, and errors in the tangential derivative

Eqs. (13) and (19) summarize the theoretical considerations that leads to a relationship between tangential derivative BIE residuals and the true errors in the tangential derivative. In this section, the relationship between nodal sensitivities and tangential derivative errors is shown.

Nodal sensitivities are defined as the rate of change of nodal solutions when the coordinates of one or a set of collocation nodes suffer an infinitesimal variation. The sensitivity formulation for the \( u \)-BIE in elastostatics can be found in [2,5] while the formulation for the sensitivity of the \( u \)-BIE and the \( q \)-BIE for potential and acoustic problems is developed in [7,6]. In all cases, nodal sensitivity values of potential (displacement) and flux (traction) are obtained using the Direct Differentiation Approach (DDA) [2].

Thus, from the \( u \)-BIE, the following BIE is obtained:
\[ \frac{1}{2} \hat{u}(y) + \int_f (q^p \hat{u} - u^p \hat{q}) d\Gamma \]  

\[ = -\frac{1}{2} \hat{u}(y) - \int_f (q^p \hat{u} - u^p \hat{q}) d\Gamma \]

where \( \hat{u}(x) \) and \( \hat{q}(x) \) are the potential and flux sensitivity, respectively, and, likewise, from the \( q \)-BIE
\[ \frac{1}{2} \hat{q}(y) + \int_f (d^i \hat{q} - s^i \hat{u}) d\Gamma = -\frac{1}{2} \hat{q}(y) - \int_f (d^i \hat{q} - s^i \hat{u}) d\Gamma \]

Considering the expressions of the residuals in Eqs. (12) and (18), the BIEs for the sensitivity can be written as
\[ \frac{1}{2} \ddot{u}(y) + \int_I (q^T \ddot{u} - u^T \ddot{q}) \, d\Gamma = \frac{1}{2} \dot{e}_u(y) \quad (21) \]

\[ \frac{1}{2} \ddot{q}(y) + \int_I (d^T \ddot{q} - s^T \ddot{u}) \, d\Gamma = \frac{1}{2} \dot{e}_q(y) \quad (22) \]

In these equations, the residuals in the right-hand side can be computed once the approximate solution is obtained. After discretization of (21) and (22) by standard Boundary Element techniques the nodal sensitivities are computed. Note that the kernels in these equations are those of the \( u \)-BIE and \( q \)-BIE, and therefore, the ensuing algebraical system of equations has the same system matrix.

To show the relationship between nodal sensitivities and tangential BIE errors, the value of the residuals in terms of the actual errors given in (13) and (19), is substituted in these equations, leading to

\[ \frac{1}{2} \dot{e}_u(y) = \frac{1}{2} \ddot{u}(y) + \int_I (q^T \ddot{u} - u^T \ddot{q}) \, d\Gamma - \int_I (q^T \dot{e}_u - u^T \dot{e}_q) \, d\Gamma \quad (23) \]

\[ \frac{1}{2} \dot{e}_q(y) = \frac{1}{2} \ddot{q}(y) + \int_I (d^T \ddot{q} - s^T \ddot{u}) \, d\Gamma - \int_I (d^T \dot{e}_q - s^T \dot{e}_u) \, d\Gamma \quad (24) \]

If the integral terms in the right-hand sides of these equations were negligible, then

\[ \dot{e}_u(y) \approx \ddot{u}(y) \quad (25) \]

\[ \dot{e}_q(y) \approx \ddot{q}(y) \quad (26) \]

Numerical experiments shown in Section 6 confirm that the integral terms in (23) and (24) are small, and therefore nodal sensitivities are truly error estimators for the tangential derivative errors. However, it has not been possible to justify theoretically that the first integral term in Eqs. (23) and (24) is always small, in comparison with the terms \( \frac{1}{2} \dot{e}_u(y) \) and \( \frac{1}{2} \dot{e}_q(y) \).

Nevertheless, this error estimator is computationally more expensive than the tangential BIE residual, since it involves the computation of this one and the solution of a system of equation, even though the system matrix were factorized. On the other hand, it will be shown in the numerical applications that the sensitivities overpredict the values of the tangential derivative errors, and is less closely correlated to the true error than the tangential BIE residuals.

5. Tangential derivative kernels

In this section, the expressions of the kernels in the \( u \)-BIE and \( q \)-BIE, both for the Laplace and Helmholtz problems are shown. In order to calculate the tangential BIE residuals, regularization techniques must be employed. The development of the theoretical considerations that leads to the tangential derivative BIEs, their singular behaviour and its regularization is not covered in this paper (see [7]).

5.1. Kernels in the \( u \)-BIE

The kernels in the \( u \)-BIE, \( u^r(x;y) \) and \( q^r(x;y) \) are obtained from the kernels of the \( u \)-BIE, \( u^T(x;y) \) and \( q^T(x;y) \) by the equations

\[ u^r(x;y) = \frac{\partial u^T}{\partial T} = - \frac{\partial u^T}{\partial y} T_i = - u^T_{yi} T_i \]

\[ q^r(x;y) = \frac{\partial q^T}{\partial T} = - \frac{\partial q^T}{\partial y} T_i = - q^T_{yi} T_i \]

where \( T \) is the unit vector tangent to the boundary at the collocation point \( y \).

5.1.1. Kernels in the \( u \)-BIE for the Laplace problem

The kernels in the standard \( u \)-BIE for the Laplace equation (1) are well known in the BEM literature:

\[ u^r(x;y) = - \frac{1}{2\pi} \ln r; \quad q^r(x;y) = - \frac{1}{2\pi r} \rho \cdot n \]

where \( r = |x - y| \); \( n \) is the outward normal vector to the boundary at \( x \); and \( \rho \) is the unit vector in the \( x - y \) direction.

The kernels in the \( u \)-BIE are readily obtained,

\[ u^r(x;y) = \frac{1}{2\pi r} \rho \cdot T; \quad q^r(x;y) = \frac{1}{2\pi r} (n \cdot T - 2 \rho \cdot t \cdot T) \]

where \( t \) is the unit vector tangent to the boundary at \( x \).

The kernel \( q^r(x;y) \) is regular, while \( u^r(x;y) \) is singular \( O(r^{-1}) \), and its Cauchy Principal Value can be evaluated by a simple zero order regularization.

5.1.2. Kernels in the \( u \)-BIE for the Helmholtz problem

The kernels in the \( u \)-BIE for the Helmholtz problem are

\[ u^r(x;y) = \frac{1}{2\pi} K_0(ikr); \quad q^r(x;y) = \frac{-ik}{2\pi} K_1(ikr) \rho \cdot n \]

where \( K(z) \) is the modified Bessel function of \( \nu \)th order.

These kernels have the same order of singularity than the kernels for Laplace’s equation. Actually, they can be decomposed in their singular part (Laplace’s kernels) plus a regular part (dynamic increment) for their numerical integration, and therefore no new regularization procedures are required.

The kernels in the \( u \)-BIE for the Helmholtz problem are

\[ u^r(x;y) = \frac{ik}{2\pi} K_1 \rho \cdot T \]

\[ q^r(x;y) = \frac{ik}{2\pi r} [K_1 n \cdot T - (2K_1 + ikrK_0) \rho \cdot T] \]
The variable \( z = ikr \) in the Bessel functions is dropped out for the sake of brevity.

Again, \( u^t(x;y) \) is singular \( O(r^{-1}) \) and \( q^t(x;y) \) is regular.

5.2. Kernels in the \( q_I \)-BIE

The kernels in the \( q_I \)-BIE, \( d^t_i(x;y) \) and \( s^t_i(x;y) \), are obtained from the kernels of the \( q \)-BIE, \( d^t(x;y) \) and \( s^t(x;y) \), respectively, by the equations

\[
d^t_i(x;y) = \frac{\partial d^t}{\partial T} = \frac{\partial d^t}{\partial y_i} T_i = -\frac{\partial d^t}{\partial x_i} T_i = -d^t_i T_i
\]

\[
s^t_i(x;y) = \frac{\partial s^t}{\partial T} = \frac{\partial s^t}{\partial y_i} T_i = -\frac{\partial s^t}{\partial x_i} T_i = -s^t_i T_i
\]

where \( T \) is the unit vector tangent to the boundary at \( y \).

5.2.1. Kernels in the \( q_I \)-BIE for the Laplace problems

The kernels in the \( q \)-BIE for the Laplace problem have the following expression:

\[
d^t(x;y) = \frac{-1}{2\pi r^2} \rho \cdot N; \quad s^t(x;y) = \frac{-1}{2\pi r^2} (n \cdot N - 2\rho \cdot n_p \cdot N)
\]

where \( N \) is the outward normal to the boundary at the point \( y \).

The kernel \( d^t(x;y) \) is regular, while \( s^t(x;y) \) is hyper-singular \( O(r^{-3}) \), and special regularization formulas are required for the computation of its Finite Part.

The kernels for the tangential \( q \)-BIE are obtained by differentiation leading to

\[
d^t_i(x;y) = \frac{1}{2\pi r^2} (\kappa_0 r - 2\rho \cdot N) \rho \cdot T
\]

\[
s^t_i(x;y) = -\frac{1}{\pi r^3} \left( \rho \cdot T(n \cdot N - 4\rho \cdot n_p \cdot N) + n \cdot Tp \cdot N - \frac{rK_0}{2} (n \cdot T - 2\rho \cdot Tp \cdot n) \right)
\]

where \( \kappa_0 \) is the boundary curvature at point \( y \).

The kernel \( d^t_i(x;y) \) is regular and \( s^t_i(x;y) \) is highly singular, \( O(r^{-3}) \), so special regularization formulas are required to compute its Finite Part.

5.2.2. Kernels in the \( q_I \)-BIE for the Helmholtz problem

The kernels in the \( q \)-BIE and in \( q_I \)-BIE are highly singular functions. Such kernels have the same order of singularity than the kernels for the \( q \)-BIE and \( q_I \)-BIE for the Laplace problem. The expressions of the kernels in the \( q \)-BIE are

\[
d^t(x;y) = -\frac{ik}{2\pi} K_1 \rho \cdot N
\]

\[
s^t(x;y) = -\frac{ik}{2\pi} [K_1 n \cdot N - (2K_1 + ikrK_0)\rho \cdot n_p \cdot N]
\]

From these kernels, differentiating with respect to the tangential vector at \( y \), the kernels in the \( q_I \)-BIE are obtained,

\[
d^t_i(x;y) = \frac{ik}{2\pi r} [\kappa_0 r K_1 - (2K_1 + ikrK_0)\rho \cdot N] \rho \cdot T
\]

\[
s^t_i(x;y) = -\frac{ik}{2\pi r} (2K_1 + ikrK_0) \left[ \rho \cdot T(n \cdot N - 4\rho \cdot n_p \cdot N) + n \cdot Tp \cdot N - \frac{rK_0}{2} (n \cdot T - 2\rho \cdot Tp \cdot n) \right]
\]

\[
+ \frac{k^2 rK_0}{4\pi} n \cdot T - \frac{ik K_1}{2\pi} \rho \cdot np \cdot Tp \cdot N
\]

6. Applications

In order to validate the formulation proposed and to show some properties of both the tangential BIE residual and nodal sensitivity as error estimators, several numerical examples are presented here. The aims of the different tests are the following:

- To show that residuals in the tangential derivative BIEs can be correlated to actual errors in the tangential derivatives of the solution. Using exact solutions, the computed residuals are compared to the values of the exact error in the tangential derivative. The slope of the linear regression between the exact error and the residual, and the correlation factor (\( R^2 \)) are computed.

- To validate that nodal sensitivities are truly error estimators. Nodal sensitivities have been obtained and contrasted with errors in tangential derivatives for mid-points of the quadratic elements. Again, the slope of the regression and the correlation factor are computed to test the accuracy of the sensitivity as error estimator.

For all of these applications there are various common numerical aspects:

1. Element type: Quadratic isoparametric elements.

2. Collocation for \( u \)-BIE and \( q \)-BIE is always performed at smooth points. At corners, out-of-node collocation is employed, shifting the collocation points inside the element but close to the end node (collocation points \( \xi = \pm 0.8 \), where \( \xi \) is the natural coordinate). Nevertheless, the interpolation nodes are at the end nodes (conforming elements).

3. Except at corners, it is assumed that the boundary normal at inter-element nodes is continuous, so these nodes are treated as smooth boundary points. It is well known in the literature that a spurious logarithmic singularity remains if the \( q \)-BIE is
collocated in such nodes. This effect is dependent on the difference of tangential vectors at the inter-element nodes, and on the difference of potential tangential derivatives. The effect is very important for isoparametric linear elements, but can be neglected for isoparametric quadratic elements if using a fine enough mesh. Nevertheless, an out-of-node collocation approach can be adopted for these nodes, avoiding altogether this minor source of computational error.

(4) Tangential BIE residuals and nodal sensitivities are collocated only at the element mid-nodes, so all continuity requirements for the boundary and the variables are fulfilled.

(5) Numerical integration with Gauss quadrature is performed with 10–15 points. For singular and hypersingular terms, after regularization, double number of points is employed.

(6) For the computation of the approximate solution a mixed formulation is used: \( u \)-BIE at nodes where the potential is unknown and \( q \)-BIE where the flux is.

The following numerical applications are included in this section:

(1) Laplace equation:
   (a) Rectangular plate with Dirichlet boundary conditions (Fig. 2, \( a = 100, b = 50 \)).
   (b) Rectangular plate with mixed boundary conditions (Fig. 2, \( a = 100, b = 50 \)).
   (c) Disk sector with mixed boundary conditions (Fig. 3, \( R_1 = 50, R_2 = 100 \)).

(2) Helmholtz equation:
   (a) Rectangular plate with mixed boundary conditions (Fig. 2, \( a = 100, b = 50 \)).
   (b) Disk sector with mixed boundary conditions (Fig. 3, \( R_1 = 50, R_2 = 100 \)).

In order to evaluate the exact errors, analytical solutions of these problems are provided. To leave out unavoidable sources of error, the boundary conditions vary quadratically at most along the boundary, so they are exactly represented with the quadratic elements.

For each case, two results are displayed:

(1) Table of values: at mid-nodes, the following values are listed:
   - Exact value of Error in Tangential Derivative (ETD).
   - Tangential Boundary Integral Equation Residual corresponding to the unknown at the point (TBIER).
   - Nodal Sensitivity for the unknown (NS).

(2) \( X-Y \) graphic: The exact error (\( X \) axis) vs. the error estimators (\( Y \) axis) are shown. The regression lines for exact tangential error vs. residual, and exact tangential error vs. nodal sensitivities are plotted as well. The better the error estimator, the closer to 1.0 the regression line slope will be. However even if the slope is not 1.0 the closer the correlation factor is to 1.0, the better the expression as error indicator.

6.1. Error estimators: application for Laplace problems

6.1.1. Rectangular plate with Dirichlet boundary conditions

The domain in the first application is a rectangular plate \( a \times b \) with Dirichlet boundary conditions. At the upper and lower sides \( u = 0 \) while at the left and right side the prescribed potential varies quadratically, attaining its maximum value \( \gamma_2 = \gamma_4 = 1.0 \) at the center of each side, as shown in Fig. 2.

The analytical solution for this problem can be computed in series form:
\[ u(x,y) = \sum_{n=1}^{\infty} (A_n e^{k_n x} + B_n e^{-k_n x}) \sin(k_n y) \]  

(27)

The coefficients \( A_n \) and \( B_n \) are obtained solving

\[ A_n + B_n = a_n \]

\[ A_n e^{k_n a} + B_n e^{-k_n a} = \beta_n \]

where

\[ a_n = \frac{b y_n}{2} \int_0^b \left[ 1 - \frac{4}{b^2} \left( y - \frac{b}{2} \right)^2 \right] \sin(k_n y) \, dy \]

\[ \beta_n = \frac{b y_n}{2} \int_0^b \left[ 1 - \frac{4}{b^2} \left( y - \frac{b}{2} \right)^2 \right] \sin(k_n y) \, dy \]

and \( k_n = \frac{\pi}{b} \), \( n = 1, 2, \ldots \).

The edges \( \Gamma_1 \) and \( \Gamma_3 \) have been divided into six quadratic elements, and \( \Gamma_2 \) and \( \Gamma_4 \) have been divided into three quadratic elements.

Table 1 lists the results at the mid nodes of the elements on \( \Gamma_1 \) and \( \Gamma_2 \).

In Fig. 4, the slope of the regression line between the Tangential BIE Residual and the exact tangential derivative error is 1.0691 while the correlation factor is 0.9958; the regression slope between the exact tangential derivative error and the nodal sensitivity is 1.319, and the correlation factor is 0.9939.

### 6.1.2. Rectangular plate with mixed boundary conditions

For this test, the rectangular domain in Fig. 2 is subject to mixed boundary conditions. At the upper and lower edge the potential is again fixed to \( u = 0 \) while on the right and left side, the flux is prescribed. The flux varies quadratically along both edges attaining its maximum at the center, as shown in figure. For this application \( \gamma_2 = 0.1 \) and \( \gamma_4 = -0.2 \).

The analytical solution for this test is given again by (27), but with coefficients \( A_n \) and \( B_n \) computed by the following equations:

\[ A_n - B_n = -a_n \]

\[ A_n e^{k_n a} - B_n e^{-k_n a} = \beta_n \]

where

\[ a_n = \frac{b y_n}{2k_n} \int_0^b \left[ 1 - \frac{4}{b^2} \left( y - \frac{b}{2} \right)^2 \right] \sin(k_n y) \, dy \]

\[ \beta_n = \frac{b y_n}{2k_n} \int_0^b \left[ 1 - \frac{4}{b^2} \left( y - \frac{b}{2} \right)^2 \right] \sin(k_n y) \, dy \]

and \( k_n = \frac{\pi}{b} \), \( n = 1, 2, \ldots \).

The boundaries \( \Gamma_1 \) and \( \Gamma_3 \) have been divided into six quadratic elements and \( \Gamma_2 \) and \( \Gamma_4 \) into three. Table 2 lists the results along boundaries \( \Gamma_1, \Gamma_2 \). Note that on \( \Gamma_1 \) the error is in the tangential derivative of \( q \), while on \( \Gamma_2 \) the error is in the tangential derivative of \( u \). Therefore, on each boundary, the corresponding residual and sensitivity is computed.

In Fig. 5 the value of the exact error vs. both error estimators, plus the corresponding regression lines are shown. The slope of the regression line between the exact error and the tangential BIE residual is 1.0047 and

---

**Table 1**

<table>
<thead>
<tr>
<th>Boundary</th>
<th>ETD</th>
<th>TBIER</th>
<th>NS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1 )</td>
<td>-3.325636298E–3</td>
<td>-3.374462729E–3</td>
<td>-4.10602836E–3</td>
</tr>
<tr>
<td></td>
<td>-4.453948290E–4</td>
<td>-4.367052738E–4</td>
<td>-5.93809616E–4</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
<td>9.422479854E–5</td>
<td>9.160759406E–5</td>
<td>1.347324937E–4</td>
</tr>
<tr>
<td></td>
<td>4.453854413E–4</td>
<td>4.366666657E–4</td>
<td>5.938263493E–4</td>
</tr>
<tr>
<td></td>
<td>-1.278448867E–7</td>
<td>5.150470520E–8</td>
<td>5.495116541E–8</td>
</tr>
<tr>
<td></td>
<td>2.703324411E–3</td>
<td>3.12473954E–3</td>
<td>3.910177098E–3</td>
</tr>
</tbody>
</table>
Table 2
Exact Error in Tangential Derivative (ETD), Tangential BIE Residual (TBIER) and Nodal Sensitivity (NS) for rectangular plate under mixed boundary conditions (Laplace’s problem)

<table>
<thead>
<tr>
<th>Boundary</th>
<th>ETD</th>
<th>TBIER</th>
<th>NS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1 )</td>
<td>4.53663888E−4</td>
<td>4.14285845E−4</td>
<td>3.18110221E−4</td>
</tr>
<tr>
<td></td>
<td>1.34142697E−4</td>
<td>1.34086832E−4</td>
<td>1.69998034E−4</td>
</tr>
<tr>
<td></td>
<td>5.18417442E−5</td>
<td>5.17010282E−5</td>
<td>1.00673969E−4</td>
</tr>
<tr>
<td></td>
<td>3.73564477E−5</td>
<td>3.73191354E−5</td>
<td>8.12908204E−5</td>
</tr>
<tr>
<td></td>
<td>7.03220870E−5</td>
<td>7.08433254E−5</td>
<td>1.05712373E−4</td>
</tr>
<tr>
<td></td>
<td>2.15149554E−4</td>
<td>2.00368683E−4</td>
<td>1.58525040E−4</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
<td>2.46090409E−3</td>
<td>2.45290635E−3</td>
<td>3.32952188E−3</td>
</tr>
<tr>
<td></td>
<td>1.17265264E−7</td>
<td>7.58850118E−8</td>
<td>1.06284234E−7</td>
</tr>
<tr>
<td></td>
<td>−2.46103689E−3</td>
<td>−2.45309681E−3</td>
<td>−3.32965020E−3</td>
</tr>
</tbody>
</table>

Fig. 5. Exact Error in Tangential Derivative vs. estimators for rectangular plate with mixed boundary conditions (Laplace’s problem).

The correlation factor 0.9999; the slope of the regression line between the exact error and the nodal sensitivity is 1.2859 while its correlation factor is 0.9977.

An enlargement of this graphics for small values of the error is presented in Fig. 6.

Fig. 6. Exact Error in Tangential Derivative vs. estimators for rectangular plate with mixed boundary conditions: range of small values (Laplace’s problem).

6.1.3. Disk sector with mixed boundary conditions

In this test, the suitability and reliability of the procedure and its implementation for curved elements is assessed. The geometry and variation of the prescribed boundary conditions is shown in Fig. 3. The edges \( \Gamma_1 \) and \( \Gamma_2 \) are both meshed with four equal size elements and \( \Gamma_3 \) and \( \Gamma_4 \) with ten elements each one. On the straight edges \( \Gamma_1 \) and \( \Gamma_4 \) the potential is prescribed to \( u = 0 \), while the flux varies quadratically along \( \Gamma_2 \) and \( \Gamma_3 \), as shown in the figure, with \( \gamma_2 = 0.1 \), and \( \gamma_4 = 0.2 \).

The analytical solution for this problem is

\[
u(r, \theta) = \sum_{n=1}^{\infty} \left( A_n r^{2n} + B_n r^{-2n} \right) \sin(2n\theta)\]

where

\[
A_n r_2^{2n-1} - B_n r_1^{-2n-1} = \alpha_n
\]

\[
A_n r_1^{2n-1} - B_n r_2^{-2n-1} = -\beta_n
\]

The right side terms are computed by

\[
\alpha_n = \frac{\pi \gamma_2}{8n} \int_0^{\pi/2} \left[ 1 - \frac{16}{\pi^2} (\theta - \frac{\pi}{4})^2 \right] \sin(2n\theta) \, d\theta
\]

\[
\beta_n = \frac{\pi \gamma_4}{8n} \int_0^{\pi/2} \left[ 1 - \frac{16}{\pi^2} (\theta - \frac{\pi}{4})^2 \right] \sin(2n\theta) \, d\theta
\]

Table 3 lists the results over boundaries \( \Gamma_1 \), \( \Gamma_2 \) for mid-nodes at each element while in Fig. 7 the same values and the regression lines between exact error and estimators are shown. The regression slopes between the exact error and the residual estimator is 0.765, while it is 1.24 for the exact error vs. sensitivity; the correlation factors are 0.9932 and 0.6085, respectively.

6.2. Error estimators: applications for Helmholtz problems

Numerical tests with Helmholtz problems have several common points:
Table 3
Exact Error in Tangential Derivative (ETD), Tangential BIE Residual (TBIER) and Nodal Sensitivity (NS) for disk sector with mixed boundary conditions (Laplace’s problem)

<table>
<thead>
<tr>
<th>Boundary</th>
<th>ETD</th>
<th>TBIER</th>
<th>NS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1 )</td>
<td>(-6.46418E-4)</td>
<td>(-4.58338E-4)</td>
<td>(-3.76454E-4)</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
<td>(6.83644E-4)</td>
<td>(5.25465E-4)</td>
<td>(-3.08525E-4)</td>
</tr>
</tbody>
</table>

Fig. 7. Exact Error in Tangential Derivative vs. estimators for disk sector with mixed boundary conditions (Laplace’s problem).

Table 4
Exact Error in Tangential Derivative (ETD), Tangential BIE Residual (TBIER) and Nodal Sensitivity (NS) for rectangular plate with mixed boundary conditions: real part

<table>
<thead>
<tr>
<th>Boundary</th>
<th>ETD</th>
<th>TBIER</th>
<th>NS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1 )</td>
<td>(-6.40312971E-6)</td>
<td>(-3.68496821E-6)</td>
<td>(-2.25227392E-5)</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
<td>(1.60055280E-4)</td>
<td>(1.41391825E-4)</td>
<td>(9.24150426E-5)</td>
</tr>
</tbody>
</table>
Table 5
Exact Error in Tangential Derivative (ETD), Tangential BIE Residual (TBIER) and Nodal Sensitivity (NS) for rectangular plate with mixed boundary conditions: imaginary part

<table>
<thead>
<tr>
<th>Boundary</th>
<th>ETD</th>
<th>TBIER</th>
<th>NS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1 )</td>
<td>(-3.10705826E-4)</td>
<td>(-2.77030678E-4)</td>
<td>(-1.84500130E-4)</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
<td>(3.83913326E-5)</td>
<td>(3.3626634E-4)</td>
<td>(1.82730614E-4)</td>
</tr>
</tbody>
</table>

The expansion functions \( f_n(x) \) and \( g_n(x) \) are harmonic or exponential functions, depending on the value of \( \lambda \) and \( n \). For \( \lambda < 2b/n \), and \( k \neq \omega_n \), they are

\[
 f_n(x) = \cos(k_n x); \quad g_n(x) = \sin(k_n x);
\]

where \( k_n = \sqrt{k^2 - \omega_n^2} \)

For values \( \lambda > 2b/n \) the functions are

\[
 f_n(x) = e^{k_n x}; \quad g_n(x) = e^{-k_n x}; \quad \text{where} \quad k_n = \sqrt{\omega_n^2 - k^2}
\]

Boundaries \( \Gamma_1 \) and \( \Gamma_3 \) are divided into eight quadratic elements, and boundaries \( \Gamma_2 \) and \( \Gamma_4 \) into four elements. Tables 4 and 5 lists the results over boundaries \( \Gamma_1 \) and \( \Gamma_2 \), for real and imaginary parts. Figs. 8 and 9 shows the exact errors in tangential derivatives vs. both error estimators.

The regression slopes and correlation factors between the true error and the estimators are shown in Table 6.

6.2.2. Disk sector with mixed boundary conditions

The domain of this test is shown in Fig. 3. Boundary conditions have been prescribed in potential and flux values. The potential is prescribed to \( u = 0 \) on \( \Gamma_1 \) and \( \Gamma_3 \), and the flux on \( \Gamma_2 \) and \( \Gamma_4 \) with \( \gamma_2 = 0.2 \) and \( \gamma_4 = 0.1i \).
The analytical solution for this problem can be obtained by a series expansion in terms of Bessel functions using polar coordinates:

\[ u(r, \theta) = \sum_{n=1}^{\infty} \left( A_n J_{2n}(kr) + B_n Y_{2n}(kr) \right) \sin(2n\theta) \]  

(30)

The coefficients in this series are the solution of the equations:

\[ A_n J'_{2n}(kr_2) + B_n Y'_{2n}(kr_2) = \alpha_n \]

\[ A_n J'_{2n}(kr_1) + B_n Y'_{2n}(kr_1) = -i \beta_n \]

where

\[ \alpha_n = \frac{\pi n}{4k} \int_{0}^{\pi} \left[ 1 - \frac{16}{\pi^2} \left( \theta - \frac{\pi}{4} \right)^2 \right] \sin(2n\theta) \, d\theta \]

\[ \beta_n = \frac{\pi n}{4k} \int_{0}^{\pi} \left[ 1 - \frac{16}{\pi^2} \left( \theta - \frac{\pi}{4} \right)^2 \right] \sin(2n\theta) \, d\theta \]

The derivatives of the Bessel functions are taken with respect to \( z = kr \).

![Fig. 10. Exact Error in Tangential Derivative vs. estimators for disk sector with mixed boundary conditions: real part (Helmholtz problem).](image)
All boundaries have been divided into six quadratic elements. Tables 7 and 8 show the results obtained for mid-nodes of each element. Figs. 10 and 11 show the values and regression lines, for real and imaginary parts. The regression slopes and correlation factors between the true error and the estimators are shown in Table 9.

Table 7 shows the results obtained for mid-nodes of each element. Figs. 10 and 11 show the values and regression lines, for real and imaginary parts.

The regression slopes and correlation factors between the true error and the estimators are shown in Table 9.

7. Conclusions

In this paper a new error estimator based on tangential BIE residual (TBIER) is proposed. Theoretical considerations lead to its relationship to the true error in the tangential derivative and several numerical applications show that the values predicted by the TBIER closely correlate to the exact error in the tangential derivative. The estimator is developed both for Laplace and Helmholtz problems, and both for the error in the potential and flux derivative.

The TBIER estimates the error in the tangential derivative, which is more significant than error in the variable itself for mesh refinement, since it avoids the refinement of parts of the boundary where the error in the variable stems from a coarse representation somewhere else in the model. Mesh adaptation schemes based on this error estimator can be devised in a general mesh adaptation solution, and various schemes can be used.

Note that for Laplace’s problem on a domain with straight edges, the correlation between the TBIER and the exact error is almost perfect. In other cases there are differences, but the source of the discrepancies may be traced to numerical inaccuracies due to the integration of the hypersingular kernels over curved elements and procedures for the dynamic kernel regularization which can be improved further.

It is also shown that nodal sensitivities are error truly estimators for the error in the tangential derivative. Again both Laplace and Helmholtz problems and both potential and flux sensitivities are considered. This error estimator, however, is more expensive than the TBIER since it requires the computation of the TBIER itself at perturbation points and the solution of a system of equations to obtain the sensitivities.

Furthermore, the TBIER is shown to be a more accurate error estimator for tangential derivative errors than the nodal sensitivities, and therefore there is no reason to use the nodal sensitivities for error estimation.

This method could be straightforwardly applied to other BEM formulations: 3D potential problems, elasticity problems, etc, and readily extended to other methods based on boundary integral equations: symmetric Galerkin, standard meshless, hypersingular boundary node method, and boundary contour method, as it has been done with the hypersingular residual estimator.

References


