BIG POINTS IN $C^*$-ALGEBRAS AND $JB^*$-TRIPLES
by JULIO BECERRA GUERRERO†
(Universidad de Granada, Facultad de Ciencias, Departamento de Matemática Aplicada, 18071-Granada, Spain)

and A. RODRÍGUEZ-PALACIOS‡
(Universidad de Granada, Facultad de Ciencias, Departamento de Análisis Matemático, 18071-Granada, Spain)

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Abstract
Let $X$ be a Banach space. For a norm-one element $u$ in $X$ we put $s(X, u) := \sup \{ \|\psi - \Pi(\psi)\| : \psi \in D(X^{**}, u) \}$, where $D(X^{**}, \cdot)$ denotes the duality mapping of $X^{**}$, and $\Pi : X^{**} \to X^*$ stands for the Dixmier projection. The element $u$ is said to be a big point of $X$ if the closed convex hull of the orbit of $u$ under the group of all surjective isometries on $X$ is the closed unit ball of $X$. We prove that, if $X$ is either a $C^*$-algebra or the predual of a von Neumann algebra, and if there is a big point $u$ of $X$ with $s(X, u) < 2$ and such that the norm of $X$ is strongly subdifferentiable at $u$ (in the sense [36]), then $X$ is finite-dimensional, and the big points of $X$ are precisely the extreme points of $B_X$. This is derived from a more general result on $JB^*$-triples and preduals of $JBW^*$-triples.

1. Introduction
Throughout this paper $\mathbb{K}$ will mean the field of real or complex numbers. Given a normed space $X$ over $\mathbb{K}$, $S_X$, $B_X$, and $X^*$ will denote, respectively, the unit sphere, the closed unit ball, and the (topological) dual of $X$, and $\mathcal{G}_X$ will stand for the group of all surjective linear isometries from $X$ to $X$. We say that an element $u$ in a normed space $X$ is a big point of $X$ if $\overline{\text{co}}(\mathcal{G}_X(u)) = B_X$, where $\overline{\text{co}}$ means closed convex hull. Note that big points of $X$ lie in $S_X$ (unless $X = 0$). The normed space $X$ is said to be convex-transitive if all elements in $S_X$ are big points of $X$. The space $X$ is said to be transitive (respectively, almost transitive) if, for every (equivalently, some) element $u$ in $S_X$, we have $\mathcal{G}_X(u) = S_X$ (respectively, $\overline{\text{co}}(\mathcal{G}_X(u)) = S_X$). The notions just defined provide us with a chain of implications

transitivity $\Rightarrow$ almost transitivity $\Rightarrow$ convex transitivity,

none of which is reversible.

The literature dealing with transitivity conditions on normed spaces is linked to the Banach–Mazur ‘rotation’ problem [2] of whether every transitive separable Banach space is linearly isometric to $\ell_2$. The reader is referred to the book of Rolewicz [49] and the survey papers of

† E-mail: juliobg@ugr.es
‡Corresponding author. E-mail: apalacio@ugr.es
Cabello [19] and the authors [9] for a comprehensive view of known results and fundamental questions in relation to this matter.

In [8] we showed that the existence in a Banach space $X$ of a big point $u$ such that

$$
\eta(X, u) := \limsup_{\|h\| \to 0} \frac{\|u + h\| + \|u - h\| - 2}{\|h\|} < 2
$$

implies that $X$ is superreflexive, and that if in fact, the norm of $X$ is Fréchet differentiable at the big point $u$ (equivalently, $\eta(X, u) = 0$) then $X$ is in addition almost transitive. These results suggest that some other ‘smooth’ behaviours of a Banach space at their big points could imply relevant properties of isomorphic or isometric type. In the present paper we choose as smooth behaviour of a Banach space $X$ at a point $u \in S_X$ the one such that $\sigma(X, u)$ is ‘small’. Here $\sigma(X, u)$ is defined by the equality

$$
\sigma(X, u) := \sup \{ \|\psi - \Pi_X(\psi)\| : \psi \in D(X^{**}, u) \},
$$

where, for any normed space $Y$, $D(Y, \cdot)$ denotes the duality mapping of $Y$, and $\Pi_Y : Y^{***} \to Y^*$ stands for the Dixmier projection. The constant $\sigma(X, u)$ is implicitly considered in [34], where it is shown that if $X$ is a Banach space, and if there is $0 \leq k < 1$ such that $\sigma(X, u) \leq k$ for every $u \in S_X$, then $X$ is Asplund and no proper closed subspace of $X^*$ is norming for $X$.

In Theorem 2.5 we shall prove that, if $X$ is a Banach space, and if there is a big point $u$ of $X$ with $\sigma(X, u) < 1$ and such that the norm of $X$ is strongly subdifferentiable at $u$, then $X$ is Asplund and no proper closed subspace of $X^*$ is norming for $X$. The strong subdifferentiability of the norm of a Banach space was introduced in [36], becoming the natural succedaneus of the Fréchet differentiability of the norm when smoothness is not required. We note that not much can be expected from the existence of a big point $u$ such that the norm of $X$ is strongly subdifferentiable at $u$. Indeed, for every unital $C^*$-algebra $X$, the unit $u$ of $X$ is a big point of $X$ (by the Russo–Dye theorem [15, Theorem 30.2]) such that the norm of $X$ is strongly subdifferentiable at $u$ (by [1, Theorem 2.7, 48, Proposition 3]). We apply Theorem 2.5 and the tools in its proof, together with some results taken from [6, 8], to obtain in Proposition 2.9 new characterizations of almost transitive superreflexive Banach spaces. Such spaces were first considered by Finet [30] (see also [26, Corollary IV.5.7]), who proves that they are uniformly smooth and uniformly convex, and have been revisited recently by Cabello [21] and the authors (in [6, 8] just quoted). Theorem 2.5 will also become one of the crucial tools in the proof of one of the main results in the paper (namely, Theorem 4.12).

Section 3 is devoted to collect other results on Banach spaces, which will be useful in the proof of the two main theorems of the paper (namely, Theorems 4.1 and 4.12). We include also a consequence of one of such auxiliary results, which will be not applied later, but has its own interest. Thus, among other things, Corollary 3.5 asserts that, if $X$ is a complex Banach space having a big point $u$ with $\sigma(X, u) < 1/e$, then the commutator of $G_X$ (in the algebra $L(X)$ of all bounded linear operators on $X$) is a reflexive Banach space. We note that there exist unital $C^*$-algebras $X$ such that the commutator of $G_X$ contains $\ell_\infty$ [10, Example 3.1].

Section 4 contains the main results of the paper. Centering in a first instance in the setting of $C^*$-algebras, we prove that if $X$ is either a $C^*$-algebra or the predual of a von Neumann algebra, and if there is a big point $u$ of $X$ with $\sigma(X, u) < 2$ and such that the norm of $X$ is strongly subdifferentiable at $u$ then $X$ is finite-dimensional and the big points of $X$ are precisely the extreme
points of \( B_X \) (Corollaries 4.6 and 4.13). In the case that \( X \) is a \( C^* \)-algebra, the assumption that the norm of \( X \) is strongly subdifferentiable at \( u \) is superabundant, and the remaining requirements on \( X \) actually characterize finite-dimensional \( C^* \)-algebras among all \( C^* \)-algebras (see again Corollary 4.6). The situation for preduals of von Neumann algebras is rather different. Indeed, although the norm of every finite-dimensional Banach space is strongly subdifferentiable at every point of its unit sphere, the dual of a finite-dimensional \( C^* \)-algebra \( X \) has big points if and only if \( X \) is a finite \( \ell_\infty \)-sum of copies of \( \mathcal{L}(H) \) for some finite-dimensional complex Hilbert space (see again Corollary 4.13). The results on \( C^* \)-algebras and preduals of von Neumann algebras just reviewed follow almost straightforwardly from more general ones on \( JB^* \)-triples and preduals of \( JBW^* \)-triples (Theorems 4.1 and 4.12, respectively). The formulations of such more general results are very similar to those already reviewed in the \( C^* \)-algebra setting. Indeed, if \( X \) is either a \( JB^* \)-triple or the predual of a \( JBW^* \)-triple, and if there is a big point \( u \) of \( X \) with \( \sigma(X,u) < 2 \) and such that the norm of \( X \) is strongly subdifferentiable at \( u \), then the Banach space of \( X \) is isomorphic to a Hilbert space, and the big points of \( X \) are precisely the extreme points of \( B_X \). As above, in the case that \( X \) is a \( JB^* \)-triple, the assumption that the norm of \( X \) is strongly subdifferentiable at \( u \) is superabundant, and the remaining requirements on \( X \) characterize the \( JB^* \)-triples whose Banach spaces are isomorphic to Hilbert spaces, whereas in the case that \( X \) is the predual of a \( JBW^* \)-triple, the requirements on \( X \) characterize the preduals of those \( JB^* \)-triples which are finite \( \ell_\infty \)-sums of copies of a simple \( JB^* \)-triple of ‘finite rank’. We note that, in the theory of \( JB^* \)-triples, the property of finite rank \([42]\) play a roll similar to that of finite-dimensionality in the \( C^* \)-algebra setting. Moreover, we point out the well-known fact that a \( JB^* \)-triple \( X \) is of finite rank if and only if the Banach space of \( X \) is isomorphic to a Hilbert space, if and only if all single-generated subtriples of \( X \) are finite-dimensional.

In the concluding section (section 5) we apply Theorems 4.1 and 4.12 just reviewed to obtain new characterizations of complex Hilbert spaces among \( JB^* \)-triples and preduals of \( JBW^* \)-triples. We prove that a complex Banach space \( X \) is a Hilbert space if (and only if) it is either a \( JB^* \)-triple or the predual of a \( JBW^* \)-triple, and there exists a big point \( u \) in \( X \) such that \( \eta(X,u) < 2 \) (Theorem 5.2). For other results in the same line the reader is referred to \([51, 7, 9]\).

2. A theorem for Banach Spaces

Let \( X \) be a normed space over \( \mathbb{K} \). For \( u \) in \( B_X \) we define the set \( D(X,u) \) of all states of \( X \) relative to \( u \) by

\[
D(X,u) := \{ f \in B_{X^*} : f(u) = 1 \},
\]

which is non-empty if and only if \( u \) belongs to \( S_X \). If this is the case then \( D(X,u) \) is a non-empty \( \omega^* \)-closed face of \( B_{X^*} \). The set-valued function \( v \mapsto D(X,v) \) on \( S_X \) is called the duality mapping of \( X \). We denote by \( \Pi_X \) the canonical projection from \( X^{**} \) onto \( X^* \) and, for \( u \) in \( S_X \), we put

\[
\sigma(X,u) := \sup \{ \| \psi - \Pi_X(\psi) \| : \psi \in D(X^{**},u) \}.
\]

Given a non-negative number \( k \), we denote by \( \sigma^k(X) \) the set of those elements \( v \) in \( S_X \) such that \( \sigma(X,v) \leq k \). We recall that a subspace \( P \) of \( X^* \) is called a norming subspace for \( X \) if for every \( x \) in \( X \) we have

\[
\| x \| = \sup \{ |f(x)| : f \in S_P \},
\]
and that $X$ is said to be *nicely smooth* if $X^*$ contains no proper closed norming subspace. Concerning nicely smooth Banach spaces, the reader is referred to [35, 3].

Most results in this section are inspired by the Giles–Gregory–Sims paper [34]. In particular, the proof of Proposition 2.1 immediately below follows the lines of that of [34, Theorem 3.3]. There it is shown that, if $X$ is a Banach space, and if there is $0 \leq k < 1$ such that $\sigma^k(X) = S_X$, then $X$ is nicely smooth and Asplund.

**Proposition 2.1** Let $X$ be a Banach space over $\mathbb{K}$ such that there is $0 \leq k < 1$ in such a way that the interior of $\sigma^k(X)$ relative to $S_X$ contains big points of $X$. Then $X$ is nicely smooth.

**Proof.** Assume that there exists a proper closed norming subspace of $X^*$ (say $P$). By Riesz’s lemma, there exists $h$ in $S_{X^*}$ satisfying $\|h + P\| > k$. Now, let $u$ and $\delta$ be a big point of $X$ and a positive number, respectively, such that $x$ belongs to $\sigma^k(X)$ whenever $x \in S_X$ and $\|x - u\| \leq \delta$. Since $u$ is a big point of $X$, the set

$$\{T^*(f) : f \in D(X,x), x \in S_X, \quad \|x - u\| \leq \delta, \quad T \in G_X\}$$

is dense in $S_{X^*}$ [9, Lemma 5.7]. It follows that there exist $T \in G_X$, $x \in \sigma^k(X)$, and $f \in D(X,x)$ satisfying $\|T^*(f) + P\| > k$. Then, putting $Q := (T^*)^{-1}(P)$, we have $\|f + Q\| > k$, and hence the Hahn–Banach theorem provides us with some $\alpha$ in $S_{X^{**}}$ such that $\alpha(Q) = 0$ and $|\alpha(f)| > k$. Since $Q$ is a norming subspace of $X^*$, we have $\|y\| \leq \|y + \beta\|$ for every $y$ in $X$ and every $\beta$ in the polar $Q^o$ of $Q$ in $X^{**}$. This implies that $X \cap Q^o = 0$ and that the unique linear extension of $f$ to $X \oplus Q^o$ which vanishes on $Q^o$ is in fact a Hahn–Banach extension (say $g$). Now, take a Hahn–Banach extension of $g$ to $X^{**}$ (say $\psi$). Since $\alpha$ belongs to $Q^o$, the equality $\psi(\alpha) = 0$ holds. On the other hand, since $\psi$ extends $f$, we have $\Pi_X(\psi) = f$. It follows that

$$\|\psi - \Pi_X(\psi)\| \geq |(\psi - \Pi_X(\psi))(\alpha)| = |f(\alpha)| > k.$$ 

Since $\psi$ belongs to $D(X^{**}, x)$, and $x$ belongs to $\sigma^k(X)$, the inequality $\|\psi - \Pi_X(\psi)\| > k$ just obtained becomes a contradiction.

Let $X$ be a Banach space fulfilling the requirements in Proposition 2.1. As a consequence of that proposition, we are provided with the relevant isomorphic property that dens $(X^*) = \text{dens } (X)$, where dens(.) means density character. In what follows we will prove that all requirements on $X$ are inherited by certain closed separable subspaces of $X$, the abundance of which is enough to deduce that $X$ becomes an Asplund space. The following lemma is a variant of [20, Theorem 1.2].

**Lemma 2.2** Let $X$ be a normed space over $\mathbb{K}$, $M$ be a separable subspace of $X$, and $u$ a big point of $X$. Then there exists a closed separable subspace $N$ of $X$ containing $\mathbb{K}u + M$ and such that $u$ becomes a big point of $N$.

**Proof.** Put $Y_1 := \mathbb{K}u + M$, and choose a countable dense subset $D_1$ of $S_{Y_1}$. For $d$ in $D_1$ and $m$ in $\mathbb{N}$, there exists a finite subset $G_{d,m}$ of $G_X$ such that the distance from $d$ to the convex hull of $G_{d,m}(u)$ is less than $1/m$. Denoting by $G_1$ the subgroup of $G_X$ generated by $\cup_{(d,m) \in D_1 \times \mathbb{N}} G_{d,m}$, $G_1$ is a countable set and we have $D_1 \subseteq \sigma(G_1(u))$. Now, denote by $Y_2$ the closed linear hull of $G_1(Y_1)$, and choose a countable dense subset $D_2$ of $S_{Y_2}$ containing $D_1$. Minor changes in the above argument allow us to show the existence of a countable subgroup $G_2$ of $G_X$ containing $G_1$ and satisfying $D_2 \subseteq \sigma(G_2(u))$. Proceeding in such a way we obtain increasing sequences $\{Y_n\}$, $\{D_n\}$, and $\{G_n\}$ such that, for every $n$ in $\mathbb{N}$, $Y_n$ is a separable subspace of $X$ containing
\[ K_u + M, \; D_n \text{ is a dense subset of } S_Y, \; G_n \text{ is a subgroup of } G_X \text{ satisfying } G_n(Y_n) \subseteq Y_{n+1}, \text{ and the inclusion } D_n \subseteq \overline{\text{co}}(G_n(u)) \text{ holds. Now, the desired subspace } N \text{ is nothing but the closure of } \bigcup_{n \in \mathbb{N}} Y_n \text{ in } X. \text{ Indeed, putting } G_\infty := \bigcup_{n \in \mathbb{N}} G_n, \; G_\infty \text{ becomes a subgroup of } G_X, \; N \text{ is } G_\infty\text{-invariant (so that } G_\infty \text{ can be regarded as a subgroup of } G_N), \text{ and we have } \overline{\text{co}}(G_\infty(u)) = B_N. \]

For background on Asplund spaces the reader is referred to [16, 26].

**Proposition 2.3** Let \( X \) be a Banach space over \( \mathbb{K} \) such that there is \( 0 \leq k < 1 \) in such a way that the interior of \( \sigma^k(X) \) relative to \( S_X \) contains big points of \( X \). Then \( X \) is Asplund.

**Proof.** Let \( M \) be a separable subspace of \( X \). We are going to show that \( M^* \) is separable. Let \( u \) be a big point of \( X \) in the interior of \( \sigma^k(X) \) relative to \( S_X \). By Lemma 2.2, there exists a closed separable subspace \( N \) of \( X \) containing \( \mathbb{K}u + M \) such that \( u \) is a big point of \( N \). On the other hand, as remarked in the proof of [34, Theorem 3.3], we have \( \sigma^k(X) \cap N \subseteq \sigma^k(N) \). Now \( u \) is a big point of \( N \) in the interior of \( \sigma^k(N) \) relative to \( S_N \), so that Proposition 2.1 applies with \( N \) instead of \( X \) to obtain that \( \text{dens}(N^*) = \text{dens}(N) \). Thus \( N^* \) (and hence \( M^* \)) is separable.

Let \( X \) be a normed space over \( \mathbb{K} \), and let \( u \) be in \( S_X \). For \( x \) in \( X \), the mapping \( \alpha \mapsto \|u + \alpha x\| \) from \( \mathbb{R} \) to \( \mathbb{R} \) is convex, and hence there exists 

\[
\tau(u,x) := \lim_{\alpha \to 0^+} \frac{\|u + \alpha x\| - 1}{\alpha}.
\]

It is well known that, for \( x \) in \( X \), the equality 

\[
\tau(u,x) = \max \{ \Re(\lambda) : \lambda \in V(X,u,x) \}
\]

holds, where \( V(X,u,x) \) (called the numerical range of \( x \) relative to \( (X,u) \)) is defined by 

\[
V(X,u,x) := \{ f(x) : f \in D(X,u) \}
\]

(see for instance [28, Theorem V.9.5]). We say that the norm of \( X \) is strongly subdifferentiable at \( u \) if 

\[
\lim_{\alpha \to 0^+} \frac{\|u + \alpha x\| - 1}{\alpha} = \tau(u,x) \quad \text{uniformly for } x \in B_X.
\]

The reader is referred to [36, 1, 31, 11] for a comprehensive view of the usefulness of the strong subdifferentiability of the norm in the theory of Banach spaces. We note that the Fréchet differentiability of the norm of \( X \) at \( u \) is nothing but the strong subdifferentiability of the norm of \( X \) at \( u \) together with the smoothness of \( X \) at \( u \).

**Lemma 2.4** Let \( X \) be Banach space over \( \mathbb{K} \), and let \( u \) be in \( S_X \) such that the norm of \( X \) is strongly subdifferentiable at \( u \). Then the mapping \( x \mapsto \sigma(X,x) \) from \( S_X \) to \( \mathbb{R} \) is upper semicontinuous at \( u \).

**Proof.** Let \( \varepsilon \) be a positive number. By [36, Corollary 4.4; 34, Corollary 2.1], there exists \( \delta > 0 \) such that the inclusion 

\[
D(X^{**},\alpha) \subseteq D(X^{**},u) + \frac{\varepsilon}{3} B_{X^{***}}
\]

holds whenever \( \alpha \) is in \( S_{X^{**}} \) with \( \|\alpha - u\| < \delta \). Let \( x \) be in \( S_X \) such that \( \|x - u\| < \delta \). Given \( \psi \) in
$D(X^{**}, x)$, there exists $\varphi$ in $D(X^{**}, u)$ satisfying $\|\psi - \varphi\| \leq \frac{1}{3} \varepsilon$, and therefore we have

$$\|\psi - \Pi_X(\psi)\| = \|\psi - \varphi - \Pi_X(\psi - \varphi) + \varphi - \Pi_X(\varphi)\| \leq \frac{2}{3} + \sigma(X, u).$$

Since $\psi$ is arbitrary in $D(X^{**}, x)$, we obtain $\sigma(X, x) < \varepsilon + \sigma(X, u)$.

According to Lemma 2.4, if $X$ is a Banach space, if $u$ is a norm-one element of $X$, and if the norm of $X$ is strongly subdifferentiable at $u$, then $u$ lies in the interior of $\sigma^k(X)$ relative to $S_X$ for every real number $k > \sigma(X, u)$. Therefore Theorem 2.5 immediately below follows from Propositions 2.1 and 2.3.

**Theorem 2.5** Let $X$ be a Banach space over $\mathbb{K}$. Assume that there exists a big point $u$ of $X$ satisfying $\sigma(X, u) < 1$ and such that the norm of $X$ is strongly subdifferentiable at $u$. Then $X$ is nicely smooth and Asplund.

Let $X$ be a normed space, and let $u$ be a norm-one element in $X$. We denote by $\delta(X, u)$ the diameter of $D(X, u)$. Since $\sigma(X, u) \leq \delta(X^{**}, u)$, the next corollary follows straightforwardly from Propositions 2.1 and 2.3.

**Corollary 2.6** Let $X$ be a Banach space over $\mathbb{K}$. Assume that there is $0 < k < 1$ such that the interior of the set $\{u \in S_X : \delta(X^{**}, u) \leq k\}$ relative to $S_X$ contains big points of $X$. Then $X$ is nicely smooth and Asplund.

Let $X$ be a normed space, let $u$ be in $S_X$, and let $\tau$ be a vector space topology on $X^*$. Following [34], we say that the duality mapping of $X$ is upper semicontinuous ($n-\tau$) at $u$ if for every $\tau$-neighbourhood of zero (say $B$) in $X^*$ there exists a norm-neighbourhood of $u$ (say $C$) in $S_X$ such that $D(X, x) \subseteq D(X, u) + B$ whenever $x$ belongs to $C$. Denote by $w$ and $n$ the weak and norm topologies on $X^*$, respectively. Corollary 4.4 of [36], applied in the proof of Lemma 2.4, asserts that the strong subdifferentiability of the norm of $X$ at $u$ is equivalent to the upper semicontinuity ($n-n$) of the duality mapping of $X$ at $u$, and hence implies the upper semicontinuity ($n-w$) of the duality mapping of $X$ at $u$. On the other hand, if $X$ is complete, then the upper semicontinuity ($n-w$) of the duality mapping of $X$ at $u$ is equivalent to the fact that $D(X, u)$ is dense in $D(X^{**}, u)$ for the $w^*$-topology of $X^{***}$ [36, Theorem 3.1], which implies $\delta(X^{**}, u) = \delta(X, u)$ and, consequently $\sigma(X, u) \leq \delta(X, u)$.

Now, let $X$ be a Banach space such that there exists a big point $u$ of $X$ in such a way that the norm of $X$ is strongly subdifferentiable at $u$. It follows from Theorem 2.5 and the above comments that, if $\delta(X, u) < 1$, then $X$ is nicely smooth and Asplund. Actually, a better result holds. Indeed, $X$ is superreflexive whenever $\delta(X, u) < 2$ (see Corollary 2.8 below). Through Lemma 2.7 which follows, this result is germinally contained in [8].

For any norm-one element $u$ in a normed space $X$, we define the roughness of $X$ at $u$, $\eta(X, u)$, by

$$\eta(X, u) := \limsup_{\|h\| \to 0} \frac{\|u + h\| + \|u - h\| - 2}{\|h\|}.$$  

The absence of roughness of $X$ at $u$ (that is, $\eta(X, u) = 0$) is nothing but the Fréchet differentiability of the norm of $X$ at $u$ [26, Lemma 1.1.10].
Let $X$ be a normed space over $\mathbb{K}$, and $u$ a norm-one element of $X$. Then $\delta(X, u) \leq \eta(X, u)$. If in addition the norm of $X$ is strongly subdifferentiable at $u$, then we have $\delta(X, u) = \eta(X, u)$.

Proof. We may assume that $\mathbb{K} = \mathbb{R}$. Let $\Gamma$ denote the mapping

$$h \rightarrow \frac{\|u + h\| + \|u - h\| - 2}{\|h\|}$$

from $X \setminus \{0\}$ into $\mathbb{R}$. For $x$ in $S_X$ we have

$$\max\{\lambda : \lambda \in V(X, u, x)\} - \min\{\mu : \mu \in V(X, u, x)\} = \lim_{\alpha \to 0^+} \frac{\|u + \alpha x\| + \|u - \alpha x\| - 2}{\alpha},$$

and hence

$$\max\{\lambda : \lambda \in V(X, u, x)\} - \min\{\mu : \mu \in V(X, u, x)\}$$

becomes a cluster point for $\Gamma$ when $\|h\| \to 0$. Since

$$\delta(X, u) = \sup\{\max\{\lambda : \lambda \in V(X, u, x)\} - \min\{\mu : \mu \in V(X, u, x)\} : x \in S_X\},$$

we deduce that $\delta(X, u)$ is also a cluster point for $\Gamma$ when $\|h\| \to 0$. Therefore we obtain $\delta(X, u) = \limsup_{\|h\| \to 0} \Gamma(h) = \eta(X, u)$.

Now suppose that the norm of $X$ is strongly subdifferentiable at $u$. Let $\varepsilon > 0$. We can find $\rho > 0$ such that

$$\frac{\|u + \alpha x\| - 1}{\alpha} < \tau(u, x) + \frac{\varepsilon}{2}$$

whenever $x$ is in $B_X$ and $0 < \alpha < \rho$. Therefore, for $0 < \|h\| < \rho$ we have

$$\Gamma(h) = \frac{\|u + h\| + \|u - h\| - 2}{\|h\|} < \tau(u, \frac{h}{\|h\|}) + \tau(u, - \frac{h}{\|h\|}) + \varepsilon$$

$$= \max\left\{\lambda : \lambda \in V\left(X, u, \frac{h}{\|h\|}\right)\right\} - \min\left\{\mu : \mu \in V\left(X, u, \frac{h}{\|h\|}\right)\right\} + \varepsilon \leq \delta(X, u) + \varepsilon.$$
member of $\mathcal{J}$ is uniformly smooth and uniformly convex. Recently, the class $\mathcal{J}$ has been revisited by Cabello [21] and the authors (see [6, 8]). According to [6, Corollary 3.3], members of $\mathcal{J}$ are nothing but convex-transitive Asplund spaces. The results in [30, 6] just quoted, together with Proposition 2.3, Theorem 2.5, Corollaries 2.6 and 2.8, and the comments after Corollary 2.6 directly yield some new characterizations of members of $\mathcal{J}$, which we list in the following proposition.

**Proposition 2.9** Let $X$ be a Banach space over $\mathbb{K}$. Each of the following conditions, added to the convex transitivity of $X$, characterizes $X$ as a member of $\mathcal{J}$.

1. There exists $0 \leq k < 1$ such that $\sigma^k(X)$ has non-empty interior relative to $S_X$.
2. There is a norm-one element $u$ in $X$ satisfying $\sigma(X, u) < 1$ and such that the norm of $X$ is strongly subdifferentiable at $u$.
3. There exists $0 \leq k < 1$ such that the set of those elements $u$ in $S_X$ satisfying $\delta(X^{**}, u) \leq k$ has non-empty interior relative to $S_X$.
4. There exists $0 \leq k < 1$ such that the set of those elements $u$ in $S_X$ where the duality mapping of $X$ is upper semicontinuous ($n-w$) at $u$ and $\delta(X, u) \leq k$ has non-empty interior relative to $S_X$.
5. There is a norm-one element $u$ in $X$ satisfying $\delta(X, u) < 2$ and such that the norm of $X$ is strongly subdifferentiable at $u$.

Of course, suitable strengthenings of conditions 1 to 5, added to the convex transitivity of a Banach space $X$, also characterize $X$ as a member of $\mathcal{J}$. For instance, a convex transitive Banach space $X$ lies in $\mathcal{J}$ if (and only if) $\sigma^0(X)$ has non-empty interior relative to $S_X$, and this is the case if (and only if) $\sigma^0(X) = S_X$ (that is, $D(X, u) = D(X^{**}, u)$ for every norm-one element $u$ of $X$). In fact, the last characterization follows straightforwardly from [34, 6]. Note that non-reflexive Banach spaces $X$ satisfying $\sigma^0(X) = S_X$ do exist. Indeed, take $X$ equal to $c_0$, or the space of all compact operators on a Hilbert space, or (more generally) any non-trivial *$M$-embedded* Banach space (see Lemma 3.1 below). Another consequence of Proposition 2.9 is that a convex-transitive Banach space $X$ lies in $\mathcal{J}$ if (and only if) there is a norm-one element $u$ in $X$ such that the norm of $X$ is Fréchet differentiable at $u$. However, as commented in section 1, a better result holds. Indeed, a Banach space $X$ is a member of $\mathcal{J}$ if (and only if) there exists a big point $u$ of $X$ such that the norm of $X$ is Fréchet differentiable at $u$ [8]. In relation to condition 4 in Proposition 2.9, it is worth mentioning that a convex-transitive Banach space $X$ lies in $\mathcal{J}$ if (and only if) the duality mapping of $X$ is upper semicontinuous ($n-w$) at every element of $S_X$. This follows from [23, 6].

3. Other auxiliary results

Let $X$ be a Banach space over $\mathbb{K}$. An $L$-projection on $X$ is a linear projection (say $\pi$) on $X$ satisfying

$$||x|| = ||\pi(x)|| + ||x - \pi||(x)||$$

for every $x \in X$. A subspace $M$ of $X$ is said to be an $L$-summand of $X$ if it is the range of an $L$-projection on $X$, and an $M$-ideal of $X$ if $M^\prime$ (the polar of $M$ in $X^{**}$) is an $L$-summand of $X^\prime$. We say that $X$ is $L$-embedded (respectively, $M$-embedded) whenever $X$ is an $L$-summand (respectively, an $M$-ideal) of $X^{**}$. According to [37, Proposition III.1.2], $X$ is $M$-embedded if
and only if $\Pi_X$ is an $L$-projection on $X^{**}$. Consequently, if $X$ is $M$-embedded, then $X^*$ is $L$-embedded.

**Lemma 3.1** Let $X$ be a Banach space over $\mathbb{K}$, and let $u$ be in $S_X$. If $X^*$ is $L$-embedded, then either $\sigma(X,u) = 0$ or $\sigma(X,u) = 2$. If actually $X$ is $M$-embedded, then $\sigma(X,u) = 0$.

**Proof.** Assume that $X^*$ is $L$-embedded and that $\sigma(X,u) \neq 0$. Then $X^*$ is the range of an $L$-projection $\pi$ on $X^{**}$, and there exists $\psi \in D(X^{**},u)X^*$. Thus we have

$$1 = \Re[(\pi(\psi)(u)) + \Re[(1 - \pi)(\psi)(u)] \leq \|\pi(\psi)\| + \|(1 - \pi)(\psi)\| = 1,$$

so $\Re[(1 - \pi)(\psi)(u)] = \|(1 - \pi)(\psi)\|$, and so $\phi := \frac{(1 - \pi)(\psi)}{\|(1 - \pi)(\psi)\|}$ belongs to $D(X^{**},u)$. Moreover, since $\phi$ belongs to $\ker(\pi)$, and $\Pi_X(\phi)$ belongs to $\pi(X^{**})$, we have

$$\|\phi - \Pi_X(\phi)\| = \|\phi\| + \|\Pi_X(\phi)\| = 2,$$

and hence $\sigma(X,u) = 2$.

Now assume that $X$ is $M$-embedded. Let $\psi$ be in $D(X^{**},u)$. Then, since $\Pi_X$ is an $L$-projection on $X^{**}$, we have

$$\Re[(1 - \Pi_X(\phi)) = 1 - \|\Pi_X(\phi)\| \leq 1 - \|\Pi_X(\psi)(u)\| = 1 - \psi(u) = 0.$$

Since $\psi$ is arbitrary in $D(X^{**},u)$, we deduce $\sigma(X,u) = 0$.

Let $X$ be a normed space over $\mathbb{K}$, and let $u$ be in $S_X$. For $x$ in $X$, the numerical radius of $x$ relative to $(X,u)$, denoted by $v(X,u,x)$, is defined by the equality

$$v(X,u,x) := \sup\{|\lambda| : \lambda \in V(X,u,x)\}.$$ 

The numerical index of $(X,u)$, denoted by $n(X,u)$, is the number

$$n(X,u) := \max\{r \geq 0 : r\|x\| \leq v(X,u,x)\} \text{ for all } x \in X.$$

**Lemma 3.2** Let $X$ be a Banach space over $\mathbb{K}$, and let $u$ be in $S_X$. Then we have $\|1 - \Pi_X\| n(X,u) \leq \sigma(X,u)$.

**Proof.** Let $(x^{**},x^{***})$ be in $X^{**} \times B_X^{***}$. Since $n(X^{**},u) = n(X,u)$ [46, Lemma 4.8], we have

$$n(X,u)\|x^{***}(x^{**})\| \leq v(X^{**},u,x^{**}).$$

Since $x^{**}$ is arbitrary in $X^{**}$, it follows from [46, Theorem 3.1] that $n(X,u)x^{***}$ belongs to the norm closure of the absolutely convex hull of $D(X^{**},u)$. Therefore, since $x^{***}$ is arbitrary in $B_X^{***}$, for every bounded linear operator $F : X^{***} \to X^{***}$ we have

$$n(X,u)\|F\| \leq \sup\{\|F(\psi)\| : \psi \in D(X^{**},u)\}.$$ 

Now the result follows by taking $F = 1 - \Pi_X$. 

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Corollary 3.3 Let $X$ be a Banach space over $\mathbb{K}$, and let $u$ be in $S_X$. If $\sigma(X,u) < 2n(X,u)$, then $X$ does not contain an isomorphic copy of $\ell_1$. If actually $\sigma(X,u) < n(X,u)$, then $X$ is reflexive.

Proof. If $\sigma(X,u) < 2n(X,u)$, then from Lemma 3.2 we deduce that $\|1 - \Pi_X\| < 2$, and therefore, by [22, Proposition 2], $X$ does not contain an isomorphic copy of $\ell_1$. If $\sigma(X,u) < n(X,u)$, then again from Lemma 3.2 we deduce that $\|1 - \Pi_X\| < 1$, so $\Pi_X = 1$, and so $X$ is reflexive.

Corollary 3.4 Let $X$ denote the real or complex space $\ell_1$, and let $u$ be any element in the canonical basis of $X$. Then $\sigma(X,u) = 2$.

Proof. From the fact that $\mathbb{K}u$ is an $L$-summand of $X$ we easily deduce that $n(X,u) = 1$. Now apply Corollary 3.3.

The next corollary will not be applied in the remaining part of this paper, but has its own interest. It follows the line of [10, Theorem 3.2], where it is shown that, if $X$ is a complex normed space having a big point $u$ with $\delta(X,u) < \sqrt{3/e}$, then $\text{Com}(G_X)$ is equal to $\mathbb{C}I_X$. Here $\text{Com}(G_X)$ stands for the set of those bounded linear operators on $X$ which commute with all elements of $G_X$, and $I_X$ denotes the identity mapping on $X$. For any normed space $X$, define the normed space numerical index, $N(X)$, of $X$ by $N(X) := n(\mathcal{L}(X), I_X)$. The arguments in the proof of [10, Theorem 3.2] actually show that $\text{Com}(G_X) = \mathbb{C}I_X$ whenever $X$ is a real (respectively, complex) normed space having a big point $u$ with $\delta(X,u) < 2N(X)$ (respectively, $\delta(X,u) < \sqrt{3N(X)}$). The original formulation of [10, Theorem 3.2] quoted above follows from the refinement just pointed out and the Bohnenblust–Karlin theorem [14, Theorem 4.1] that $N(X) \geq 1/e$ when $X$ is a complex normed space.

Corollary 3.5 Let $X$ be a Banach space over $\mathbb{K}$ having a big point $u$. Then we have the following.

(a) If $\sigma(X,u) < 2N(X)$, then $\text{Com}(G_X)$ does not contain an isomorphic copy of $\ell_1$.
(b) If $\sigma(X,u) < N(X)$, then $\text{Com}(G_X)$ is reflexive.
(c) If $\mathbb{K} = \mathbb{C}$, and if $\sigma(X,u) < 2e$, then $\text{Com}(G_X)$ does not contain an isomorphic copy of $\ell_1$.
(d) If $\mathbb{K} = \mathbb{C}$, and if $\sigma(X,u) < 1/e$, then $\text{Com}(G_X)$ is reflexive.
(e) If $X$ is $M$-embedded, and if $N(X) > 0$, then $\text{Com}(G_X)$ is reflexive.
(f) If $\mathbb{K} = \mathbb{C}$, and if $X$ is $M$-embedded, then $\text{Com}(G_X)$ is reflexive.

Proof. We have clearly

$$N(X) = n(\mathcal{L}(X), I_X) \leq n(\text{Com}(G_X), I_X).$$

On the other hand, for $F \in \text{Com}(G_X)$, the set $\{x \in X : \|F(x)\| \leq \|F(u)\|\}$ is closed, convex, and $G_X$-invariant, so that the bigness of $u$ gives $\|F\| = \|F(u)\|$. Therefore the mapping $F \mapsto F(u)$ from $\text{Com}(G_X)$ to $X$ is a linear isometry sending $I_X$ into $u$, and hence we have

$$\sigma(\text{Com}(G_X), I_X) = \sigma(\text{Com}(G_X)(u), u) \leq \sigma(X,u).$$

Now, (a) and (b) follow from Corollary 3.3, whereas (c) and (d) follow from (a) and (b), respectively, and the Bohnenblust–Karlin theorem. Finally, keeping in mind Lemma 3.1, (e) and (f) follow from (b) and (d), respectively.
The following lemma is known in the case $\tau = n$ (see [31, Theorem 1.2 and Proposition 3.1]).

**Lemma 3.6** Let $X$ be a Banach space over $\mathbb{K}$, let $u, v$ be in $S_X$ such that $D(X,u) = D(X,v)$, and let $\tau$ be a vector space topology on $X^*$. If the duality mapping of $X$ is upper semicontinuous $(n-\tau)$ at $u$, then the duality mapping of $X$ is upper semicontinuous $(n-\tau)$ at $v$.

**Proof.** Assume that the duality mapping of $X$ is upper semicontinuous $(n-\tau)$ at $u$ but not at $v$. Then, by [34, Theorem 2.1], there exists a $\tau$-neighbourhood $N$ of zero in $X^*$ such that for every $n \in \mathbb{N}$ we can find $f_n \in B_{X^*}$ satisfying

$$\Re(f_n(v)) > \frac{n}{n+1}$$

and

$$f_n \notin D(X,v) + N,$$ (3.2)

and there exists $0 < \delta < 1$ such that

$$\{g \in B_{X^*} : \Re(g(u)) > \delta\} \subseteq D(X,u) + N.$$ (3.3)

Take a cluster point $f$ to the sequence $\{f_n\}$ in the $w^*$-topology of $X^*$. Then, by (3.1), we have $f \in D(X,v) = D(X,u)$, so $1 = f(u)$ is a cluster point of $\{\Re(f_n(u))\}$, and so there is $m \in \mathbb{N}$ with $\Re(f_m(u)) > \delta$. By (3.3), $f_m$ belongs to $D(X,u) + N$. But, applying again that $D(X,u) = D(X,v)$, this contradicts (3.2).

In Corollaries 3.7 and 3.8 which follow we emphasize the cases $\tau = n$ and $\tau = w$ of Lemma 3.6, respectively, adding some peculiar information for such cases. We recall that strong subdifferentiability of the norm of a normed space $X$ at a point $u \in S_X$ is nothing but the upper semicontinuity $(n-n)$ of the duality mapping of $X$ at $u$ [36, Corollary 4.4].

**Corollary 3.7** Let $X$ be a Banach space over $\mathbb{K}$, and let $u, v$ be in $S_X$ such that $D(X,u) = D(X,v)$. If the norm of $X$ is strongly subdifferentiable at $u$, then the norm of $X$ is strongly subdifferentiable at $v$. If in addition $u$ is a big point of $X$, then $v$ is also a big point of $X$.

**Proof.** Assume that the norm of $X$ is strongly subdifferentiable at $u$ and $v$, and that $u$ is a big point of $X$. Then, by the equality $D(X,u) = D(X,v)$ and the equivalence $2 \leftrightarrow 6$ in [13, Corollary 3.6], $v$ is a big point of $X$.

**Corollary 3.8** Let $X$ be a Banach space over $\mathbb{K}$, and let $u, v$ be in $S_X$ such that $D(X,u) = D(X,v)$. If the duality mapping of $X$ is upper semicontinuous $(n-w)$ at $u$, then the duality mapping of $X$ is upper semicontinuous $(n-w)$ at $v$, and we have $\sigma(X,u) = \sigma(X,v)$.

**Proof.** We recall that, since $X$ is complete, the upper semicontinuity $(n-w)$ of the duality mapping of $X$ at a point $x \in S_X$ is equivalent to the fact that $D(X,x)$ is dense in $D(X^{**}, x)$ for the $w^*$-topology of $X^{**}$ [34, Theorem 3.1]. Therefore, if the duality mapping of $X$ is upper semicontinuous $(n-w)$ at $u$ and $v$, then, by the assumption $D(X,u) = D(X,v)$, we have $D(X^{**}, u) = D(X^{**}, v)$, which implies $\sigma(X,u) = \sigma(X,v)$.

**Corollary 3.9** Let $X$ be a Banach space over $\mathbb{K}$, and let $u,v$ be in $S_X$ such that $D(X,u) = D(X,v)$. If $\sigma(X,u) = 0$, then also $\sigma(X,v) = 0$. 
Proof. Assume that $\sigma(X, u) = 0$. Then we have $D(X^{**}, u) = D(X, u)$, and hence, by [34, Theorem 3.1] just applied in the proof of Corollary 3.8, the duality mapping of $X$ is upper semicontinuous at $u$. Now apply Corollary 3.8.

4. The main results

We recall that a JB*-triple is a complex Banach space $X$ with a continuous triple product $\{\cdot, \cdot, \cdot\}: X \times X \times X \to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies the following.

1. For all $x$ in $X$, the mapping $y \mapsto \{x, x, y\}$ from $X$ to $X$ is a hermitian operator on $X$ and has non-negative spectrum.
2. The main identity

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$$

holds for all $a, b, x, y, z$ in $X$.
3. $\|\{x, x, x\}\| = \|x\|^3$ for every $x$ in $X$.

Concerning condition 1 above, we also recall that a bounded linear operator $T$ on a complex Banach space $X$ is said to be hermitian if $\exp(itT)$ belongs to $G_X$ for every $r$ in $\mathbb{R}$.

The JB*-triple is of capital importance in the study of bounded symmetric domains in complex Banach spaces. Indeed, open balls in JB*-triples are bounded symmetric domains, and every symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a suitable JB*-triple (see [41, 43]). Examples of JB*-triples are all C* algebras under the triple product

$$\{x, y, z\} := \frac{1}{2}(xy^*z + zy^*x), \quad (4.1)$$

the spaces $L(H_1, H_2)$ (bounded linear operators) for arbitrary complex Hilbert spaces $H_1$ and $H_2$ (with triple product formally defined as in (4.1)), and the so-called spin factors. These are constructed from an arbitrary complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ of hilbertian dimension at least 3, by taking a conjugate-linear involutive isometry $\sigma$ on $H$, and then by defining the triple product and the norm by

$$\{x, y, z\} := (x|y)z + (z|y)x - (x|\sigma(z))\sigma(y)$$

and

$$\|x\|^2 := (x|x) + \sqrt{(x|x)^2 - |(x|\sigma(x))|^2},$$

respectively, for all $x, y, z$ in $H$. Other examples can be obtained by noticing that the class of JB*-triples is closed under $\ell_\infty$- and $c_0$-sums.

Our first main result reads as follows.
THEOREM 4.1 Let $X$ be a $JB^*$-triple. Then the following conditions are equivalent.

1. There exists a big point $v$ of $X$ such that $\sigma(X, v) < 2$.
2. The Banach space of $X$ is isomorphic to a Hilbert space.
3. $X$ is $M$-embedded and has big points.

Moreover, if the above conditions are satisfied, then big points of $X$, denting points of $B_X$, and extreme points of $B_X$ coincide.

The proof of Theorem 4.1 involves a big amount of background on $JB^*$-triples, a part of which is recalled before formally attacking such a proof. First of all, we recall that linear mappings preserving triple products between $JB^*$-triples are called triple homomorphisms, and that triple isomorphisms (that is, bijective triple homomorphisms) between $JB^*$-triples are nothing but surjective linear isometries.

Let $X$ be a $JB^*$-triple. A subtriple (respectively, triple ideal) of $J$ is a subspace $M$ of $J$ such that $\{u,v,w\} \subseteq M$ (respectively, $\{u,v,w\} \subseteq M$). We say that $X$ is simple (respectively, topologically simple) if there are no triple ideals (respectively, closed triple ideals) of $X$ other than $\{0\}$ and $X$. Now let $x$ be in $X$, and denote by $X_x$ the closed subtriple of $X$ generated by $x$. It is well known that there is a unique couple $(S_x, \phi_x)$, where $S_x$ is a locally compact subset of $[0,\infty[$ such that $S_x \cup \{0\}$ is compact, and $\phi_x$ is a triple isomorphism from $X_x$ onto the $C^*$-algebra $C_0(S_x)$ (all complex-valued continuous functions on $S_x$ vanishing at infinity), such that $\phi_x(x)$ is the inclusion mapping $S_x \hookrightarrow \mathbb{C}$ (see [41, 4.8; 43, 1.15; 32]). Following [17], we say that the $JB^*$-triple $X$ is elementary if it is of one of the following types: $K(H_1,H_2)$ (compact operators) for complex Hilbert spaces $H_1$ and $H_2$, $\{x \in K(H) : x = -\theta * \theta\}$ for a complex Hilbert space $H$ and a conjugation $\theta$ on $H$, $\{x \in K(H) : x = \theta * \theta\}$ for $H$ and $\theta$ as above, a spin factor, the $JB^*$-triple consisting of all $1 \times 2$ matrices over the complex Cayley numbers, or the $JB^*$-triple consisting of all hermitian $3 \times 3$ matrices over the complex Cayley numbers. An element $x$ of $X$ is said to be weakly compact if the conjugate-linear operator $\{x, \cdot, \cdot \}$ is weakly compact, and the $JB^*$-triple $X$ is called weakly compact whenever every element of $X$ is weakly compact.

Weakly compact $JB^*$-triples are well understood thanks to the results in [18]. For instance, elementary $JB^*$-triples are nothing but topologically simple weakly compact $JB^*$-triples. Actually, keeping in mind [4, Theorem 3.2], the following characterizations of weakly compact $JB^*$-triples follow from [18].

LEMMA 4.2 Let $X$ be a $JB^*$-triple. Then the following assertions are equivalent:

1. $X$ is weakly compact,
2. $S_x$ is discrete for every $x \in X$,
3. $X$ is $M$-embedded,
4. $X$ is the $c_0$-sum of a suitable family of elementary $JB^*$-triples.

Let $X$ be a $JB^*$-triple. An element $u$ of $X$ is said to be a tripotent if $\{u,u,u\} = u$. Given a tripotent $u$ in $X$, we have $X = X_0(u) \oplus X_1(u) \oplus X_2(u)$, where, for $j \in \{0,1,2\}$, $X_j(u)$ denotes the eigenspace of the operator $\{u,u,\cdot\}$ corresponding to the eigenvalue $\frac{1}{4}j$. Following [42], the tripotent $u$ is said to be maximal (respectively, minimal) if $X_0(u) = 0$ (respectively, if $u \neq 0$ and $X_2(u) = \mathbb{C}u$). Two tripotents $u,v$ in $X$ are said to be orthogonal if $\{u,v,X\} = \{v,u,X\} = 0$. The results of [18], together with [32, Lemma 2.11; 5, p. 270], lead to the next lemma.
LEMMA 4.3 Let $X$ be a weakly compact JB*-triple. Then we have the following.

1. There exists a canonical bijection from the set of all extreme points of $B_X$ to the set of all minimal tripotents of $X$, which extends to an injective and contractive and conjugate-linear mapping $\pi : X^* \to X$.
2. The mapping $(f, g) \mapsto (f\langle g \rangle) : X^* \times X^* \to \mathbb{C}$ becomes an inner product on $X^*$.
3. For $x$ in $X$ there are (possibly finite) sequences $\{\lambda_n\}$ of positive numbers and $\{u_n\}$ of pairwise orthogonal minimal tripotents of $X$ such that $x = \sum_n \lambda_n u_n$ (which implies $\|x\| = \max_n \{\lambda_n\}$). Moreover, $x$ lies in the range of $\pi$ if and only if $\sum_n \lambda_n < \infty$, and if this is the case, then, taking $f$ in $X^*$ with $\pi(f) = x$, we have $\|f\| = \sum_n \lambda_n$ and $(f\langle f \rangle) = \sum_n \lambda_n^2$.

Let $X$ be a JB*-triple. By a frame in $X$ we mean a family $E$ of pairwise orthogonal minimal tripotents of $X$ such that $\cap_{u \in E} X_0(u) = 0$. We say that $X$ is of finite rank if there exists a finite frame in $X$. As a by-product of Lemma 4.4 immediately below, we realize that JB*-triples of finite rank are weakly compact. For the proof of Lemma 4.4 the reader is referred to [42, (2.15) and (4.10); 18, Proposition 4.5. (iii) and its proof], noticing that the implication $4 \Rightarrow 5$ in the lemma is clear.

LEMMA 4.4 Let $X$ be a JB*-triple. Then the following assertions are equivalent:

1. $X$ is of finite rank,
2. $X$ has the Radon–Nikodym property,
3. $S_x$ is finite for all $x \in X$,
4. $X$ is a finite $\ell_\infty$-sum of closed simple triple ideals which are either finite-dimensional, spin factors, or of the form $\mathcal{L}(H_1, H_2)$ for suitable complex Hilbert spaces $H_1$ and $H_2$ with $\dim(H_2) < \infty$,
5. The Banach space of $X$ is isomorphic to a Hilbert space.

In relation to Lemma 4.4 just formulated, it is worth mentioning that certain requirements on a JB*-triple $X$, much weaker than the Radon–Nikodym property, also imply that the Banach space of $X$ is isomorphic to a Hilbert space [12].

Let $X$ be a JB*-triple. Since the elements of $\mathcal{G}_{X}$ are nothing but the triple automorphisms of $X$, the set of all maximal tripotents of $X$ is $\mathcal{G}_{X}$-invariant. If $X$ is finite-dimensional, then in fact $\mathcal{G}_{X}$ acts transitively on the set of all maximal tripotents of $X$ [45, Theorem 5.3. (b)]. But, actually the same is true if $X$ is an arbitrary JB*-triple of finite rank. Indeed, with the help of Lemma 4.4 above, we can reduce to the case that $X$ is either finite-dimensional (where, as we have just seen, the result is known), a spin factor, or of the form $\mathcal{L}(H_1, H_2)$ for suitable complex Hilbert spaces $H_1$ and $H_2$ with $\dim(H_2) < \infty$, and, in the two last cases, the result follows by a direct inspection (see [42, Section 3;38]). Therefore we have the following.

LEMMA 4.5 Let $X$ be a JB*-triple of finite rank. Then $G_{X}$ acts transitively on the set of all maximal tripotents of $X$.

We define JBW*-triples as those JB*-triples having a (complete) predual. The bidual $X^{**}$ of every JB*-triple $X$ is a JBW*-triple under a suitable triple product which extends the one of $X$ [27]. Now, let $X$ be a JBW*-triple. Then the predual of $X$ (denoted by $X_0$) is unique, and the triple product of $X$ becomes $w^*$-continuous in each of its variables [4, Thm 2.1]. On the other hand, for $x$ in $S_{X}$, $D(X, x) \cap X_*$ is a (possibly empty) proper closed face of $B_{X_*}$, and therefore, by [29, Theorem 4.4],
there is a unique tripotent $u$ (possibly equal to zero) such that $D(X, x) \cap X_s = D(X, u) \cap X_s$. Such a tripotent $u$ is called the support of $x$ in $X$, and will be denoted by $u(X, x)$.

**Proof of Theorem 4.1.** 1 $\Rightarrow$ 2. Let $v$ be the big point of $X$ whose existence is assumed in condition 1. Since $\sigma(X, v) < 2$, and $X^*$ is $L$-embedded [4, Proposition 3.4], Lemma 3.1 applies, giving that $\sigma(X, v) = 0$. Now we proceed in several steps.

**Step (a).** There exists a tripotent $u$ in $X$ with the same properties as $v$. Indeed, since $\sigma(X, v) = 0$, the duality mapping of $X$ is upper semicontinuous $(n-w)$ at $v$ (see the proof of Corollary 3.9), and, by [11, Theorem 2.7], this last fact is equivalent to the one that $u := u(X^*, v)$ lies in $X$. Since for $x$ in $B_X$ we have $D(X^*, x) \cap X^* = D(X, x)$, the definition of $u(X^*, v)$ and the above lead to $D(X, v) = D(X, u)$, and hence, from Corollary 3.9 we deduce $\sigma(X, u) = 0$. Since the norm of $X$ is strongly subdifferentiable at $u$ [11, Corollary 2.4], and $v$ is a big point of $X$, and $D(X, v) = D(X, u)$, Corollary 3.7 applies, giving that $u$ is a big point of $X$.

**Step (b).** $X$ is weakly compact. Let $u$ be the tripotent of $X$ given by step (a). It is well known that $X_2(u)$, endowed with the product $x \triangle y := \{x, u, y\}$, becomes a norm unital complete normed (possibly non-associative) algebra whose unit is precisely $u$, and hence we have $n(X_2(u), u) > 0$ [46, p. 617]. Since $0 \leq \sigma(X_2(u), u) \leq \sigma(X, u) = 0$, it follows from Corollary 3.3 that $X_2(u)$ is a reflexive Banach space. Therefore, since $\{uXu\}$ is contained in $X_2(u)$, we realize that $u$ is a weakly compact element of $X$. Now, recalling that elements of $G_X$ are triple automorphisms, it follows that every element of $G_X(u)$ is weakly compact. Since the set of all weakly compact elements of $X$ is a closed subspace of $X$ [18, Proposition 4.7], and $u$ is a big point of $X$, we obtain that all elements of $X$ are weakly compact, that is, $X$ is a weakly compact $JB^*$-triple.

Accordingly to step (b) and Lemma 4.3, the tripotent $u$ of $X$ given by step (a) satisfies $u = u_1 + \ldots + u_m$ for some $m \in \mathbb{N}$ and suitable pairwise orthogonal minimal tripotents $u_1, \ldots, u_m \in X$.

**Step (c).** Every set of pairwise orthogonal minimal tripotents of $X$ has at most $m$ elements. Assume on the contrary that we have found pairwise orthogonal minimal tripotents $v_1, \ldots, v_{m+1}$ in $X$. Let $T$ be in $G_X$. Since $T(u_1), \ldots, T(u_m)$ are pairwise orthogonal minimal tripotents in $X$ with $\sum_{i=1}^{m} T(u_i) = T(u)$, step (b) and Lemma 4.3 apply to find $f, g \in X^*$ satisfying $\pi(f) = T(u)$, $\pi(g) = \sum_{i=1}^{m} v_i$, $\|f\| = (f[f]) = m$, and $\|g\| = (g|g) = m + 1$. Then we have

$$|g(T(u))| = |g(\pi(f))| = |(g|f)| \leq \sqrt{(g|g)\sqrt{(f[f])}} = \sqrt{(m + 1)m}.$$ 

Since $T$ is arbitrary in $G_X$, and $u$ is a big point of $X$, the above shows that

$$(m + 1) = \|g\| = \sup\{|g(T(u))| : T \in G_X\} \leq \sqrt{(m + 1)m},$$

a contradiction.

**Step (d).** $X$ satisfies condition 2 in the theorem. By steps (b) and (c), and Lemma 4.3, every element of $X$ belongs to the linear hull of a finite set of pairwise orthogonal tripotents of $X$. Since the linear hull of any set of pairwise orthogonal tripotents of $X$ is a subtriple of $X$, it follows that $X_x$ is finite-dimensional (equivalently, $S_x$ is finite) for all $x \in X$. By Lemma 4.4, $X$ satisfies condition 2.

2 $\Rightarrow$ 3. The assumption 2 clearly implies that $X$ is weakly compact. On the other hand, by the assumption 2 and the Krein–Milman theorem, $B_X$ is the norm-closed convex hull of its extreme points. Since extreme points of $B_X$ are precisely the maximal tripotents of $X$ [44, Proposition...
3.5], and $G_X$ acts transitively on the set of all maximal tripotents of $X$ (by the assumption 2 and Lemmas 4.4 and 4.5), it follows that each extreme point of $B_X$ is a big point of $X$.

$3 \Rightarrow 1$. By Lemma 3.1.

Now, the equivalence of conditions 1 to 3 in the theorem has been established. Assume that $X$ satisfies condition 2. We have just shown that extreme points of $B_X$ are big points of $X$. Take an extreme point $u$ of $B_X$. Then, since $u$ is a big point of $X$, and $B_X$ is dentable, Theorem 4.1 applies, giving that big points of $X$ and denting points of $B_X$ coincide, and that $G_X(u)$ is dense in the set of all big points of $X$. Since $G_X(u)$ consists only of extreme points of $B_X$, and the set of all extreme points of $B_X$ is closed (since they are the maximal tripotents of $X$), it follows that big points of $X$ are extreme points of $B_X$.

Since $C^*$-algebras are $JB^*$-triples, and they are finite-dimensional whenever their Banach spaces are reflexive [50], the next corollary follows straightforwardly from Theorem 4.1.

**Corollary 4.6** Let $X$ be a $C^*$-algebra. Then the following conditions are equivalent.

1. There exists a big point $v$ of $X$ such that $\sigma(X,v) < 2$.
2. $X$ is finite-dimensional.
3. $X$ is $M$-embedded and has big points.

Moreover, if the above conditions are satisfied, then the big points of $X$ are precisely the extreme points of $B_X$.

It is well known that $M$-embedded $C^*$-algebras are precisely those of the form $(\oplus_{\lambda \in \Lambda} K(H_{\lambda}))_{\ell_0}$ for some family $\{H_{\lambda}\}_{\lambda \in \Lambda}$ of complex Hilbert spaces. It is also known that extreme points of the closed unit ball of a finite-dimensional $C^*$-algebra $X$ are the unitary elements of $X$.

Now, we are going to determine those $JBW^*$-triples $X$ such that there exists a big point $f$ of $X$ with $\sigma(X,f) < 2$ and such that the norm of $X^*$ is strongly subdifferentiable at $f$.

**Proposition 4.7** Let $X$ be a weakly compact $JB^*$-triple. Then the following conditions are equivalent.

1. $X^*$ has big points.
2. $X$ is the $c_0$-sum of a suitable family of copies of some elementary $JB^*$-triple.
3. $G_X$ acts transitively on the set of all minimal tripotents of $X$.
4. $G_X$ acts transitively on the set of all extreme points of $B_X$.

Moreover, if the above conditions are satisfied, then big points of $X^*$, denting points of $B_X$, and extreme points of $B_X$, coincide.

**Proof.** 1 $\Rightarrow$ 2. By Lemma 4.2, we have $X = (\oplus_{\lambda \in \Lambda} X_{\lambda})_{\ell_0}$ for some family $\{X_{\lambda}\}_{\lambda \in \Lambda}$ of elementary $JB^*$-triples, so that we have $X^* = (\oplus_{\lambda \in \Lambda} X_{\lambda}^*)_{\ell_0}$ in the natural manner. Let $\lambda$ and $\mu$ be in $\Lambda$. Since $X_{\lambda}$ and $X_{\mu}$ are $M$-embedded (by Lemma 4.2), they are Asplund spaces (by [37, Theorem III.3.1]), so $X_{\lambda}^*$ and $X_{\mu}^*$ have the Radon–Nikodym property, and so we may choose $f_{\lambda}$ and $f_{\mu}$ denting points of $B_{X_{\lambda}^*}$ and $B_{X_{\mu}^*}$, respectively, which are also denting points of $B_{X_{\lambda}^*}$ (since $X_{\lambda}^*$ and $X_{\mu}^*$ are $L$-summands of $X^*$). Now the assumption 1 and [13, Proposition 4.3] provide us with some $T \in G_{X^*}$ satisfying $\|T(f_{\lambda}) - f_{\mu}\| < 2$. On the other hand, since $X$ is $M$-embedded, we
have $T = F^*$ for some $F$ in $G_X$ (by [37, Proposition III.2.2]), and, since $\{X_\lambda\}_{\lambda \in \Lambda}$ is the family of all minimal closed ideals of $X$, we must have $F(X_\lambda) = X_\rho$ for some $\rho \in \Lambda$. It follows that $T(X^*_\lambda) = X^*_\rho$, which together with $f_\lambda \in S_{X^*_\lambda}$, $f_\mu \in S_{X^*_\mu}$, and $\|T(f_\lambda) - f_\mu\| < 2$, implies that $\rho = \mu$. Now $X_\mu$ is a copy of $X_\lambda$ by means of the restriction of $F$ to $X_\lambda$.

$2 \Rightarrow 3$. Since a minimal tripotent in a $c_0$-sum of $JB^*$-triples must lie in some of the summands, and, by the assumption 2, all summands are identical in our case, we may actually assume that the $JB^*$-triple $X$ is elementary. Then condition 3 follows from [42, (4.6)] if $X$ is of finite rank, and by a direct inspection from the few remaining examples otherwise (compare [42, section 3]).

$3 \Rightarrow 4$. Let $\pi$ be the mapping from $X^*$ to $X$ given by Lemma 4.3. Since $\pi$ is canonical, we have $\pi^*(F^*)^{-1} = F^*\pi$ for every $F \in G_X$. Now, let $f$ and $g$ be extreme points of $B_X$. By the assumption 3, there exists $F \in G_X$ satisfying $F(\pi(f)) = \pi(g)$. Since $\pi$ is injective, it follows that $T(f) = g$ with $T := (F^*)^{-1} \in G_{X^*}$.

$4 \Rightarrow 1$. We know that $X^*$ has the Radon–Nikodym property, and hence $B_{X^*}$ is the norm-closed convex hull of its denting points. Since denting points of $B_{X^*}$ are extreme points of $B_{X^*}$, it follows from the assumption 4 that denting points of $B_{X^*}$ and extreme points of $B_{X^*}$ actually coincide, and that each extreme point of $B_{X^*}$ is a big point of $B_{X^*}$. Finally, since $B_{X^*}$ is dentable, it follows from [13, Proposition 4.3] that big points of $B_{X^*}$ are denting points of $B_{X^*}$.

**Lemma 4.8** Let $X$ be a $JBW^*$-triple, and let $\{u_\lambda\}_{\lambda \in \Lambda}$ be a family of pairwise orthogonal non-zero tripotents of $X$. Then, for $\phi \in c_0(\Lambda, \mathbb{C})$, the family $\{\phi(\lambda)u_\lambda\}_{\lambda \in \Lambda}$ is $w^*$-sumable in $X$, and the mapping $\phi \mapsto \sum_{\lambda \in \Lambda} \phi(\lambda)u_\lambda$ becomes a triple isomorphism from $c_0(\Lambda, \mathbb{C})$ onto the smallest $w^*$-closed subtriple of $X$ containing $\{u_\lambda : \lambda \in \Lambda\}$.

**Proof.** Put $Y := c_0(\Lambda, \mathbb{C})$. Then, for $\psi \in Y$, the family $\{\psi(\lambda)u_\lambda\}_{\lambda \in \Lambda}$ is norm-sumable in $X$, and the mapping $\Psi : \psi \mapsto \sum_{\lambda \in \Lambda} \psi(\lambda)u_\lambda$ becomes a triple homomorphism from $c_0(\Lambda, \mathbb{C})$ to $X$. By [47, Lemma 1.5] and the separate $w^*$-continuity of the triple product of $X$, $\Psi$ extends uniquely to a $w^*$-continuous triple homomorphism $\Phi : Y^{**} = \ell_\infty(\Lambda, \mathbb{C}) \rightarrow X$. Denote by $\{e_\lambda\}_{\lambda \in \Lambda}$ the canonical generalized basis of $Y$. For $\phi \in \ell_\infty(\Lambda, \mathbb{C})$, the family $\{\phi(\lambda)e_\lambda\}_{\lambda \in \Lambda}$ is $w^*$-sumable in $Y^{**}$ with sum $\phi$, and hence, since $\Phi(e_\lambda) = u_\lambda$ for all $\lambda \in \Lambda$, the family $\{\phi(\lambda)u_\lambda\}_{\lambda \in \Lambda}$ is $w^*$-sumable in $X$ with sum $\Phi(\phi)$. If $\Phi(\phi) = 0$ for some $\phi$ in $\ell_\infty(\Lambda, \mathbb{C})$, then for every $\mu \in \Lambda$ we have

$$0 = \{\Phi(\phi), u_\mu, u_\mu\} = \left\{\sum_{\lambda \in \Lambda} \phi(\lambda)u_\lambda, u_\mu, u_\mu\right\} = \sum_{\lambda \in \Lambda} \{\phi(\lambda)u_\lambda, u_\mu, u_\mu\} = \phi(\mu)u_\mu,$$

and hence $\phi = 0$. Finally note that, by [40, Proposition 1.2; 47, Lemma 1.3], $w^*$-continuous triple homomorphisms between $JBW^*$-triples have $w^*$-closed range.

**Proposition 4.9** Let $X$ be a $JBW^*$-triple. Then the Banach space of $X$ is isomorphic to a Hilbert space if (and only if) $X$ is a $JBW^*$-triple whose predual does not contain an isomorphic copy of $\ell_1$.

**Proof.** Assume that $X$ is a $JBW^*$-triple whose Banach space is not isomorphic to a Hilbert space. By Lemma 4.4, the Banach space of $X$ is not reflexive. By [39, Theorem 3.23], this implies the existence of a sequence $\{u_n\}_{n \in \mathbb{N}}$ of pairwise orthogonal non-zero tripotents of $X$. By Lemma 4.8, $X$ contains a $w^*$-closed subspace linearly isometric to $\ell_\infty$. Since $\ell_1$ is the unique predual of $\ell_\infty$, this is equivalent to the existence of a closed subspace of $P$ of $X$, such that
\(X_*/P = \ell_1\) isometrically. By lifting the canonical basis of \(\ell_1\) to a bounded subset of \(X_*\), and passing to the closed linear hull, we obtain an isomorphic copy of \(\ell_1\) in \(X_*\).

**Corollary 4.10** Let \(X\) be a JBW*-triple, and let \(f\) be in \(S_X\) such that the duality mapping of \(X_*\) is upper semicontinuous \((n-w)\) at \(f\). Then either \(\sigma(X_*, f) = 0\) or \(\sigma(X_*, f) = 2\).

**Proof.** By [29, Theorem 4.6], there exists a non-zero tripotent \(u\) in \(X\) such that \(D(X_*, f) = u + B_{X_0(u)}\). Therefore we have \(D(X_*, f) \supseteq u + B_{X_0(u)}\). We identify \(X_0(u)\) with \(X_0(u)\) and note that \(X_0(u)\) is a JBW*-triple, and that, in the above identification, the restriction of \(\Pi_X\) to \(X_0(u)\) converts into \(\Pi_{X_0(u)}\). Let \(g\) be in \(B_{X_0(u)}\). Then \(u + g\) lies in \(D(X_*, f)\), and the equality \(\Pi_X(u + g) = u + \Pi_{X_0(u)}(g)\) holds. Therefore we have

\[
\| (1 - \Pi_{X_0(u)})(g) \| = \| (1 - \Pi_X)(u + g) \| \leq \sigma(X_*, f)
\]

and, since \(g\) is arbitrary in \(B_{X_0(u)}\), we deduce \(\| 1 - \Pi_{X_0(u)} \| \leq \sigma(X_*, f)\). Assume that \(\sigma(X_*, f) < 2\). Then we have \(\| 1 - \Pi_{X_0(u)} \| < 2\), and hence, by [22, Proposition 2], \(X_0(u)\) does not contain an isomorphic copy of \(\ell_1\). By Proposition 4.9, \(X_0(u)\) is a reflexive Banach space. Since \(D(X_*, f) = u + B_{X_0(u)}\), and the duality mapping of \(X_*\) is upper semicontinuous \((n-w)\) at \(f\), it follows from [34, Theorem 3.1] that \(D(X_*, f) = D(X_*, f)\), i.e. \(\sigma(X_*, f) = 0\).

Given a normed space \(X\) and a subset \(U\) of \(S_X\), we say that the norm of \(X\) is uniformly strongly subdifferentiable on \(U\) if

\[
\lim_{\alpha \to 0^+} \frac{\| u + \alpha x \| - 1}{\alpha} = \tau(u, x) \text{ uniformly for } (u, x) \in U \times B_X.
\]

**Proposition 4.11** Let \(X\) be a JB*-triple of finite rank. Then the norm of \(X^*\) is uniformly strongly subdifferentiable on the set of all extreme points of \(B_X\).

**Proof.** Since \(X\) is a finite \(\ell_\infty\)-sum of simple JB*-triples of finite rank (by Lemma 4.4), we may assume that \(X\) is simple.

Assume in addition that \(X\) is finite-dimensional. Then the norm of \(X^*\) is strongly subdifferentiable at every point of \(S_X^*\) [1, p. 123]. But, since \(G_X^*\) acts transitively on the set of all extreme points of \(B_X\) (by Proposition 4.7), we have in fact that the norm of \(X^*\) is uniformly strongly subdifferentiable on such a set.

Now assume that \(X = \mathcal{L}(H_1, H_2)\) for suitable complex Hilbert spaces \(H_1\) and \(H_2\) with \(\dim(H_1) = \infty\) and \(\dim(H_2) < \infty\). Let \(K\) be the complex Hilbert space of dimension equal to \(\dim(H_2) + 1\), and let \(Y\) stand for \(\mathcal{L}(K, H_2)\). Let \(\epsilon > 0\). By the above paragraph, there exists \(\delta > 0\) such that

\[
\frac{\| \hat{f} + \alpha \hat{g} \| - 1}{\alpha} - \tau(\hat{f}, \hat{g}) < \epsilon
\]

whenever \(\hat{f}\) is an extreme point of \(B_{Y^*}\), \(\hat{g}\) belongs to \(B_{Y^*}\), and \(0 < \alpha < \delta\). Now, let \(f\) be an extreme point of \(B_{X^*}\), and let \(g\) be in \(B_Y\). Since \(X^*\) equals the projective tensor product \(H_1 \otimes \pi H_2\), and \(f = x_1 \otimes x_2\) for some \((x_1, x_2) \in H_1 \times H_2\), there exists a copy of \(K\) in \(H_1\) such that \(K \otimes H_2\) contains \(f\) and \(g\). Moreover, since \(K\) is one-complemented in \(H_1\), the natural mapping \(K \otimes \pi H_2\to
big points in $C^*$-algebras
every set of pairwise orthogonal minimal tripotents of \( Y \) is finite. Then, applying again Lemma 4.3, we deduce that \( S_y \) is finite for every \( y \in Y \). Now Condition 4 follows from Lemma 4.4 and Proposition 4.7.

4. By Lemma 4.4 and Propositions 4.7 and 4.11.

When conditions 1 to 4 are satisfied, the coincidence of big points of \( X_* \), denting points of \( B_{X_*} \), and extreme points of \( B_{X_*} \) follows from Proposition 4.7.

**Corollary 4.13** Let \( X \) be a von Neumann algebra. Then the following conditions are equivalent.

1. There exists a big point \( f \) of \( X_* \) with \( \sigma(X_*,f) < 2 \) and such that the norm of \( X_* \) is strongly subdifferentiable at \( f \).
2. \( X \) is finite-dimensional, and \( X_* \) has big points.
3. \( X \) is purely atomic and there exists a big point \( f \) of \( X_* \) satisfying \( \sigma(X_*,f) < 2 \).
4. \( X \) is a finite \( \ell_\infty \)-sum of copies of \( \mathcal{L}(H) \) for some finite-dimensional complex Hilbert space \( H \).

Moreover, if the above conditions are fulfilled, then big points of \( X_* \) and extreme points of \( B_{X_*} \) coincide.

**5. Characterizing Hilbert spaces**

Every complex Hilbert space \( H \) becomes a simple \( JB^* \)-triple of finite rank (indeed, we have \( H = \mathcal{L}(H,\mathbb{C}) \)), whose triple product is given by

\[
\{x,y,z\} = \frac{1}{2}((x|y)z + (z|y)x)
\]

for all \( x,y,z \in H \). More precisely, it follows from [42] that all frames in a \( JB^* \)-triple \( X \) of finite rank have the same cardinal (called the rank of \( X \)), and that complex Hilbert spaces are precisely the \( JB^* \)-triples of rank one.

In [7] we obtained several characterizations of complex Hilbert spaces, among either the \( JB^* \)-triples or the preduals of \( JBW^* \)-triples, in terms of transitivity conditions. A detailed review of such characterizations and other new characterizations in the same line can be seen in [9, section 5]. On the other hand, since almost transitive \( JBW^* \)-triples are Hilbert spaces [7, Corollary 2.6], all characterizations obtained in [6, 8] of members \( X \) of the class \( \mathcal{J} \) (of all almost transitive superreflexive Banach spaces), as well as the new ones shown in Proposition 2.9 above, become in fact characterizations of complex Hilbert spaces when we require that \( X \) is in addition a \( JB^* \)-triple. We are going to conclude this paper showing deeper characterizations of complex Hilbert spaces, among either the \( JB^* \)-triples or the preduals of \( JBW^* \)-triples, again in terms of transitivity conditions.

\( JB^* \)-algebras are defined as those complete normed Jordan complex algebras \( X \) endowed with a conjugate-linear algebra involution \( * \) satisfying \( \|U_x(x^*)\| = \|x\|^3 \) for every \( x \) in \( X \), where, for \( x \) in \( X \), the operator \( U_x : X \to X \) is defined by \( U_x(y) = 2x(x^*y) - x^2y \). It is well known that, if \( X \) is a \( JB^* \)-triple, and if \( u \) is a tripotent of \( X \), then \( X_2(u) \) becomes a \( JB^* \)-algebra with unit \( u \) for suitable product an involution.

**Lemma 5.1** Let \( X \) be a non-zero \( JB^* \)-algebra with a unit 1 such that \( X \neq \mathbb{C}1 \). Then we have \( \delta(X,1) = 2 \).
Proof. Take \( x = \hat{x} \in X \cap \mathbb{R}^1 \). Then, since Jordan algebras are power-associative, the closed subalgebra of \( X \) generated by \( \{ x, 1 \} \) (say \( Y \)) is associative and \(*\)-invariant, and hence it is a commutative \( C^* \)-algebra different from \( \mathbb{C} \cdot 1 \). Write \( Y = C(\Omega) \) for some Hausdorff compact topological space \( \Omega \), choose \( a, b \in \Omega \) such that \( x(a) = \min \{ x(t) : t \in \Omega \} \) and \( x(b) = \max \{ x(t) : t \in \Omega \} \), and put
\[
y := \frac{2x - x(b) - x(a)}{x(b) - x(a)} \in SY.
\]
Since \( y(a) = 1 \) and \( y(b) = -1 \) the unit point measures at \( a \) and \( b \) on \( \Omega \) are elements of \( D(Y, 1) \) whose distance is equal to 2. Since \( \delta(X, 1) = \delta(Y, 1) \), we have \( \delta(X, 1) = 2 \).

**Theorem 5.2** For a complex Banach space \( X \), the following assertions are equivalent:

1. \( X \) is a \( JB^* \)-triple and there exists a big point \( u \) of \( X \) with \( \eta(X, u) < 2 \);
2. \( X \) is a \( JB^* \)-triple and there exists a big point \( u \) of \( X \) with \( \delta(X, u) < 2 \) and such that the duality mapping of \( X \) is upper semicontinuous \((n-w)\) at \( u \);
3. \( X \) is the predual of a \( JBW^* \)-triple and there exists a big point \( f \) of \( X \) with \( \eta(X, f) < 2 \);
4. \( X \) is the predual of a \( JBW^* \)-triple and there exists a big point \( f \) of \( X \) with \( \delta(X, f) < 2 \) and such that the norm of \( X \) is strongly subdifferentiable at \( f \);
5. \( X \) is a Hilbert space.

Proof. \( 1 \Rightarrow 2. \) By \([8, Lemma 1; 24, Theorem VII.4.4]\), the assumption 1 implies that \( X \) is a superreflexive Banach space. As a consequence, \( X \) is \( M \)-embedded. Since, by the assumption 1, \( X \) has big points, Theorem 4.1 applies giving that such big points are extreme points of \( B_X \), and hence maximal tripotents. Since the norm of every \( JB^* \)-triple is strongly subdifferentiable at any non-zero tripotent \([11, Corollary 2.4]\), and, for the big point \( u \) of \( X \) whose existence is assumed in 1, we have \( \delta(X, u) < 2 \) (by Lemma 2.7), assertion 2 follows.

\( 2 \Rightarrow 5. \) Let \( u \) be the big point of \( X \) whose existence is assumed in 2. By the comments after Corollary 2.6, the assumption 2 implies that \( \sigma(X, u) < 2 \). Then, by Theorem 4.1, \( u \) is a maximal tripotent of \( X \). Now, since \( \delta(X, u) < 2 \), \( X_2(u) \) is a \( JB^* \)-algebra with unit \( u \) such that \( \delta(X_2(u), u) < 2 \). By Lemma 5.1, we have \( X_2(u) = C_u \), that is, \( u \) is a minimal tripotent of \( X \). It follows that \( \{ u \} \) is a frame in \( X \), and therefore \( X \) is a \( JB^* \)-triple of rank one.

\( 5 \Rightarrow 1. \) This implication is clear.

\( 3 \Rightarrow 4. \) By \([8, Lemma 1; 24, Theorem VII.4.4]\) the assumption 3 implies that \( X \) is a superreflexive Banach space. Then, by Lemma 4.4, the Banach space of the \( JBW^* \)-triple \( X^* \) is isomorphic to a Hilbert space. Since, by the assumption 3, \( X \) has big points, Theorem 4.12 applies giving that such big points are extreme points of \( B_X \). Since the norm of \( X \) is strongly subdifferentiable at every extreme point of \( B_X \) (by Proposition 4.11), and, for the big point \( f \) of \( X \) whose existence is assumed in 3, we have \( \delta(X, f) < 2 \) (by Lemma 2.7), assertion 4 follows.

\( 4 \Rightarrow 5. \) Let \( f \) be the big point of \( X \) whose existence is assumed in 4. By the comments after Corollary 2.6, the assumption 4 implies that \( \sigma(X, f) < 2 \). Then, by Theorem 4.12, the \( JBW^* \)-triple \( X^* \) is of finite rank, and \( f \) is an extreme point of \( B_X \). Since \( G_X \) acts transitively on the set of all extreme points of \( B_X \) (by Proposition 4.7), and \( \delta(X, f) < 2 \), it follows that \( \delta(X, g) < 2 \) for every extreme point \( g \) of \( B_X \). Now, assume that \( 5 \) does not hold. Then \( X^* \) is of finite rank greater than 1, and hence there are two orthogonal minimal tripotents in \( X^* \). With the help of Lemma 4.3, we can find extreme points \( g_1, g_2 \) of \( B_X \) such that the linear hull of \( \{ g_1, g_2 \} \) (say \( Y \)) becomes an isometric copy of \( \ell_1^2 \) in such a way that \( \{ g_1, g_2 \} \) converts into the canonical basis. Therefore
we have
\[ 2 = \delta(Y, g_1) \leq \delta(X, g_1) < 2, \]
a contradiction.

5 \Rightarrow 3. This implication is clear.

**Corollary 5.3** For a complex Banach space $X$, the following assertions are equivalent:

1. $X$ is a $C^\ast$-algebra and there exists a big point $u$ of $X$ with $\eta(X, u) < 2$;
2. $X$ is a $C^\ast$-algebra and there exists a big point $u$ of $X$ with $\delta(X, u) < 2$ and such that the duality mapping of $X$ is upper semicontinuous (n–w) at $u$;
3. $X$ is the predual of a von Neumann algebra and there exists a big point $f$ of $X$ with $\eta(X, f) < 2$;
4. $X$ is the predual of a von Neumann algebra and there exists a big point $f$ of $X$ with $\delta(X, f) < 2$ and such that the norm of $X$ is strongly subdifferentiable at $f$;
5. $X = \mathbb{C}$.

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