A HEIGHT ESTIMATE FOR \( H \)-SURFACES AND EXISTENCE OF \( H \)-GRAPHS

By SEBASTIÁN MONTIEL

Abstract. Let \( \Omega \) be a bounded planar domain which is convex (although not necessarily strictly convex) with area \( A \). We prove that, for each real number \( H \) satisfying \( AH^2 < \rho^2 \pi \), with \( \rho = (\sqrt{3} - 1)/2 \), there exists a graph on \( \Omega \) with constant mean curvature \( H \) and boundary \( \partial \Omega \). This existence theorem is deduced as a consequence of an \( L^\infty \) estimate for compact constant mean curvature surfaces with planar boundary, in terms of the \( L^1 \) norm of a component of its Gauss map, which will be obtained in this paper.

1. Introduction and statement of results. We shall consider the problem of determining the existence of smooth solutions to the equation of nonparametric surfaces of constant mean curvature

\[ \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = -2H, \quad H \in \mathbb{R}, \]

on a domain \( \Omega \) in \( \mathbb{R}^2 \). In that equation, the constant \( H \) is the mean curvature of the graph corresponding to the function \( u \), provided that we have chosen the downwards orientation on that graph.

Different kinds of boundary conditions have been imposed to study that mean curvature equation (being \( H \) constant or not). For example, one can prescribe the angle between the graph surface of the solution \( u \) of (1) and the boundary of the solid cylinder \( \Omega \times \mathbb{R} \) in \( \mathbb{R}^3 \), that is, the value of

\[ \left\langle \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, n \right\rangle, \]

where \( n \) is the inner unit normal along \( \partial \Omega \) in \( \mathbb{R}^2 \). Solutions corresponding to this boundary condition represent capillary surfaces in the absence of gravity and [Fi2] is a complete reference about this boundary problem.

Another boundary condition is just this: Do not consider any boundary condition whatsoever. For example, in [Gi], Giusti looks for generalized solutions...
of (1) without imposing boundary constraints and finds necessary and sufficient conditions on the domain $\Omega$ in order to obtain those types of solutions.

In this paper, we shall consider a certain Dirichlet problem for the equation (1), which consists of finding solutions taking prescribed values along the boundary $\partial \Omega$. This is equivalent, from a geometric point of view, to constructing graphs on $\Omega$ with constant mean curvature $H$ spanning arbitrary given space curves with single valued projection on $\partial \Omega$. The first existence result is due to Serrin. He proved in [Se] that

*If the domain $\Omega$ is bounded, then there exists a smooth solution to (1) for each arbitrary Dirichlet condition on $\partial \Omega$ if and only if $k \geq 2H > 0$, where $k$ is the curvature of the planar curve $\partial \Omega$. In particular, the domain $\Omega$ must be strictly convex.*

(Notice that we can always suppose $H \geq 0$ because changing $u$ by $-u$ we pass from a solution to (1) corresponding to $H$ to another one corresponding to $-H$.)

This theorem was a natural extension of the result by Finn [Fi1] concerning to minimal graphs, that is, concerning the case $H = 0$. He showed that

*There is a minimal graph over $\Omega$ for each Dirichlet condition along $\partial \Omega$ if and only if $\Omega$ is convex.*

The results above refer to arbitrary boundary values. We shall consider the particular case of zero boundary data. That is, we shall study the following Dirichlet problem

$$
\begin{align*}
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) &= -2H & \text{on } \Omega \\
u &= 0 & \text{along } \partial \Omega,
\end{align*}
$$

where $H$ is a nonnegative number. The graph $\Sigma$ of a solution $u$ to the problem (2) describes a surface in $\mathbb{R}^3$ with constant mean curvature $H$ (with respect to the downwards orientation) whose boundary is $\partial \Sigma = \partial \Omega \times \{0\}$. An immediate application of the maximum principle gives us that $u \geq 0$, that is, $\Sigma$ is above the plane where its boundary lies. In this case, the sufficient condition found by Serrin can be improved. In fact, one can easily see by using as barriers half-spheres with radius $1/H$ that

*If $\Omega$ is a bounded planar domain such that the curvature $k$ along its boundary curve $\partial \Omega$ satisfies $k \geq H > 0$, then there exists a solution to the Dirichlet problem with zero boundary data (2).*
Although all the results referred to above deal with convex domains, in the case of zero boundary condition, we can obtain some information for an arbitrary (not necessarily convex) \( \Omega \). It is clear that, if \( H = 0 \), then \( u = 0 \) is the only solution to (2). That is, the very planar domain \( \Omega \) is a minimal graph over itself with the required boundary. Now, we can inflate this zero solution by using the inverse function theorem for Banach spaces to obtain solutions of (2) for positive \( H \) small enough. This process can be continued, until reaching a unique value \( H_{\text{max}} > 0 \), depending only on the domain \( \Omega \), such that, if \( 0 < H < H_{\text{max}} \), there is a unique solution \( u_H \) to (2) and, if \( H > H_{\text{max}} \), then (2) has no solution. Properties of these solutions \( u_H \) have been studied in [Sa] and [Mc].

An interesting problem is to estimate that critical value \( H_{\text{max}} \) in terms of the geometric features of the domain \( \Omega \). In [LM], the author in collaboration with R. Lópész proved that

If \( \Omega \) is a convex bounded planar domain, \( L \) is the length of its boundary and \( H \) is a nonnegative real number such that \( LH < \sqrt{3}\pi \), then there exists a graph over \( \Omega \) with constant mean curvature \( H \) and boundary \( \partial \Omega \).

In other words, we proved that, if \( \Omega \) is convex, then \( H_{\text{max}} \geq \sqrt{3}\pi / L \). This result was a consequence of an \( L^\infty \) estimate for constant mean curvature compact surfaces with planar boundary obtained by the authors (see also [T]). The bizarre constant \( \sqrt{3}\pi \) (notice that if \( \Omega \) is a disc, then the best constant is \( 2\pi \)) probably comes from some limitation in the techniques that we used in [LM] in order to obtain gradient estimates from our acurated height estimate. This is why we think that the following natural conjecture is true:

Conjecture 1. If \( \Omega \) is a convex bounded planar domain and the length of its boundary is \( L \), then \( H_{\text{max}} \geq 2\pi / L \) and the equality holds if and only if \( \Omega \) is a disc.

In this paper, we are going to obtain the following area version of the theorem in [LM] mentioned above.

Theorem 1. Let \( \Omega \) be a convex (although not necessarily strictly convex) bounded planar domain with area \( A \). For each real number \( H \) such that \( AH^2 < \rho^2\pi \), there exists a smooth solution to the Dirichlet problem (2). That is, there exists a smooth graph over \( \Omega \) which is a surface in \( \mathbb{R}^3 \) with constant mean curvature \( H \) and whose boundary is \( \partial \Omega \).

That is, this theorem assures that if \( \Omega \) is convex, \( H_{\text{max}} \geq \rho \sqrt{\pi} / \sqrt{A} \). Notice that, since the classical isoperimetric inequality in the plane says that \( 4\pi A \leq L^2 \), the theorem above implies a (weaker) result in the spirit of [LM]. In another sense, this result improves [LM] because the condition imposed on the area in Theorem 1 above allows one to consider domains with arbitrarily long boundaries, which were not included in the result of [LM]. Taking into account Theorem 1, Con-
jecture 1 quoted above and the connection between Theorem 1 and the existence theorem in [LM], it seems to us that the following assertion should be true:

**Conjecture 2.** Theorem 1 must be true even if $AH^2 < \pi$ and so $H_{\text{max}} \geq \sqrt{\pi}/A$ for each convex bounded planar domain, with equality only for discs.

It is important to point out here that R. López has found in [L] a necessary and sufficient condition in order to have solutions of the same Dirichlet problem (2) on a convex *unbounded* planar domain.

Theorem 1 above will be a consequence of a height estimate that we shall obtain for compact surfaces in $\mathbb{R}^3$ with constant mean curvature and planar boundary. In fact, we shall estimate the height of such a surface in terms of the $L^1$ norm of the vertical component of its Gauss map, and shall prove the following result.

**Theorem 2.** Let $\Sigma$ be a compact surface immersed in the three-dimensional Euclidean space with constant mean curvature $H > 0$ whose boundary belongs to a plane $P = \{x \in \mathbb{R}^3 \mid \langle x, a \rangle = 0\}$, where $a \in \mathbb{R}^3$ is a unit vector. Denote by $\Sigma^+$ the region of $\Sigma$ which is above the plane $P$ and suppose that the Gauss map $N$ of the immersion satisfies

$$
\int_{\Sigma^+} |\langle N, a \rangle| d\Sigma \leq \rho^2 \frac{\pi}{H^2}.
$$

Then, if $h^+$ is the maximum height of $\Sigma^+$ with respect to $P$, we have

$$
h^+ \leq \frac{1}{H} \left( 1 - \sqrt{1 - \frac{H^2}{\pi} \int_{\Sigma^+} |\langle N, a \rangle| d\Sigma} \right)
$$

and the equality holds if and only if $\Sigma$ is a small spherical cap.

Notice that, when the surface $\Sigma$ is a graph over the plane $P$, since the function $\langle N, a \rangle$ is nothing but the Jacobian of the orthogonal projection of the surface onto the plane $P$, it does not change its sign and its integral on $\Sigma^+ = \Sigma$ is the area of the planar domain $\Omega$ determined by the curve $\partial \Sigma$ in the plane $P$. Then we deduce the following consequence, which yields a very natural estimate for the height of constant mean curvature graphs, with equality characterizing the spherical ones.

**Corollary 3.** Let $\Omega$ be a bounded planar domain with area $A$ and consider a smooth graph $\Sigma$ over $\Omega$ with zero boundary values and constant mean curvature $H > 0$ such that $AH^2 \leq \rho^2 \pi$. If we denote by $h$ the maximum height of $\Sigma$ with respect to the plane containing $\Omega$, then we have

$$
h \leq \frac{1}{H} \left( 1 - \sqrt{1 - \frac{AH^2}{\pi}} \right)
$$

and the equality is attained if and only if the graph $\Sigma$ is a small spherical cap.
This is a clear improvement of the classical height estimate $1/H$ due to Serrin (see [Se]) when the area of the domain $\Omega$ is less than or equal to the area $k\pi / H^2$ of the half-sphere with the same constant mean curvature as the graph.

2. Proof of the results. Denote by $\phi: \Sigma \to \mathbb{R}^3$ an immersion from a compact surface $\Sigma$ into the Euclidean space of dimension three with nonzero constant mean curvature. Then $\Sigma$ is orientable because we can choose a Gauss map $N: \Sigma \to \mathbb{R}^3$ for the immersion such that the corresponding mean curvature is a positive constant $H > 0$. Suppose also that the image $\phi(\partial \Sigma)$ of the boundary of $\Sigma$ is contained in a plane $P$. After a translation, one can assume that

$$P = \{ x \in \mathbb{R}^3 \mid \langle x, a \rangle = 0 \} \quad \text{for some unit } a \in \mathbb{R}^3.$$ 

Consider the height function $h = \langle \phi, a \rangle: \Sigma \to \mathbb{R}^3$. It is clear that the function $h$ vanishes along the boundary $\partial \Sigma$. Also, for each $p \in \Sigma$ and each $v \in T_p \Sigma$,

$$\langle (\nabla h)_p, v \rangle = \langle (d\phi)_p(v), a \rangle,$$

and so a point $p \in \Sigma$ is critical for $h$ if and only if $N_p = \pm a$. Then the set $C$ of the critical points of $h$ is contained in the intersection

$$\{ p \in \Sigma \mid \langle N_p, b \rangle = 0 \} \cap \{ p \in \Sigma \mid \langle N_p, c \rangle = 0 \},$$

where $\{a, b, c\}$ form an orthonormal basis of the space $\mathbb{R}^3$. Hence, the set $C$ is contained in the intersection of nodal sets of two solutions $u_1 = \langle N, b \rangle$ and $u_2 = \langle N, c \rangle$ of the Schrödinger equation

$$\Delta u + |\sigma|^2 u = 0,$$

$\sigma$ being the second fundamental form of the immersion $\phi$ and $\Delta$ is the Laplacian operator corresponding to the metric on the surface $\Sigma$ induced from the usual metric on $\mathbb{R}^3$. If both solutions were identically zero, then we would have that either $N = a$ or $N = -a$ everywhere on the surface. So, the surface would be planar and hence minimal, which is not the case. Thus, either $u_1$ or $u_2$ is nonidentically zero and the corresponding nodal set consists of a finite number of immersed circles (see, for example, [Che]). As a conclusion, the set $C$ of the critical points of the height function $h$ has zero measure in $\Sigma$. In this situation, for almost every real number $t$, the set

$$\Omega(t) = \{ p \in \Sigma \mid h(p) \geq t \}$$
is a compact surface with smooth boundary
\[ \partial \Omega(t) = \Gamma(t) = \{ p \in \Sigma | h(p) = t \} \subset P_t = \{ x \in \mathbb{R}^3 | \langle x, a \rangle = t \}, \]
when \( t \geq 0 \) and
\[ \partial \Omega(t) = \partial \Sigma \cup \Gamma(t), \]
when \( t < 0 \).
We will define a function \( F: \mathbb{R} \to \mathbb{R} \) by the equation
\begin{equation}
F(t) = \int_{\Omega(t)} |\langle N, a \rangle| \, d\Sigma.
\end{equation}
This function \( F \) is continuous, and the coarea formula (see [Fe] or Chapter IV in [Cha]) says that
\[ F(t) = \int_{t}^{\infty} \int_{\Gamma(t)} \frac{|\langle N, a \rangle|}{|\nabla h|} \, ds_t, \]
where \( ds_t \) is the arc element along \( \Gamma(t) \). Hence \( F \) is absolutely continuous and so its derivative exists for almost every \( t \in \mathbb{R} \) and its value is
\[ F'(t) = -\int_{\Gamma(t)} \frac{|\langle N, a \rangle|}{|\nabla h|} \, ds_t. \]
But, from (3), we have that along the curve \( \Gamma(t) \),
\[ |\nabla h|^2 = 1 - \langle N, a \rangle^2 = \langle \nu_t, a \rangle^2, \]
where \( \nu_t \) is the inner conormal along \( \Gamma(t) \). Moreover, since the image of \( \Omega(t) \) is above the plane \( P_t \), we know that \( \langle \nu_t, a \rangle \geq 0 \) along \( \Gamma(t) \). Hence
\[ |\nabla h|_{|\Gamma(t)} = \langle \nu_t, a \rangle \quad |\langle N, a \rangle|_{|\Gamma(t)} = \sqrt{1 - \langle \nu_t, a \rangle^2}. \]
Bringing together all this information, we have that
\[ F'(t) = -\int_{\Gamma(t)} \frac{\sqrt{1 - \langle \nu_t, a \rangle^2}}{\langle \nu_t, a \rangle} \, ds_t \quad \text{for a.e. } t \in \mathbb{R}. \]
On the other hand, consider the continuous function \( \psi: [0, 1[ \to \mathbb{R} \) given by
\[ \psi(x) = -\frac{\sqrt{1 - x^2}}{x} \quad \text{for all } x \in [0, \rho] \]
and being linear on $[\rho, 1]$ with $\psi(1) = 0$, where $\rho > 0$ satisfies $\rho^2 + \rho - 1 = 0$. One can easily check that $\psi$ is a concave function. Then, by using the Jensen inequality, we have for almost every $t \in \mathbb{R}$ that

$$F'(t) \leq \int_{\Gamma(t)} \psi \left( \langle \nu_t, a \rangle \right) \, ds_t \leq L(t)\psi \left( \frac{1}{L(t)} \int_{\Gamma(t)} \langle \nu_t, a \rangle \, ds_t \right),$$

where $L(t)$ is the length of the planar curve $\Gamma(t)$ and the equality holds if and only if $\langle \nu_t, a \rangle$ is a constant function along the curve $\Gamma(t)$. On the other hand, it is straightforward to see that

$$\Delta \phi = 2HN.$$

Integrating this equality on the surface $\Omega(t)$ when $t \geq 0$ and using the divergence theorem, we obtain

$$\int_{\Gamma(t)} \langle \nu_t, a \rangle \, ds_t = 2H\mathcal{A}(t),$$

where we have put

$$\mathcal{A}(t) = -\int_{\Omega(t)} \langle N, a \rangle \, d\Sigma \geq 0,$$

which is the so called algebraic area of the planar curve $\phi(\Gamma(t))$. (Notice that the function $\langle N, a \rangle$ is the Jacobian of the projection of $\Sigma$ onto a plane orthogonal to the vector $a$ and, so the integral on $\Omega(t)$ depends only on the boundary $\Gamma(t)$.) As we are supposing that $F(t) \leq F(0) \leq \rho^2 \pi / H^2$, taking into account the definition (4), we have

$$\mathcal{A}(t) \leq F(t) \leq \rho^2 \frac{\pi}{H^2},$$

and the first equality holds if and only if $\langle N, a \rangle \leq 0$ on the domain $\Omega(t)$. Hence we obtain

$$\frac{1}{L(t)} \int_{\Gamma(t)} \langle \nu_t, a \rangle \, ds_t = \frac{2H\mathcal{A}(t)}{L(t)} \leq \rho,$$

because we also have, for this algebraic area, a version of the classical isoperimetric inequality (see p. 590 in [LM] for details), namely

$$L(t)^2 \geq 4\pi \mathcal{A}(t) \quad \text{for each } t \geq 0,$$
where the equality is attained only when \( \phi(\Gamma(t)) \) is an embedded circle. Then we have

\[
F'(t) \leq -L(t) \frac{\sqrt{L(t)^2 - 4H^2\bar{A}(t)^2}}{2H\bar{A}(t)}
\]

for a.e. \( t \geq 0 \)

and the equality holds if and only if \( \langle \nu_t, a \rangle \) is a constant function along \( \Gamma(t) \). As the function on the right side of (5) is decreasing in the variable \( L(t) \), we have that

\[
F'(t) \leq -2\sqrt{\pi - H^2\bar{A}(t)}
\]

which is valid for almost every \( t \geq 0 \). Hence we deduce the following differential inequality for the function \( F \) which was defined in (4)

\[
F'(t) \leq -2\sqrt{\pi - H^2F(t)}
\]

for a.e. \( t \geq 0 \),

and the equality is attained if and only if the curve \( \phi(\Gamma(t)) \) is an embedded circle, the function \( \langle \nu_t, a \rangle \) is constant along it and the function \( \langle N, a \rangle \) is nonpositive on the domain \( \Omega(t) \). But the first inequality is equivalent to

\[
\left( \sqrt{\pi - H^2F(t)} \right)' \geq \sqrt{\pi}H
\]

for a.e. \( t \geq 0 \).

Notice that, as the function \( F \) is absolutely continuous, then \( \sqrt{\pi - H^2F} \) is also absolutely continuous and its a.e. derivative is weak. Then, we can use a corresponding version for the fundamental theorem of calculus. So, if we integrate this inequality between \( t = 0 \) and \( t = h^+ \), where this \( h^+ \) is the maximum height of the surface \( \Sigma \) above the plane which contains its boundary, we obtain that

\[
h^+ \leq \frac{1}{H} \left( 1 - \sqrt{1 - \frac{H^2F(0)}{\pi}} \right).
\]

and the equality holds if and only if each level curve \( \phi(\Gamma(t)) \) is an embedded circle, each function \( \langle \nu_t, a \rangle \) is constant and the function \( \langle N, a \rangle \) is nonpositive on the region of the surface being above of the plane \( P \). Thus, in the case of the equality, our immersion is a constant mean curvature embedding of \( \Sigma \) whose image lies above the plane \( P \) and whose boundary is a circle in \( P \). Then the reflection principle of Alexandrov [A] implies that our surface is umbilic and so a spherical cap. Moreover, as \( \langle N, a \rangle \) does not change sign, the surface must be a graph over the plane \( P \) and then it has to be a small spherical cap. This proves Theorem 2 and as an immediate consequence Corollary 3.
Now, we are going to prove Theorem 1. We want to show the existence of a graph over the convex bounded planar domain $\Omega$ with constant mean curvature $H > 0$, provided that $AH^2 < \rho^2\pi$, where $A$ is the area of $\Omega$. This is equivalent to finding a solution of the Dirichlet problem (2) and this can be done by using the continuity method. According to the usual Leray-Schauder approach (see Theorem 13.8 in [GT]), since we have the trivial solution $u_0 = 0$ for the case $H = 0$, existence of a solution follows from existence of a priori $C^1(\overline{\Omega})$ estimates independent of $H$ for the solutions $u_H$ of (2) with $0 < H \leq H_0$, where $H_0$ is a fixed positive number such that $AH^2_0 < \rho^2\pi$. But under these hypotheses we also have that $AH^2 < \rho^2\pi$ and so we can apply Corollary 3 to the graph of $u_H$ and obtain

$$u_H \leq \frac{1}{H} \left( 1 - \sqrt{1 - \frac{AH^2}{\pi}} \right) \leq \frac{1}{H_0} \left( 1 - \sqrt{1 - \frac{AH^2_0}{\pi}} \right).$$

That is, we have an a priori $C^0(\overline{\Omega})$ bound for the solution $u_H$ of (2). So it remains to look for global gradient estimates for these $u_H$ on $\overline{\Omega}$. But either standard considerations about quasilinear elliptic equations (see 11.3 in [GT]) or a detailed analysis of our concrete case assure us that it suffices to obtain these gradient estimates for the $u_H$ along the boundary $\partial\Omega$. In order to find them, notice that since we are supposing that $AH^2 < \rho^2\pi < 3\pi/4$, inequality (6) implies that

$$u_H \leq \frac{1}{H} \left( 1 - \sqrt{1 - \frac{AH^2}{\pi}} \right) \leq \frac{1}{2H} - \varepsilon(H_0)$$

for a certain positive number $\varepsilon(H_0)$ depending only on $H_0$. From this, our reasoning will exactly follow the proof of Corollary 4 in [LM]. In fact, we choose a slab $S$ of width $1/2H$ parallel to the plane $P$ where the domain $\Omega$ is. Now, we move the graph $\Sigma_H$ corresponding to the solution $u_H$ in such a way that the plane $P$ is at height $\varepsilon(H_0)/2$ over the lowest limiting plane $Q$ of the slab $S$. Given a point $p \in \partial\Omega$, we shall represent by $C_p$ a quarter-cylinder of radius $1/2H$ whose axis belongs to $Q$ and is parallel to the line tangent to the convex curve $\partial\Omega$ at the point $p$. We shall choose $C_p$ also for it not to intersect the graph $\Sigma_H$ and with its concave side in front of $\Sigma_H$. This piece of cylinder is another graph surface with the same constant mean curvature $H$ as our $\Sigma_H$. Now we slide $C_p$ toward the surface $\Sigma_H$ until they first touch each other. The maximum principle forbids that the first contact point be upon the plane $P$ and the convexity of $\Omega$ implies that this first contact point is nothing but the very $p$. Then the slope of $\Sigma_H$ at the point $p$ is less than the slope of the corresponding $C_p$ and this occurs for every $p \in \partial\Omega$. Since all these quarter-cylinders $C_p$ cut the plane $P$ with a same angle which does not depend on $p$, which is less than $\pi/2$ and which depends only on $H_0$, we have obtained an a priori upper bound for $|\nabla u_H|$ along $\partial\Omega$, as we were looking for. This finishes the proof of Theorem 1.
REFERENCES


