GROTHENDIECK’S INEQUALITIES FOR
REAL AND COMPLEX JBW*-TRIPLES

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Introduction

This paper relies on the important works of T. Barton and Y. Friedman [3] and C.-H. Chu, B. Iochum and G. Loupias [8] on the generalization of ‘Grothendieck’s inequalities’ to complex JB*-triples. Of course, the Barton–Friedman–Chu–Iochum–Loupias techniques are strongly related to those of A. Grothendieck [15], G. Pisier (see [27, 28, 29]), and U. Haagerup [16], leading to the classical ‘Grothendieck inequalities’ for C*-algebras. One of the most important ideas contained in the Barton–Friedman paper is the construction of ‘natural’ prehilbertian seminorms \( k \cdot k \), associated to norm-one continuous linear functionals \( J \) on complex JB*-triples, in order to play, in Grothendieck’s inequalities, the same role as that of the prehilbertian seminorms derived from states in the case of C*-algebras. This is very relevant because JB*-triples need not have a natural order structure.

Section 1 of the present paper is devoted to reviewing the main results in [3]. In fact we observe some gaps in the proofs of those results (some of which are also subsumed in [8]), and provide the reader with quick partial amendments of such gaps by applying theorems of J. Lindenstrauss [24] and V. Zizler [37] (see Theorems 1 and 2, respectively).

We begin § 2 by establishing a deeper amendment of the Barton–Friedman ‘Little Grothendieck Theorem’ for complex JB*-triples [3, Theorem 1.3] (see Theorem 3). Roughly speaking, our result assures that the assertion in [3, Theorem 1.3] is true whenever we replace the prehilbertian seminorm \( \| \cdot \|_\phi \) arising in that assertion with

\[
\| \cdot \|_{\phi_1, \phi_2} := \sqrt{\| \cdot \|_{\phi_1}^2 + \| \cdot \|_{\phi_2}^2},
\]

where \( \phi_1 \) and \( \phi_2 \) are suitable norm-one continuous linear functionals. It is worth mentioning that in fact our Theorem 3 deals with complex JBW*-triples and weak*-continuous operators, and that, in such a case, the functionals \( \phi_1 \) and \( \phi_2 \) above can be chosen to be weak*-continuous. Among the consequences of Theorem 3 we emphasize appropriate ‘Little Grothendieck inequalities’ for JBW-algebras and von Neumann algebras (see Corollary 3 and Remark 1, respectively).

Corollary 3 allows us to adapt an argument in [26] in order to extend Theorem 3 to the real setting (Theorem 5).

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Section 3 contains the main results of the paper, namely the ‘Big Grothendieck inequalities’ for complex and real JBW*-triples (Theorems 6 and 7, respectively). Indeed, given $M > 4(1 + 2\sqrt{3})$ (respectively, $M > 4(1 + 2\sqrt{3})(1 + 3\sqrt{2})^2$), $\varepsilon > 0$, $V$ and $W$ complex (respectively, real) JBW*-triples, and a separately weak*-continuous bilinear form $U$ on $V \times W$, there exist norm-one functionals $\varphi_1, \varphi_2 \in V_\ast$ and $\psi_1, \psi_2 \in W_\ast$ satisfying

$$|U(x, y)| \leq M \|U\| (\|x\|^2_{\varphi_2} + \varepsilon^2 \|x\|^2_{\varphi_1})^{1/2} \|y\|^2_{\psi_2} + \varepsilon^2 \|y\|^2_{\psi_1})^{1/2}$$

for all $(x, y) \in V \times W$.

The concluding section of the paper (§ 4) deals with some applications of the results previously obtained. We give a complete solution to a gap in the proof of the results of [31] on the strong* topology of complex JBW*-triples, and extend those results to the real setting. We also extend to the real setting the fact proved in [32] that the strong* topology of a complex JBW*-triple $\mathscr{W}$ and the Mackey topology $m(\mathscr{W}, \mathscr{W}_\ast)$ coincide on bounded subsets of $\mathscr{W}$. From this last result we derive a Jarchow-type characterization of weakly compact operators from (real or complex) JB*-triples to arbitrary Banach spaces.

1. Discussing previous results

We recall that a complex JB*-triple is a complex Banach space $\mathscr{E}$ with a continuous triple product $\{\cdot, \cdot, \cdot\}: \mathscr{E} \times \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ which is bilinear and symmetric in the outer variables and conjugate linear in the middle variable, and satisfies:

(i) $L(a, b)\{x, y, z\} = \{L(a, b)x, y, z\} = \{x, L(b, a)y, z\} + \{x, y, L(a, b)z\}$ for all $a, b, c, x, y, z$ in $\mathscr{E}$, where $L(a, b)x := \{a, b, x\}$ (Jordan Identity);

(ii) the map $L(a, a)$ from $\mathscr{E}$ to $\mathscr{E}$ is an hermitian operator with non-negative spectrum for all $a$ in $\mathscr{E}$;

(iii) $\|\{a, a, a\}\| = \|a\|^3$ for all $a$ in $\mathscr{E}$.

Complex JB*-triples were introduced by W. Kaup in order to provide an algebraic setting for the study of bounded symmetric domains in complex Banach spaces (see [21, 22, 34]).

If $\mathscr{E}$ is a complex JB*-triple and $e \in \mathscr{E}$ is a tripotent ($\{e, e, e\} = e$), it is well known that there exists a decomposition of $\mathscr{E}$ into the eigenspaces of $L(e, e)$, the Peirce decomposition,

$$\mathscr{E} = \mathscr{E}_0(e) \oplus \mathscr{E}_1(e) \oplus \mathscr{E}_2(e),$$

where $\mathscr{E}_k := \{x \in \mathscr{E}: L(e, e)x = \frac{1}{k}kx\}$. The natural projection $P_k(e): \mathscr{E} \to \mathscr{E}_k(e)$ is called the Peirce $k$-projection. A tripotent $e \in \mathscr{E}$ is called complete if $\mathscr{E}_0(e) = 0$.

By [23, Proposition 3.5] we know that the complete tripotents in $\mathscr{E}$ are exactly the extreme points of its closed unit ball.

By a complex JBW*-triple we mean a complex JB*-triple which is a dual Banach space. We recall that the triple product of every complex JBW*-triple is separately weak*-continuous [5], and that the bidual $\mathscr{E}^{**}$ of a complex JB*-triple $\mathscr{E}$ is a JBW*-triple whose triple product extends the one of $\mathscr{E}$ [11].

Given a complex JBW*-triple $\mathscr{W}$ and a norm-one element $\varphi$ in the predual $\mathscr{W}_\ast$ of $\mathscr{W}$, we can construct a prehilbert seminorm $\|\cdot\|_\varphi$ as follows (see [3, Proposition 1.2]). By the Hahn–Banach theorem there exists $z \in \mathscr{W}$ such that
\( \varphi(z) = \|z\| = 1 \). Then \((x, y) \mapsto \varphi(x, y, z)\) becomes a positive sesquilinear form on \(W\) which does not depend on the point of support \(z\) for \(\varphi\). The prehilbert seminorm \(\| \cdot \|_\varphi\) is then defined by \(\|x\|_\varphi^2 := \varphi(x, x, z)\) for all \(x \in W\). If \(\mathcal{E}\) is a complex JB*-triple and \(\varphi\) is a norm-one element in \(\mathcal{E}^*\), then \(\| \cdot \|_\varphi\) acts on \(\mathcal{E}^{**}\), and hence, in particular, it acts on \(\mathcal{E}\).

In [3, Theorem 1.4], J. T. Barton and Y. Friedman claim that for every pair of complex JB*-triples \(\mathcal{E}, \mathcal{F}\), and every bounded bilinear form \(V\) on \(\mathcal{E} \times \mathcal{F}\), there exist norm-one functionals \(\varphi \in \mathcal{E}^*\) and \(\psi \in \mathcal{F}^*\) such that the inequality

\[
|V(x, y)| \leq (3 + 2\sqrt{3}) \|x\|_\varphi \|y\|_\psi
\]

holds for every \((x, y) \in \mathcal{E} \times \mathcal{F}\). This result is called ‘Grothendieck’s inequality for JB*-triples’. However, the beginning of the Barton–Friedman proof assumes that the two following assertions are true.

(i) For \(\mathcal{E}, \mathcal{F}\) and \(V\) as above, there exists a separately weak*-continuous extension of \(V\) to \(\mathcal{E}^{**} \times \mathcal{F}^{**}\).

(ii) Again for \(\mathcal{E}, \mathcal{F}\) and \(V\) as above, every separately weak*-continuous extension of \(V\) to \(\mathcal{E}^{**} \times \mathcal{F}^{**}\) attains its norm (at a pair of complete tripotents).

We have been able to verify Assertion (i), but only by applying the fact, later proved by C.-H. Chu, B. Iochum and G. Loupias [8, Lemma 5], that every bounded linear operator from a complex JB*-triple to the dual of another complex JB*-triple factors through a complex Hilbert space. Actually, this fact is also claimed in the Barton–Friedman paper (see [3, Corollary 3.2]), but their proof relies on their alleged [3, Theorem 1.4].

**Lemma 1.** Let \(\mathcal{E}\) and \(\mathcal{F}\) be complex JB*-triples. Then every bounded bilinear form \(V\) on \(\mathcal{E} \times \mathcal{F}\) has a separately weak*-continuous extension to \(\mathcal{E}^{**} \times \mathcal{F}^{**}\).

**Proof.** Let \(V\) be a bounded bilinear form on \(\mathcal{E} \times \mathcal{F}\). Let \(F\) denote the unique bounded linear operator from \(\mathcal{E}\) to \(\mathcal{F}^*\) which satisfies

\[
V(x, y) = \langle F(x), y \rangle
\]

for every \((x, y) \in \mathcal{E} \times \mathcal{F}\). By [8, Lemma 5], \(F\) factors through a Hilbert space, and hence is weakly compact. By [18, Lemma 2.13.1], we have \(F^{**}(\mathcal{E}^{**}) \subset \mathcal{F}^*\). Then the bilinear form \(V\) on \(\mathcal{E}^{**} \times \mathcal{F}^{**}\) given by

\[
\tilde{V}(\alpha, \beta) = \langle F^{**}(\alpha), \beta \rangle
\]

extends \(V\) and is weak*-continuous in the second variable. But \(\tilde{V}\) is also weak*-continuous in the first variable because, for \((\alpha, \beta) \in \mathcal{E}^{**} \times \mathcal{F}^{**}\), the equality

\[
\langle F^{**}(\alpha), \beta \rangle = \langle \alpha, F^*(\beta) \rangle
\]

holds.

Unfortunately, as the next example shows, Assertion (ii) above is not true.

**Example 1.** Take \(\mathcal{E}\) and \(\mathcal{F}\) equal to the complex \(\ell_2\) space, and consider the bounded bilinear form on \(\mathcal{E} \times \mathcal{F}\) defined by \(V(x, y) := \langle S(x), \sigma(y) \rangle\) where \(S\) is...
the bounded linear operator on $\ell_2$ whose associated matrix is
\[
\begin{pmatrix}
\frac{1}{2} & 0 & \cdots & 0 & \cdots \\
0 & \frac{2}{3} & 0 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \frac{n}{n+1} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},
\]
and $\sigma$ is the conjugation on $\ell_2$ fixing the elements of the canonical basis. Then $V$ does not attain its norm.

It is worth mentioning that, although the bilinear form $V$ above does not attain its norm, it satisfies inequality (1.1) for every $x, y \in \ell_2$ and all norm-one-elements $\varphi, \psi \in \ell^*_2$. Therefore it does not become a counterexample to the Barton–Friedman claim. In fact we do not know if Theorem 1.4 of [3] is true.

Now that we know that Assertion (ii) is not true, we prove that it is ‘almost’ true.

**Lemma 2.** Let $\mathcal{E}$ and $\mathcal{F}$ be complex JB*-triples. Then the set of bounded bilinear forms on $\mathcal{E} \times \mathcal{F}$ whose separately weak*-continuous extensions to $\mathcal{E}^{**} \times \mathcal{F}^{**}$ attain their norms is norm-dense in the space $\mathcal{L}(^2(\mathcal{E} \times \mathcal{F}))$ of all bounded bilinear forms on $\mathcal{E} \times \mathcal{F}$.

**Proof.** Let $V$ be in $\mathcal{L}(^2(\mathcal{E} \times \mathcal{F}))$. Denote by $\tilde{V}$ the (unique) separately weak*-continuous extension of $V$ to $\mathcal{E}^{**} \times \mathcal{F}^{**}$. By the proof of Lemma 1, we can assure the existence of a bounded linear operator $F_V: \mathcal{E} \rightarrow \mathcal{F}^*$ satisfying $F_V^{**} (\mathcal{E}^{**}) \subset \mathcal{F}^*$ and
\[
\tilde{V}(\alpha, \beta) = \langle F_V^{**}(\alpha), \beta \rangle
\]
for every $(\alpha, \beta) \in \mathcal{E}^{**} \times \mathcal{F}^{**}$. It follows that $\tilde{V}$ attains its norm whenever $F_V^{**}$ does. Since the mapping $V \mapsto F_V$, from $\mathcal{L}(^2(\mathcal{E} \times \mathcal{F}))$ into the Banach space of all bounded linear operators from $\mathcal{E}$ to $\mathcal{F}^*$, is a surjective isometry, the result follows from [24, Theorem 1]. \(\Box\)

An alternative proof of the above lemma can be given taking as a key tool [1, Theorem 1].

Now note that, if $X$ and $Y$ are dual Banach spaces, and if $U$ is a separately weak*-continuous bilinear form on $X \times Y$ which attains its norm, then $U$ actually attains its norm at a pair of extreme points of the closed unit balls of $X$ and $Y$ (and hence at a pair of complete tripotents in the case that $X$ and $Y$ are complex JB*-triples). Since the Barton–Friedman proof of their claim actually shows that the inequality (1.1) holds (for suitable norm-one functionals $\varphi \in \mathcal{E}^*$ and $\psi \in \mathcal{F}^*$) whenever the separately weak*-continuous extension of $V$ given by Lemma 1 attains its norm at a pair of complete tripotents, the next theorem follows from Lemma 2.

**Theorem 1.** Let $\mathcal{E}$ and $\mathcal{F}$ be complex JB*-triples. Then the set of all bounded bilinear forms $V$ on $\mathcal{E} \times \mathcal{F}$ such that there exist norm-one functionals $\varphi \in \mathcal{E}^*$ and

...
Another proof of the Barton–Friedman claim [3, Theorem 1.4] (with constant \(3 + 2\sqrt{3}\) replaced by \(4(1 + 2\sqrt{3})\)) appears in the Chu–Iochum–Loupias paper already quoted (see [8, Theorem 6]). However, such a proof relies on the Barton–Friedman version of the so-called ‘Little Grothendieck Theorem’ for complex JB*-triples [3, Theorem 1.3], and the Barton–Friedman argument for this result also has a gap (see [26]).

Several authors (the second author of the present paper among others) subsumed the gap in the proof of Theorem 1.3 of [3] just commented, and formulated daring claims like the following (see [31, Proposition 1] and the proof of Lemma 4 of [9]). For every complex JB*-triple \(\mathcal{W}\), every complex Hilbert space \(\mathcal{K}\), and every weak*-continuous linear operator \(T: \mathcal{W} \rightarrow \mathcal{K}\), there exists a norm-one functional \(\varphi \in \mathcal{K}^*\) such that the inequality

\[
\|T(x)\| \leq \sqrt{2} \|T\|\|x\|\varphi
\]

holds for all \(x \in \mathcal{W}\). As in the case of the Barton–Friedman Big Grothendieck inequality, we do not know whether the above claim is true. In any case, the next lemma is implicitly shown in the proof of Theorem 1.3 of [3].

**Lemma 3.** Let \(\mathcal{W}\) be a complex JB*-triple, \(\mathcal{K}\) a complex Hilbert space, and \(T\) a weak*-continuous linear operator from \(\mathcal{W}\) to \(\mathcal{K}\) which attains its norm. Then \(T\) satisfies inequality \((1.2)\) for a suitable norm-one functional \(\varphi \in \mathcal{K}^*\).

We note that, for \(\mathcal{W}\) and \(\mathcal{K}\) as in the above lemma, weak*-continuous linear operators from \(\mathcal{W}\) to \(\mathcal{K}\) need not attain their norms (see the introduction of [26]).

Now, from Lemma 3 and [37] we obtain the following result.

**Theorem 2.** Let \(\mathcal{W}\) be a complex JB*-triple and \(\mathcal{K}\) a complex Hilbert space. Then the set of weak*-continuous linear operators \(T\) from \(\mathcal{W}\) to \(\mathcal{K}\) such that there exists a norm-one functional \(\varphi \in \mathcal{K}^*\) satisfying

\[
\|T(x)\| \leq \sqrt{2} \|T\|\|x\|\varphi
\]

for all \(x \in \mathcal{W}\), is norm dense in the space of all weak*-continuous linear operators from \(\mathcal{W}\) to \(\mathcal{K}\).

2. **The Little Grothendieck Theorem for JBW*-triples**

In this section we prove appropriate versions of the ‘Little Grothendieck inequality’ for real and complex JBW*-triples. We begin by considering the complex case, where the key tools are the Barton–Friedman result collected in Lemma 3, and a fine principle on approximation of operators by operators attaining their norms, due to R. A. Poliquin and V. E. Zizler [30].

**Theorem 3.** Let \(K > \sqrt{2}\) and \(\epsilon > 0\). Then, for every complete JBW*-triple \(\mathcal{W}\), every complex Hilbert space \(\mathcal{K}\), and every weak*-continuous linear operator
Let $T : \mathcal{W} \to \mathcal{H}$, there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{W}$, such that the inequality
\[ \| T(x) \| \leq K \| T \| (\| x \|_{\varphi_2}^2 + \varepsilon^2 \| x \|_{\varphi_1}^2)^{1/2} \]
holds for all $x \in \mathcal{W}$.

**Proof.** Without loss of generality we can suppose $\| T \| = 1$. Take $\delta > 0$ such that $\delta \leq \varepsilon^2$ and $\sqrt{2((1 + \delta)^2 + \delta)} \leq K$. By [30, Corollary 2] there is a rank-one weak*-continuous linear operator $T_1 : \mathcal{W} \to \mathcal{H}$ such that $\| T_1 \| \leq \delta$ and $T - T_1$ attains its norm. Since $T_1$ is of rank 1 and weak*-continuous, it also attains its norm. By Lemma 3, there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{W}$, such that
\[ \| T_1(x) \| \leq \sqrt{2} \| T_1 \| \| x \|_{\varphi_1}, \]
\[ \| (T - T_1)(x) \| \leq \sqrt{2} \| T - T_1 \| \| x \|_{\varphi_2} \]
for all $x \in \mathcal{W}$. Therefore for $x \in \mathcal{W}$ we have
\[ \| T(x) \| \leq \| (T - T_1)(x) \| + \| T_1(x) \| \]
\[ \leq \sqrt{2} \| T - T_1 \| \| x \|_{\varphi_2} + \sqrt{2} \| T_1 \| \| x \|_{\varphi_1} \]
\[ \leq \sqrt{2}(1 + \delta)\| x \|_{\varphi_2} + \sqrt{2}\delta \| x \|_{\varphi_1} \]
\[ \leq \sqrt{2}((1 + \delta)^2 + \delta)(\| x \|_{\varphi_2}^2 + \delta \| x \|_{\varphi_1}^2)^{1/2} \]
\[ \leq K(\| x \|_{\varphi_2}^2 + \varepsilon^2 \| x \|_{\varphi_1}^2)^{1/2}. \]

Given a complex JBW*-triple $\mathcal{W}$ and norm-one elements $\varphi_1, \varphi_2 \in \mathcal{W}$, we denote by $\| \cdot \|_{\varphi_1, \varphi_2}$ the prehilbert seminorm on $\mathcal{W}$ given by $\| x \|_{\varphi_1, \varphi_2}^2 := \| x \|_{\varphi_1}^2 + \| x \|_{\varphi_2}^2$. The next result follows straightforwardly from Theorem 3.

**Corollary 1.** Let $\mathcal{W}$ be a complex JBW*-triple and $T$ a weak*-continuous linear operator from $\mathcal{W}$ to a complex Hilbert space. Then there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{W}$, such that, for every $x \in \mathcal{W}$, we have
\[ \| T(x) \| \leq 2 \| T \| \| x \|_{\varphi_1, \varphi_2}. \]

We recall that a JB*-algebra is a complete normed Jordan complex algebra (say $\mathcal{A}$) endowed with a conjugate-linear algebra involution $*$ satisfying $\| U_x(x^*) \| = \| x \|^3$ for every $x \in \mathcal{A}$. Here, for every Jordan algebra $\mathcal{A}$, and every $x \in \mathcal{A}$, $U_x$ denotes the operator on $\mathcal{A}$ defined by $U_x(y) := x \circ (x \circ y) - y^2 \circ y$, for all $y \in \mathcal{A}$. We note that every JB*-algebra can be regarded as a complex JB*-triple under the triple product given by
\[ \{ x, y, z \} := (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^* \]
(see [6] and [36]). By a JBW*-algebra we mean a JB*-algebra which is a dual Banach space. Every JBW*-algebra $\mathcal{A}$ has a unit $1$ [36], so that the binary product of $\mathcal{A}$ can be obtained from the triple product by means of the equality $x \circ y = \{ x, 1, y \}$. 
**Theorem 4.** Let $M > 2$. Then, for every JB*-algebra $\mathcal{A}$, every complex Hilbert space $\mathcal{H}$, and every weak*-continuous linear operator $T: \mathcal{A} \rightarrow \mathcal{H}$, there exists a norm-one positive linear functional $\xi \in \mathcal{A}$ such that the inequality
\[
\|T(x)\| \leq M \|T(\xi(x \circ x^*))\|^{1/2}
\]
holds for all $x \in \mathcal{A}$.

**Proof.** Taking $K := \sqrt{M}$ and $\varepsilon := \sqrt{\frac{1}{2}(M - 2)}$ in Theorem 3, we find norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{A}$, such that
\[
\|T(x)\| \leq K \|T\| \left(\|x\|^2_{\varphi_2} + \varepsilon^2 \|x\|^2_{\varphi_1}\right)^{1/2}
\]
for all $x \in \mathcal{A}$. Let $i = 1, 2$. We choose $e_i \in \mathcal{A}$ with $\varphi_i(e_i) = \|e_i\| = 1$, and denote by $\xi_i$ the mapping $x \mapsto \varphi_i(x \circ e_i)$ from $\mathcal{A}$ to $\mathbb{C}$. Clearly $\xi_i$ is a norm-one weak*-continuous linear functional on $\mathcal{A}$. Moreover, from the identity
\[
\{x, x, e_i\} + \{x^*, x^*, e_i\} = 2e_i \circ (x \circ x^*)
\]
we see that $\xi_i$ is positive and that the equality $\|x\|^2_{\varphi_i} + \|x^*\|^2_{\varphi_i} = 2\xi_i(x \circ x^*)$ holds. Therefore we have $\|x\|^2_{\varphi_i} \leq 2\xi_i(x \circ x^*)$ and hence
\[
\|T(x)\| \leq \sqrt{2} K \|T\| \left(\xi_1(x \circ x^*) + \varepsilon^2 \xi_1(x \circ x^*)\right)^{1/2}.
\]
Finally, putting
\[
\xi := \frac{1}{1 + \varepsilon^2} \left(\xi_1 + \varepsilon^2 \xi_1\right),
\]
we find that $\xi$ becomes a norm-one positive functional in $\mathcal{A}$, and for $x \in \mathcal{A}$ we have
\[
\|T(x)\| \leq \sqrt{2(1 + \varepsilon^2) K \|T\| \left(\xi(x \circ x^*)\right)^{1/2} = M \|T\| \left(\xi(x \circ x^*)\right)^{1/2}. \quad \square
\]

We recall that the bidual of every JB*-algebra $\mathcal{A}$ is a JBW*-algebra containing $\mathcal{A}$ as a JB*-subalgebra.

**Corollary 2.** Let $\mathcal{A}$ be a JB*-algebra and $T$ a bounded linear operator from $\mathcal{A}$ to a complex Hilbert space. Then there exists a norm-one positive functional $\xi \in \mathcal{A}^*$ satisfying
\[
\|T(x)\| \leq 2 \|T\| \left(\xi(x \circ x^*)\right)^{1/2}
\]
for all $x \in \mathcal{A}$.

**Proof.** By Theorem 4, for $n \in \mathbb{N}$ there is a norm-one positive functional $\xi_n \in \mathcal{A}^*$ satisfying
\[
\|T(x)\| \leq \left(2 + \frac{1}{n}\right) \|T\| \left(\xi_n(x \circ x^*)\right)^{1/2}
\]
for all $x \in \mathcal{A}$. Take in $\mathcal{A}^*$ a weak* cluster point $\eta$ of the sequence $\xi_n$. Then $\eta$ is a positive functional with $\|\eta\| \leq 1$, and the inequality
\[
\|T(x)\| \leq 2 \|T\| \left(\eta(x \circ x^*)\right)^{1/2}
\]
holds for all $x \in \mathcal{A}$. If $\eta = 0$, then $T = 0$ and nothing has to be proved. Otherwise take $\xi := (1/\|\eta\|)\eta$. \quad \square
For background about JB- and JBW-algebras the reader is referred to [17]. We recall that JB-algebras and JBW-algebras are nothing but the self-adjoint parts of JB*-algebras and JBW*-algebras, respectively (see [35] and [13] respectively).

**Corollary 3.** Let $K > 2\sqrt{2}$. Then, for every JBW-algebra $A$, every real Hilbert space $H$, and every weak*-continuous linear operator $T: A \to H$, there exists a norm-one positive functional $\xi \in A_+$ such that

$$
\|T(x)\| \leq K\|T\|(\xi(x^2))^{1/2}
$$

for all $x \in A$.

**Proof.** Let $\hat{A}$ denote the JBW*-algebra whose self-adjoint part is equal to $A$, and $\hat{H}$ be the Hilbert space complexification of $H$. Consider the complex-linear operator $\hat{T}: \hat{A} \to \hat{H}$, which extends $T$. Clearly we have $\|\hat{T}\| \leq \sqrt{2}\|T\|$. By Theorem 4 there exists a norm-one positive functional $\xi \in \hat{A}_+$ such that

$$
\|T(x)\| = \|\hat{T}(x)\| \leq \frac{K}{\sqrt{2}} \|\hat{T}\|(\xi(x^2))^{1/2} \leq K\|T\|(\xi(x^2))^{1/2}
$$

for all $x \in A$. Since $\xi$ is positive, $\xi|_A$ is in fact a norm-one positive functional in $A_+$. \hfill $\Box$

The next result follows from the above corollary in the same way that Corollary 2 was derived from Theorem 4.

**Corollary 4 [26, Theorem 3.2].** Let $A$ be a JB-algebra, $H$ a real Hilbert space, and $T: A \to H$ a bounded linear operator. Then there is a norm-one positive linear functional $\varphi \in A^*$ such that

$$
\|T(x)\| \leq 2\sqrt{2}\|T\|(\varphi(x^2))^{1/2}
$$

for all $x \in A$.

**Remark 1.** (1) Since every C*-algebra becomes a JB*-algebra under the Jordan product $x \circ y := \frac{1}{2}(xy + yx)$, it follows from Theorem 4 that, given $M > 2$, a von Neumann algebra $\mathcal{A}$, and a weak*-continuous linear operator $T$ from $\mathcal{A}$ to a complex Hilbert space, there exists a norm-one positive functional $\varphi \in \mathcal{A}^*$ satisfying

$$
\|T(x)\| \leq M\|T\|(\varphi(\frac{1}{2}(xx^* + x^*x)))^{1/2}
$$

for all $x \in \mathcal{A}$. A slightly stronger result can be derived from [16, Proposition 2.3].

(2) As is asserted in [8], Corollary 2 can be proved by translating verbatim Pisier’s arguments for the case of C*-algebras [28, Theorem 9.4]. We note that actually Corollary 2 contains Pisier’s result. Moreover, it is worth mentioning that our proof of Corollary 2 avoids any use of ultraproducts techniques.

Following [19], we define real JB*-triples as norm-closed real subtriples of complex JB*-triples. In [19] it is shown that every real JB*-triple $E$ can be regarded as a real form of a complex JB*-triple. Indeed, given a real JB*-triple $E$ there exists a unique complex JB*-triple structure on the complexification $\tilde{E} = E \oplus iE$, and a unique conjugation (that is, conjugate-linear isometry of period
2) $\tau$ on $\hat{E}$ such that $E = \hat{E}^\tau := \{ x \in \hat{E} : \tau(x) = x \}$. The class of real JB$^*$-triples includes all JB-algebras [17], all real C$^*$-algebras [14], and all JB-algebras [2].

By a real JBW$^*$-triple we mean a real JB$^*$-triple whose underlying Banach space is a dual Banach space. As in the complex case, the triple product of every real JBW$^*$-triple is separately weak$^*$-continuous [25], and the bidual $\delta^{**}$ of a real JB$^*$-triple $\delta$ is a real JBW$^*$-triple whose triple product extends that of $\delta$ [19]. Noticing that every real JBW$^*$-triple is a real form of a complex JBW$^*$-triple [19], we see easily that, if $W$ is a real JBW$^*$-triple and if $\varphi$ is a norm-one element in $W_*$, then, for $z \in W$ such that $\varphi(z) = \|z\| = 1$, the mapping $x \mapsto (\varphi\{x, x, z\})^{1/2}$ is a prehilbert seminorm on $W$ (not depending on $z$). Such a seminorm will be denoted by $\| \cdot \|_{\varphi}$.

Now we proceed to deal with the ‘Little Grothendieck inequality’ for real JBW$^*$-triples. We begin by showing the appropriate version of Lemma 3 for real JBW$^*$-triples. Such a version is obtained by adapting the proof of a recent result of the first author for real JB$^*$-triples (see [26]) to the setting of real JBW$^*$-triples.

**Lemma 4.** Let $M > 1 + 3\sqrt{2}$. Then, for every real JBW$^*$-triple $W$, every real Hilbert space $H$, and every weak$^*$-continuous linear operator $T : W \to H$ which attains its norm, there exists a norm-one functional $\varphi \in W_*$ such that

$$\| T(x) \| \leq M \| x \|_{\varphi}$$

for all $x \in W$.

**Proof.** We follow, with minor changes, the line of the proof of [26, Theorem 4.3]. Without loss of generality, we can suppose $\| T \| = 1$. Write

$$K = \left[ 2\sqrt{2} \left( \frac{M^2}{1 + 3\sqrt{2}} - (1 + \sqrt{2}) \right) \right]^{1/2} > 2\sqrt{2}$$

and $\rho = 2\sqrt{2}/(1 + \sqrt{2})$. By [19, Lemma 3.3], there exists a complete tripotent $e \in W$ with $1 = \| T(e) \|$. Then denoting by $\xi$ the linear functional on $W$ given by $\xi(x) := (T(x) | T(e))$ for every $x \in W$, we see that $\xi$ belongs to $W_*$ and satisfies $\| \xi \| = \xi(e) = 1$. Moreover, when, in the proof of [26, Theorem 4.3], Corollary 3 replaces [26, Theorem 3.2], we obtain the existence of a norm-one functional $\psi \in W_*$ with $\psi(e) = 1$ such that

$$\| T(x) \| \leq K \| x \|_{\psi} + (1 + \sqrt{2}) \| x \|_{\xi}$$

for all $x \in W$. Setting $\varphi := (1 + \rho)^{-1}(\xi + \rho \psi)$, we find that $\varphi$ is a norm-one functional in $W_*$ with $\varphi(e) = 1$, and we have

$$\| T(x) \| \leq \sqrt{(1 + \sqrt{2})^2 + \frac{K^2}{\rho} \| x \|_{\xi}^2 + \rho \| x \|_{\psi}^2}$$

$$= \left[ (1 + \sqrt{2})^2 + \frac{K^2}{\rho} \right]^{1/2} \| x \|_{\varphi} = M \| x \|_{\varphi}$$

for all $x \in W$.  

When, in the proof of Theorem 3, Lemma 4 replaces Lemma 3, we arrive at the following result.
**Theorem 5.** Let $K > 1 + 3\sqrt{2}$ and $\varepsilon > 0$. Then, for every real JBW$^*$-triple $W$, every real Hilbert space $H$, and every weak$^*$-continuous linear operator $T: W \to H$, there exist norm-one functionals $\varphi_1, \varphi_2 \in W_*$ such that the inequality

$$\|T(x)\| \leq K \|T\| \left(\|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2 \right)^{1/2}$$

holds for all $x \in W$.

For norm-one elements $\varphi_1$ and $\varphi_2$ in the predual of a given real JBW$^*$-triple $W$, we define the prehilbert seminorm $\|\cdot\|_{\varphi_1, \varphi_2}$ on $W$ verbatim as in the complex case.

**Corollary 5.** Let $W$ be a real JBW$^*$-triple and $T$ a weak$^*$-continuous linear operator from $W$ to a real Hilbert space. Then there exist norm-one functionals $\varphi_1, \varphi_2 \in W_*$ such that, for every $x \in W$, we have

$$\|T(x)\| \leq 6 \|T\| \|x\|_{\varphi_1, \varphi_2}.$$  

3. Grothendieck’s Theorem for JBW$^*$-triples

In this section we prove ‘Grothendieck’s inequality’ for separately weak$^*$-continuous bilinear forms defined on the cartesian product of two JBW$^*$-triples.

**Theorem 6.** Let $M > 4(1 + 2\sqrt{3})$ and $\varepsilon > 0$. For every pair $(\mathcal{V}^*, \mathcal{W}^*)$ of complex JBW$^*$-triples and every separately weak$^*$-continuous bilinear form $V$ on $\mathcal{V}^* \times \mathcal{W}^*$, there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{V}_*$, and $\psi_1, \psi_2 \in \mathcal{W}_*$, satisfying

$$|V(x, y)| \leq M \|V\| \left(\|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2 \right)^{1/2} \left(\|y\|_{\psi_2}^2 + \varepsilon^2 \|y\|_{\psi_1}^2 \right)^{1/2}$$

for all $(x, y) \in \mathcal{V}^* \times \mathcal{W}^*$.

**Proof.** We begin by noticing that a bilinear form $U$ on $\mathcal{V}^* \times \mathcal{W}^*$ is separately weak$^*$-continuous if and only if there exists a weak$^*$-to-weak-continuous linear operator $F_U: \mathcal{V}^* \to \mathcal{W}^*$, such that the equality

$$U(x, y) = \langle F_U(x), y \rangle$$

holds for every $(x, y) \in \mathcal{V}^* \times \mathcal{W}^*$.

Put $T := F_U: \mathcal{V}^* \to \mathcal{W}^*$, in the sense of the above paragraph. By [8, Lemma 5] there exist a Hilbert space $\mathcal{H}$ and bounded linear operators $S: \mathcal{V}^* \to \mathcal{H}$ and $R: \mathcal{H} \to \mathcal{W}^*$, satisfying $T = RS$ and $\|R\| \|S\| \leq 2(1 + 2\sqrt{3})\|T\|$. Notice that in fact we can form such a factorization in such a way that $R$ is injective. Indeed, take $\mathcal{H}'$ equal to the orthogonal complement of $\text{Ker}(R)$ in $\mathcal{H}$, $R' := R|_{\mathcal{H}'}$, and $S' := \pi_{\mathcal{H}'} S$, where $\pi_{\mathcal{H}'}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}'$, to have $T = R'S'$ with $R'$ injective and $\|R'\| \|S'\| \leq 2(1 + 2\sqrt{3})\|T\|$.

Next we show that $S$ is weak$^*$-continuous. By [12, Corollary V.5.5] it is enough to prove that $S$ is weak$^*$-continuous on bounded subsets of $\mathcal{V}^*$. Let $x_\lambda$ be a bounded net in $\mathcal{V}^*$ weak$^*$-convergent to zero. Take a weak cluster point $h$ of $S(x_\lambda)$ in $\mathcal{H}$. Then $R(h)$ is a weak cluster point of $T(x_\lambda) = RS(x_\lambda)$ in $\mathcal{W}^*$. Moreover, since $T$ is weak$^*$-to-weak-continuous, we have $T(x_\lambda) \to 0$ weakly. It follows that $R(h) = 0$ and hence $h = 0$ by the injectivity of $R$. Now, zero is the unique weak cluster point in $\mathcal{H}$ of the bounded net $S(x_\lambda)$, and therefore we have $S(x_\lambda) \to 0$ weakly.
Now that we know that the operator $S$ is weak*-continuous, we apply Theorem 3 with
\[ K = \sqrt{\frac{M}{2(1+2\sqrt{3})}} > \sqrt{2} \]
to find norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{V}_*$, and $\psi_1, \psi_2 \in \mathcal{W}_*$, satisfying
\[ \| S(x) \| \leq K \| S \| (\| x \|_{\varphi_2}^2 + \varepsilon^2 \| x \|_{\psi_1}^2)^{1/2} \]
and
\[ \| R^*(y) \| \leq K \| R^* \| (\| y \|_{\varphi_2}^2 + \varepsilon^2 \| y \|_{\psi_1}^2)^{1/2} \]
for all $x \in \mathcal{V}$ and $y \in \mathcal{W}$. Therefore
\[ |V(x,y)| = |\langle T(x), y \rangle| = |\langle S(x), R^*(y) \rangle| \leq \frac{M}{2(1+2\sqrt{3})} \| R \| \| S \| (\| x \|_{\varphi_2}^2 + \varepsilon^2 \| x \|_{\psi_1}^2)^{1/2} (\| y \|_{\varphi_2}^2 + \varepsilon^2 \| y \|_{\psi_1}^2)^{1/2} \]
\[ \leq M \| V \| (\| x \|_{\varphi_2}^2 + \varepsilon^2 \| x \|_{\psi_1}^2)^{1/2} (\| y \|_{\varphi_2}^2 + \varepsilon^2 \| y \|_{\psi_1}^2)^{1/2}, \]
for all $(x,y) \in \mathcal{V} \times \mathcal{W}$. Therefore
\[ |V(x,y)| \leq M \| V \| (\varphi(x \circ x^*))^{1/2} (\psi(y \circ y^*))^{1/2} \]
for all $(x,y) \in \mathcal{A} \times \mathcal{B}$. As a relevant particular case we obtain the following result.

**Corollary 6.** Let $M > 8(1 + 2\sqrt{3})$. For every pair $(\mathcal{A}, \mathcal{B})$ of von Neumann algebras and every separately weak*-continuous bilinear form $V$ on $\mathcal{A} \times \mathcal{B}$, there exist norm-one positive functionals $\varphi \in \mathcal{A}_*$ and $\psi \in \mathcal{B}_*$ satisfying
\[ |V(x,y)| \leq M \| V \| (\varphi(\frac{1}{2}(xx^* + x^*x)))^{1/2} (\psi(\frac{1}{2}(yy^* + y^*y)))^{1/2} \]
for all $(x,y) \in \mathcal{A} \times \mathcal{B}$. A refined version of the above corollary can be found in [16, Proposition 2.3].

Now we proceed to deal with Grothendieck’s Theorem for real JBW*-triples. The following lemma generalizes [8, Lemma 5] to the real case.

**Lemma 5.** Let $E$ and $F$ be real JB*-triples and $T: E \to F^*$ a bounded linear operator. Then $T$ has a factorization $T = RS$ through a real Hilbert space with $\| R \| \| S \| \leq 4(1 + 2\sqrt{3})\| T \|$. 

**Proof.** Let us consider the JB*-complexifications $\tilde{E}$ and $\tilde{F}$ of $E$ and $F$, respectively, and denote by $\tilde{T}: E \to \tilde{F}^*$ the complex linear extension of $T$, so that we easily check that $\| \tilde{T} \| \leq 2\| T \|$. As we have mentioned before, $\tilde{T}$ has a factorization $\tilde{T} = \tilde{R}\tilde{S}$ through a complex Hilbert space $\mathcal{H}$, with $\| \tilde{R} \| \| \tilde{S} \| \leq 2(1 + 2\sqrt{3})\| T \|$. 

Since \( \hat{T} \) is the complex linear extension of \( T \), the inclusion \( \hat{T}(E) \subseteq F^* \) holds. Put \( H := \overline{S(E)} \), the closure of \( S(E) \) in \( \mathcal{H} \). Then \( H \) is a real Hilbert space and we have \( \overline{R(H)} \subseteq \overline{S(E)} = \overline{T(E)} \subseteq F^* \).

Finally we define the bounded linear operators \( S := \hat{S} |_E : E \to H \) and \( R := \overline{R} |_H : H \to F^* \). It is easy to see that \( T = RS \) and
\[
\|R\| \|S\| \leq \|\hat{R}\| \|\hat{S}\| \leq 2(1 + 2\sqrt{3})\|\hat{T}\| \leq 4(1 + 2\sqrt{3})\|T\|.
\]

If in the proof of Theorem 6 we replace Lemma 5 and Theorem 5 by \([8, \text{Lemma 5}]\) and Theorem 3, then we obtain the following theorem.

**Theorem 7.** Let \( M > 4(1 + 2\sqrt{3})(1 + 3\sqrt{2})^2 \) and \( \varepsilon > 0 \). For every pair \( (V, W) \) of real \( JBW \)-triples and every separately weak*-continuous bilinear form \( U \) on \( V \times W \), there exist norm-one functionals \( \varphi_1, \varphi_2 \in V^* \), and \( \psi_1, \psi_2 \in W^* \) satisfying
\[
|U(x, y)| \leq M \|U\| \|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2 \)\] \[\times \left( \|y\|_{\psi_2}^2 + \varepsilon^2 \|y\|_{\psi_1}^2 \right)^{1/2}
\]
for all \( (x, y) \in V \times W \).

Thanks to Lemma 5, Lemma 1 remains true when real \( JB^* \)-triples replace complex ones. Then Theorems 7 and 6 give rise to the real and complex cases, respectively, of the result which follows.

**Corollary 7.** Let
\[
M > 4(1 + 2\sqrt{3})(1 + 3\sqrt{2})^2 \quad (\text{respectively, } M > 4(1 + 2\sqrt{3}))
\]
and \( \varepsilon > 0 \). Then for every pair \( (E, F) \) of real (respectively, complex) \( JB^* \)-triples and every bounded bilinear form \( U \) on \( E \times F \) there exist norm-one functionals \( \varphi_1, \varphi_2 \in E^* \) and \( \psi_1, \psi_2 \in F^* \) satisfying
\[
|U(x, y)| \leq M \|U\| \|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2 \)\] \[\times \left( \|y\|_{\psi_2}^2 + \varepsilon^2 \|y\|_{\psi_1}^2 \right)^{1/2}
\]
for all \( (x, y) \in E \times F \).

**Remark 2.** In the complex case of the above corollary, the interval of variation of the constant \( M \) can be enlarged by arguing as follows. Let \( M > 3 + 2\sqrt{3}, \varepsilon > 0, \delta \) and \( \mathcal{F} \) be complex \( JB^* \)-triples, and \( U \) be a norm-one bounded bilinear form on \( \delta \times \mathcal{F} \). Consider the separately weak*-continuous bilinear form \( \hat{U} \) on \( \delta^{**} \times \mathcal{F}^{**} \) which extends \( U \), and take a weak*-to-weak continuous linear operator \( T : E^{**} \to F^* \) satisfying
\[
\hat{U}(\alpha, \beta) = \langle T(\alpha), \beta \rangle
\]
for all \( (\alpha, \beta) \in \delta^{**} \times \mathcal{F}^{**} \). Choose \( \delta > 0 \) such that \( \delta \leq \varepsilon^2 \) and
\[
(3 + 2\sqrt{3})(1 + \delta) \leq M.
\]
By \([30, \text{Corollary 2}]\) there is a rank-one weak*-to-weak continuous linear operator \( T_1 : \delta^{**} \to \mathcal{F}^* \) such that \( \|T_1\| \leq \delta \) and \( T_2 := T - T_1 \) attains its norm. Since \( T_1 \) is of rank 1 and weak*-continuous, it also attains its norm. For \( i = 1, 2 \), consider the separately weak*-continuous bilinear form \( \hat{U}_i \) on \( \delta^{**} \times \mathcal{F}^{**} \) defined by
\[
\hat{U}_i(\alpha, \beta) = \langle T_i(\alpha), \beta \rangle,
\]
and put $U_i = \tilde{U}_i|_{\mathcal{E} \times \mathcal{F}}$, so that $U_i$ is a bounded bilinear form on $\mathcal{E} \times \mathcal{F}$ whose separately weak$^*$-continuous extension to $\mathcal{E}^{**} \times \mathcal{F}^{**}$ attains its norm. By the proof of [3, Theorem 1.4], there exist norm-one functionals $\varphi_1, \varphi_2 \in \mathcal{E}^*$ and $\psi_1, \psi_2 \in \mathcal{F}^*$ such that

$$|U_i(x, y)| \leq (3 + 2 \sqrt{3}) \|U_i\| \|x\|_{\varphi_i} \|y\|_{\psi_i},$$

for all $(x, y) \in \mathcal{E} \times \mathcal{F}$ and $i = 1, 2$.

Therefore

$$|U(x, y)| \leq |U_2(x, y)| + |U_1(x, y)|
\leq (3 + 2 \sqrt{3})((1 + \delta)\|x\|_{\varphi_2} \|y\|_{\psi_2} + \|x\|_{\varphi_1} \|y\|_{\psi_1})
\leq (3 + 2 \sqrt{3})(1 + \delta)(\|x\|_{\varphi_2} \|y\|_{\psi_2} + \|x\|_{\varphi_1} \|y\|_{\psi_1})
\leq (3 + 2 \sqrt{3})(1 + \delta)\sqrt{\|x\|_{\varphi_2}^2 + \|x\|_{\varphi_1}^2} \sqrt{\|y\|_{\psi_2}^2 + \|y\|_{\psi_1}^2}
\leq M(\|x\|_{\varphi_2}^2 + \varepsilon^2 \|x\|_{\varphi_1}^2)^{1/2}(\|y\|_{\psi_2}^2 + \varepsilon^2 \|y\|_{\psi_1}^2)^{1/2}$$

for all $(x, y) \in E \times F$.

We do not know if the value $\varepsilon = 0$ is allowed in Theorems 6 and 7. In any case, as the next result shows, the value $\varepsilon = 0$ is allowed for a ‘large number’ of separately weak$^*$-continuous bilinear forms.

**Theorem 8.** Let $M > 4(1 + 2 \sqrt{3})(1 + 3 \sqrt{2})^2$ (respectively, $M > 4(1 + 2 \sqrt{3})$) and $V$ and $W$ be real (respectively, complex) JBW$^*$-triples. Then the set of all separately weak$^*$-continuous bilinear forms $U$ on $V \times W$ such that there exist norm-one functionals $\varphi \in V_*$ and $\psi \in W_*$ satisfying

$$|U(x, y)| \leq M \|U\| \|x\|_{\varphi} \|y\|_{\psi}$$

for all $(x, y) \in V \times W$, is norm dense in the set of all separately weak$^*$-continuous bilinear forms on $V \times W$.

**Proof.** Let $U$ be a non-zero separately weak$^*$-continuous bilinear form on $V \times W$. By the proof of Theorem 7 (respectively, Theorem 6) there exists a real (respectively, complex) Hilbert space $H$ such that for all $(x, y) \in V \times W$ we have

$$U(x, y) := \langle F(x), G(y) \rangle,$$

where $F: V \to H$ and $G: W \to H^*$ are weak$^*$-continuous linear operators satisfying $\|F\| \|G\| \leq L \|U\|$ with $L = 4(1 + 2 \sqrt{3})$ (respectively, $L = 2(1 + 2 \sqrt{3})$).

By [37], there are sequences $\{F_n: V \to H\}$ and $\{G_n: W \to H^*\}$ of weak$^*$-continuous linear operators, converging in norm to $F$ and $G$, respectively, and such that $F_n$ and $G_n$ attain their norms for every $n$. Then, putting

$$U_n(x, y) := \langle F_n(x), G_n(y) \rangle \quad ((n, x, y) \in \mathbb{N} \times V \times W),$$

we see that $\{U_n\}$ becomes a sequence of separately weak$^*$-continuous bilinear forms on $V \times W$, converging in norm to $U$. Take $\sqrt{M/L} > K > 1 + 3 \sqrt{2}$ (respectively, $\sqrt{M/L} > K > \sqrt{2}$). Applying Lemma 4 (respectively, Lemma 3),
for \( n \in \mathbb{N} \) we find norm-one functionals \( \varphi_n \in V_* \) and \( \psi_n \in W_* \) satisfying
\[
|F_n(x)| \leq K \|F_n\| \|x\|_{\varphi_n} \quad \text{and} \quad |G_n(y)| \leq K \|G_n\| \|y\|_{\psi_n}
\]
for all \((x, y) \in V \times W\).

Set
\[
\delta = \frac{M / K^2 - L}{1 + L} \frac{\|U\|}{2} > 0,
\]
and take \( m \in \mathbb{N} \) such that the inequalities
\[
\|F_n\| \|G_n\| - \|F\| \|G\| < \delta, \\
\|U_n\| - \|U\| < \delta \quad \text{and} \quad \|U_n\| \geq \frac{1}{2} \|U\|
\]
hold for every \( n \geq m \).

Now for \( n \geq m \) and \((x, y) \in V \times W\) we have
\[
|U_n(x, y)| \leq K^2 \|F_n\| \|G_n\| \|x\|_{\varphi_n} \|y\|_{\psi_n}
\]
\[
\leq K^2 (\|F\| \|G\| + \delta) \|x\|_{\varphi_n} \|y\|_{\psi_n}
\]
\[
\leq K^2 (L \|U\| + \delta) \|x\|_{\varphi_n} \|y\|_{\psi_n}
\]
\[
\leq K^2 \left( L \|U_n\| + (M / K^2 - L) \frac{\|U\|}{2} \right) \|x\|_{\varphi_n} \|y\|_{\psi_n}
\]
\[
\leq M \|U_n\| \|x\|_{\varphi_n} \|y\|_{\psi_n}.
\]

As we observed before Corollary 7, Lemma 1 remains true in the real setting. Then, given real or complex JB*-triples \( E \) and \( F \), the mapping sending each element \( U \in \mathcal{L}(2(E \times F)) \) to its unique separately weak*-continuous bilinear extension \( \tilde{U} \) to \( E^{**} \times F^{**} \) is an isometry from \( \mathcal{L}(2(E \times F)) \) onto the Banach space of all separately weak*-continuous bilinear forms on \( E^{**} \times F^{**} \). Therefore we obtain the following corollary.

**Corollary 8.** Let \( M > 4(1 + 2 \sqrt{3})(1 + 3 \sqrt{2})^2 \) (respectively, \( M > 4(1 + 2 \sqrt{3}) \)) and \( E \) and \( F \) be real (respectively, complex) JB*-triples. Then the set of all bounded bilinear forms \( U \) on \( E \times F \) such that there exist norm-one functionals \( \varphi \in E^* \) and \( \psi \in F^* \) satisfying
\[
|U(x, y)| \leq M \|U\| \|x\|_{\varphi} \|y\|_{\psi}
\]
for all \((x, y) \in E \times F\), is norm dense in \( \mathcal{L}(2(E \times F)) \).

We note that Theorem 1 is finer than the complex case of the above corollary. However, since Theorem 1 depends on the proof of [3, Theorem 1.4], it is much more difficult.

**Remark 3.** We do not know if the value \( \varepsilon = 0 \) is allowed in Theorems 3 and 5 (respectively, in Theorems 6 and 7) for some value of the constant \( K \) (respectively, \( M \)). Concerning this question, it is worth mentioning that the following three assertions are equivalent.
(i) There is a universal constant \( G \) such that, for every real (respectively, complex) JBW*-triple \( W \) and every pair \( (\varphi, \varphi_2) \) of norm-one functionals in \( W \times W \), we can find a norm-one functional \( \varphi \in W \) satisfying
\[
\| x \|_{\varphi_i} \leq G \| x \|_{\varphi_i}
\]
for every \( x \in W \) and \( i = 1, 2 \).

(ii) There is a universal constant \( \hat{G} \) such that for every pair of real (respectively, complex) JBW*-triples \( (V, W) \) and every separately weak*-continuous bilinear form \( U \) on \( V \times W \), there are norm-one functionals \( \varphi \in V \) and \( \psi \in W \) satisfying
\[
\| U(x, y) \| \leq \hat{G} \| U \| \| x \|_{\varphi} \| y \|_{\psi}
\]
for all \( (x, y) \in V \times W \).

(iii) There is a universal constant \( \hat{G} \) such that for every real (respectively, complex) JBW*-triple \( W \) and every weak*-continuous linear operator \( T \) from \( W \) to a real (respectively, complex) Hilbert space, there exists a norm-one functional \( \varphi \in W \) satisfying
\[
\| T(x) \| \leq \hat{G} \| T \| \| x \|_{\varphi}
\]
for all \( x \in W \).

The implication (i) \( \Rightarrow \) (ii) follows from Theorems 6 and 7.

Assume that Assertion (ii) above is true. Let \( W \) be a real (respectively, complex) JBW*-triple, \( H \) a real (respectively, complex) Hilbert space, and \( T: W \to H \) a weak*-continuous linear operator. Consider the separately weak*-continuous bilinear form \( U \) on \( W \times H \) given by \( U(x, y) = (T(x), y) \) (respectively, \( U(x, y) = (T(x), \sigma(y)) \), where \( \sigma \) is a conjugation on \( H \)). Regarding \( H \) as a JBW*-triple under the triple product \( \{ x, y, z \} := \frac{1}{2} ((x, y)z + (z, y)x) \), and applying the assumption, we find norm-one functionals \( \varphi \in W \) and \( \psi \in H \) satisfying
\[
\| U(x, y) \| \leq \hat{G} \| U \| \| x \|_{\varphi} \| y \|_{\psi}
\]
for all \( (x, y) \in W \times H \). Taking \( y = T(x) \) (respectively, \( y = \sigma(T(x)) \)) we obtain
\[
\| T(x) \| \leq \hat{G} \| T \| \| x \|_{\varphi}
\]
for all \( x \in W \). In this way Assertion (iii) holds.

Finally let us assume that Assertion (iii) is true. Let \( W \) be a real (respectively, complex) JBW*-triple and \( \varphi_1 \) and \( \varphi_2 \) norm-one functionals in \( W \). Since \( \| \cdot \|_{\varphi_1, \varphi_2} \), comes from a suitable separately weak*-continuous positive sesquilinear form \( \langle \cdot, \cdot \rangle \) on \( W \) by means of the equality \( \| x \|_{\varphi_1, \varphi_2} = \langle x, x \rangle \), it follows from the proof of [31, Corollary] that there exists a weak*-continuous linear operator \( T \) from \( W \) to a real (respectively, complex) Hilbert space satisfying \( \| x \|_{\varphi_1, \varphi_2} = \| T(x) \| \) for all \( x \in W \) (which implies \( \| T \| \leq \sqrt{2} \)). Now applying the assumption we find a norm-one functional \( \varphi \in W \) such that
\[
\| x \|_{\varphi_1, \varphi_2} = \| T(x) \| \leq \hat{G} \| T \| \| x \|_{\varphi} \leq \sqrt{2} \hat{G} \| x \|_{\varphi}
\]
for all \( x \in W \). As a consequence, for \( i = 1, 2 \) we have
\[
\| x \|_{\varphi_i} \leq \sqrt{2} \hat{G} \| x \|_{\varphi_i}
\]
for all \( x \in W \).
4. Some applications

We define the strong\(^*\)-topology \(S^*(W, W_*^t)\) of a given real or complex JBW\(^*\)-triple \(W\) as the topology on \(W\) generated by the family of seminorms \(\{ \| \cdot \|_{\varphi^*} : \varphi \in W_*, \| \varphi \| = 1 \}\). In the complex case, the above notion has been introduced by T. J. Barton and Y. Friedman in [4]. When a JBW\(^*\)-algebra \(\mathcal{A}\) is regarded as a complex JBW\(^*\)-triple, \(S^*(\mathcal{A}, \mathcal{A}_*)\) coincides with the so-called ‘algebra-strong’ topology’ of \(\mathcal{A}\), namely the topology on \(\mathcal{A}\) generated by the family of seminorms of the form \(x \mapsto \sqrt{\xi(x \circ x^*)}\) when \(\xi\) is any positive functional in \(\mathcal{A}_*\) [31, Proposition 3]. As a consequence, when a von Neumann algebra \(\mathcal{H}\) is regarded as a complex JBW\(^*\)-triple, \(S^*(\mathcal{H}, \mathcal{H}_*)\) coincides with the familiar strong\(^*\)-topology of \(\mathcal{H}\) (compare [33, Definition 1.8.7]).

We note that, if \(\mathcal{H}\) is a complex JBW\(^*\)-triple, then, denoting by \(\mathcal{H}_R\) the realization of \(\mathcal{H}\) (that is, the real JBW\(^*\)-triple obtained from \(\mathcal{H}\) by restriction of scalar to \(\mathbb{R}\)), we have \(S^*(\mathcal{H}, \mathcal{H}_*) = S^*(\mathcal{H}_R, (\mathcal{H}_R)_*)\). Indeed, the mapping \(\varphi \mapsto \text{Re}\varphi\) identifies \(\mathcal{H}_*\) with \((\mathcal{H}_R)_*\), and, when \(\varphi\) has norm 1, the equality \(\|x\|_\varphi = \|x\|_{\text{Re}\varphi}\) holds for every \(x \in \mathcal{H}\).

**Proposition 1.** Let \(W\) be a real (respectively, complex) JBW\(^*\)-triple. The following topologies coincide in \(W\):

- (i) the strong\(^*\)-topology of \(W\);
- (ii) the topology on \(W\) generated by the family of seminorms of the form \(x \mapsto \sqrt{\langle x, x \rangle}\), where \(\langle \cdot, \cdot \rangle\) is any separately weak\(^*\)-continuous positive sesquilinear form on \(W\);
- (iii) the topology on \(W\) generated by the family of seminorms \(x \mapsto \|T(x)\|\), when \(T\) runs over all weak\(^*\)-continuous linear operators from \(W\) to arbitrary real (respectively, complex) Hilbert spaces.

**Proof.** Let us denote by \(\tau_1, \tau_2, \text{ and } \tau_3\) the topologies arising in (i), (ii), and (iii), respectively. The inequality \(\tau_1 \supseteq \tau_3\) follows from Corollary 5 (respectively, Corollary 1). Since the proof of [31, Corollary 1] shows that for every separately weak\(^*\)-continuous positive sesquilinear form \(\langle \cdot, \cdot \rangle\) on \(W\) there exists a weak\(^*\)-continuous linear operator \(T\) from \(W\) to a real (respectively, complex) Hilbert space satisfying \(\sqrt{\langle x, x \rangle} = \|T(x)\|\) for all \(x \in W\), we have \(\tau_3 \supseteq \tau_2\). Finally, since for every norm-one functional \(\varphi \in W_*\) there is a separately weak\(^*\)-continuous positive sesquilinear form \(\langle \cdot, \cdot \rangle\) satisfying \(\|x\|_\varphi = \sqrt{\langle x, x \rangle}\) for all \(x \in W\), the inequality \(\tau_2 \supseteq \tau_1\) follows.

For every dual Banach space \(X, B_X\) will stand for the closed unit ball of \(X\). For every dual Banach space \(X\) (with a fixed predual denoted by \(X_*\)), we denote by \(m(X, X_*)\) the Mackey topology on \(X\) relative to its duality with \(X_*\).

**Corollary 9.** Let \(W\) be a real or complex JBW\(^*\)-triple. Then the strong\(^*\)-topology of \(W\) is compatible with the duality \((W, W_*)\).

**Proof.** We apply the characterization of \(S^*(W, W_*^t)\) given by (iii) in Proposition 1. Clearly \(S^*(W, W_*^t)\) is stronger than the weak\(^*\)-topology \(\sigma(W, W_*^t)\) of \(W\). On the other hand, if \(T\) is a weak\(^*\)-continuous linear operator from \(W\) to a
Hilbert space $H$, and if we put $T = S^*$ for a suitable bounded linear operator $S: H \to W$, then $S(B_{B_{H^*}})$ is an absolutely convex and weakly compact subset of $W$, and we have $\|T(x)\| = \sup \langle x, S(B_{B_{H^*}}) \rangle$. This shows that $S^*(W, W)$ is weaker than $m(W, W)$. □

The complex case of the above corollary is due to T. J. Barton and Y. Friedman [4]. The complex case of Proposition 1 is claimed in [31, Corollary 2] (see also [32, Proposition D.17]), but the proof relies on [31, Proposition 1], which subsumes a gap from [3] (see the comments before Lemma 3). Now that we have saved [31, Corollary 2], all subsequent results in [31] concerning the strong*-topology of complex JBW*-triples are valid. Moreover, keeping in mind Proposition 1 and Corollary 9, we note that some of those results remain true for real JBW*-triples with verbatim proof. For instance, the following assertions hold.

(i) Linear mappings between real JBW*-triples are strong*-continuous if and only if they are weak*-continuous (compare [31, Corollary 3]).

(ii) If $W$ is a real JBW*-triple, and if $V$ is a weak*-closed subtriple, then the inequality $S^*(W, W)_V = S^*(V, V)$ holds, and in fact $S^*(W, W)_V$ and $S^*(V, V)$ coincide on bounded subsets of $V$ (compare [31, Proposition 2]).

It follows from the first part of Assertion (ii) above and a new application of Proposition 1 that, if $W$ is a real JBW*-triple, and if $V$ is a weak*-complemented subtriple of $W$, then we have $S^*(W, W)_V = S^*(V, V)$. Since every real JBW*-triple $V$ is weak*-complemented in the realification of a complex JBW*-triple $\mathcal{W}$ (see $V$ as a real form of its JB*-complexification), and $S^*(\mathcal{W}, \mathcal{W}) = S^*(\mathcal{W}_R, \mathcal{W}_R)$, the results [31, Theorem] and [32, Theorem D.21] for complex JBW*-triples can be transferred to the real setting, providing the following result.

**Theorem 9.** Let $W$ be a real JBW*-triple. Then the triple product of $W$ is jointly $S^*(W, W)$-continuous on bounded subsets of $W$, and the topologies $m(W, W)$ and $S^*(W, W)$ coincide on bounded subsets of $W$.

Our concluding goal in this paper is to establish, in the setting of real JB*-triples, a result on weakly compact operators originally due to H. Jarchow [20] in the context of $C^*$-algebras, and later extended to complex JB*-triples by C.-H. Chu and B. Iochum [7]. This could be made by transferring the complex results to the real setting by a complexification method. However, we prefer to do it in a more intrinsic way, by deriving the result from the second assertion in Theorem 9 according to some ideas outlined in [32, pp. 142, 143].

**Proposition 2.** Let $X$ be a dual Banach space (with a fixed predual $X_*$). Then the Mackey topology $m(X, X_*)$ coincides with the topology on $X$ generated by the family of seminorms $x \mapsto \|T(x)\|$, where $T$ is any weak*-continuous linear operator from $X$ to a reflexive Banach space.

**Proof.** Let us denote by $\tau$ the second topology arising in the statement. As in the proof of Corollary 9, if $T$ is a weak*-continuous linear operator from $X$ to a reflexive Banach space, then there exists an absolutely convex and weakly compact subset $D$ of $X$, such that the equality

$$\|T(x)\| = \sup \langle x, D \rangle$$

holds for every $x \in X$. This shows that $\tau \equiv m(X, X_*)$. 


Let $D$ be an absolutely convex and weakly compact subset of $X$. Consider the Banach space $\ell_1(D)$ and the bounded linear operator 

$$F: \ell_1(D) \to X$$

given by 

$$F(\{\lambda_\varphi\}_{\varphi \in D}) := \sum_{\varphi \in D} \lambda_\varphi \varphi.$$ 

Then we have $F(B_{\ell_1(D)}) = D$, and hence $F$ is weakly compact. By [10] there exists a reflexive Banach space $Y$, together with bounded linear operators $S: \ell_1(D) \to Y$ and $R: Y \to X$, such that $F = RS$. Then, for $x \in X$, we have

$$\sup \| \langle x, D \rangle \| = \sup \| \langle x, F(B_{\ell_1(D)}) \rangle \|$$

$$= \sup \| \langle x, R(S(B_{\ell_1(D)}) \rangle \|$$

$$\leq \| S \| \sup \| \langle x, R(B_Y) \rangle \|$$

$$= \| S \| \| R^*(x) \|.$$

Since $D$ is an arbitrary absolutely convex and weakly compact subset of $X$, and $R^*$ is a weak$^*$-continuous linear operator from $X$ to the reflexive Banach space $Y^*$, the inequality $m(X, X^*) \leq \tau$ follows.

Let $X$ be a dual Banach space (with a fixed predual $X$). In agreement with Proposition 1, we define the strong$^*$-topology of $X$, denoted by $S^*(X, X^*)$, as the topology on $X$ generated by the family of seminorms $x \mapsto \| T(x) \|$, where $T$ is any weak$^*$-continuous linear operator from $X$ to a Hilbert space.

**Proposition 3.** Let $X$ be a dual Banach space (with a fixed predual $X$). Then the following assertions are equivalent:

(i) the topologies $m(X, X^*)$ and $S^*(X, X^*)$ coincide on bounded subsets of $X$;

(ii) for every weak$^*$-continuous linear operator $F$ from $X$ to a reflexive Banach space, there exists a weak$^*$-continuous linear operator $G$ from $X$ to a Hilbert space satisfying $\| F(x) \| \leq \| G(x) \| + \| x \|$ for all $x \in X$;

(iii) for every weak$^*$-continuous linear operator $F$ from $X$ to a reflexive Banach space, there exist a weak$^*$-continuous linear operator $G$ from $X$ to a Hilbert space and a mapping $N: (0, \infty) \to (0, \infty)$ satisfying

$$\| F(x) \| \leq N(\varepsilon) \| G(x) \| + \varepsilon \| x \|$$

for all $x \in X$ and $\varepsilon > 0$.

**Proof.** (i) $\Rightarrow$ (ii) Let $F$ be a weak$^*$-continuous linear operator from $X$ to a reflexive Banach space. Then, by Proposition 2,

$$\mathbb{V} := \{ y \in B_X : \| F(y) \| \leq 1 \}$$

is an $m(X, X^*)_{B_X}$-neighborhood of zero in $B_X$. By assumption, there exist Hilbert spaces $H_1, \ldots, H_n$ and weak$^*$-continuous linear operators $G_i: X \to H_i$ ($i = 1, \ldots, n$) such that

$$\mathbb{V} \supseteq \bigcap_{i=1}^n \{ y \in B_X : \| G_i(y) \| \leq 1 \}.$$
Now set $H := (\bigoplus_{n=1}^\infty H_n)_{\ell_2}$, and consider the weak*‐continuous linear operator $G: X \to H$ defined by $G(x) := (G_1(x), \ldots, G_n(x))$. Notice that

$$\{ y \in B_X: \| G(y) \| \leq 1 \} \subseteq \bigcap_{n=1}^\infty \{ y \in B_X: \| G_i(y) \| \leq 1 \} \subseteq \emptyset.$$

Finally, if $x \in X \setminus \{0\}$, then $(1/\|x\| + \| G(x) \|) x$ lies in $\{ y \in B_X: \| G(y) \| \leq 1 \} \subseteq \emptyset$, and hence

$$\left\| F \left( \frac{1}{\|x\| + \| G(x) \|} x \right) \right\| \leq 1.$$

(ii) $\Rightarrow$ (iii) Let $F$ be a weak*‐continuous linear operator from $X$ to a reflexive Banach space. By assumption, for every $n \in \mathbb{N}$ there exist a Hilbert space $H_n$ and a weak*‐continuous linear operator $G_n$ from $X$ to $H_n$ such that $\| nF(x) \| \leq \| G_n(x) \| + \| x \|$ for all $x \in X$. Now set $H := (\bigoplus_{n=1}^\infty H_n)_{\ell_2}$, and consider the bounded linear operator $G: X \to H$ defined by

$$G(x) := \left\{ \frac{1}{n \| G_n \|} G_n(x) \right\}$$

and the mapping $N: \varepsilon \to \| G_n(\varepsilon) \|$ (where $n(\varepsilon)$ denotes the smallest natural number satisfying $n > 1/\varepsilon$). Then $G$ is weak*‐continuous. Indeed, given $y = \{ h_n \} \in H$, we can take, for $n \in \mathbb{N}$, $\alpha_n$ in $X_*$ satisfying $(G_n(x) \mid h_n) = \langle x, \alpha_n \rangle$ for every $x \in X$, so that we have

$$\sum_{n \in \mathbb{N}} \frac{\alpha_n}{n \| G_n \|} \leq \sum_{n \in \mathbb{N}} \frac{\| h_n \|}{n} \leq \sqrt{\sum_{n \in \mathbb{N}} \| h_n \|^2} \sqrt{\sum_{n \in \mathbb{N}} \frac{1}{n^2}} < \infty,$$

and hence

$$\alpha := \sum_{n \in \mathbb{N}} \frac{\alpha_n}{n \| G_n \|}$$

is an element of $X_*$ satisfying $(G(x) \mid h) = \langle x, \alpha \rangle$ for all $x \in X$. Moreover, for all $\varepsilon > 0$ and $x \in X$ we have

$$\| F(x) \| \leq \frac{1}{n(\varepsilon)} \| G_n(\varepsilon) (x) \| + \frac{1}{n(\varepsilon)} \| x \|$$

$$\leq \| G_n(\varepsilon) \| \| G(x) \| + \frac{1}{n(\varepsilon)} \| x \|$$

$$\leq N(\varepsilon) \| G(x) \| + \varepsilon \| x \|.$$

(iii) $\Rightarrow$ (i) Let $x_\lambda$ be a net in $B_X$ converging to zero in the topology $S^*(X, X_*)$. Let $F$ be a weak*‐continuous linear operator from $X$ to a reflexive Banach space, and $\varepsilon > 0$. By assumption, there exist a weak*‐continuous linear operator $G$ from $X$ to a Hilbert space and a mapping $N: (0, \infty) \to (0, \infty)$ satisfying

$$\| F(x) \| \leq N(\lambda) \| G(x) \| + \frac{1}{2} \| x \|$$

for all $x \in X$. Take $\lambda_0$ such that $\| G(x_\lambda) \| \leq \varepsilon / 2N(\lambda_0)$ whenever $\lambda \geq \lambda_0$. Then we have $\| F(x_\lambda) \| \leq \varepsilon$ for all $\lambda \geq \lambda_0$. By Proposition 2, $x_\lambda m(X, X_*)$‐converges to zero. □

We can now state the following characterization of weakly compact operators on JB*‐triples.
Theorem 10. Let $E$ be a real (respectively, complex) $JB^*$-triple, $X$ a real (respectively, complex) Banach space, and $T: E \to X$ a bounded linear operator. The following assertions are equivalent:

(i) $T$ is weakly compact;

(ii) there exist a bounded linear operator $G$ from $E$ to a real (respectively, complex) Hilbert space and a function $N: (0, +\infty) \to (0, +\infty)$ such that

$$\|T(x)\| \leq N(\varepsilon)\|G(x)\| + \varepsilon\|x\|$$

for all $x \in E$ and $\varepsilon > 0$;

(iii) there exist norm-one functionals $\varphi_1, \varphi_2 \in E^*$ and a function $N: (0, +\infty) \to (0, +\infty)$ such that

$$\|T(x)\| \leq N(\varepsilon)\|x\|_{\varphi_1, \varphi_2} + \varepsilon\|x\|$$

for all $x \in E$ and $\varepsilon > 0$.

Proof. The implication (ii) $\Rightarrow$ (iii) follows from Corollary 5 (respectively, Corollary 1). The implication (iii) $\Rightarrow$ (ii) holds because, for norm-one functionals $\varphi_1, \varphi_2 \in E^*$, $\|\cdot\|_{\varphi_1, \varphi_2}$ is a prehilbert seminorm on $E$, and hence there exists a bounded linear operator $G$ from $E$ to a Hilbert space satisfying $\|G(x)\| = \|x\|_{\varphi_1, \varphi_2}$ for all $x \in E$. On the other hand, the implication (ii) $\Rightarrow$ (i) is known to be true, even if $E$ is an arbitrary Banach space (see for instance [20, Theorem 20.7.3]). To conclude the proof, let us show that (i) implies (ii). Assume that Assertion (i) holds. Then, by [10], there exist a reflexive Banach space $Y$ and bounded linear operators $F: E \to Y$ and $S: Y \to X$ such that $T = SF$ and $\|S\| \leq 1$. By Theorem 9 and Proposition 3, there exist a weak* continuous linear operator $\tilde{G}$ from $E^{**}$ to a Hilbert space and a mapping $N: (0, \infty) \to (0, \infty)$ satisfying

$$\|F^{**}(\alpha)\| \leq N(\varepsilon)\|\tilde{G}(\alpha)\| + \varepsilon\|\alpha\|$$

for all $\alpha \in E^{**}$ and $\varepsilon > 0$. By putting $G := \tilde{G}|_E$, the inequality in Assertion (ii) follows.

The complex case of the above theorem is established in [7, Theorem 11], with $\|\cdot\|_{\varphi_1, \varphi_2}$ in Assertion 3 replaced with $\|\cdot\|_{\varphi}$ for a single norm-one functional $\varphi \in E^*$. As we have noticed on similar occasions, we do not know if such a replacement is correct.

References

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