The notion of FTF ring (see Definition 1.1 or Proposition 1.2) captures homological and finiteness properties shared by several classes of rings. Thus coherent rings with left flat-dominant dimension ≥ 1 [3, Corolario 2.2.11] or rings having quasi-Frobenius two-sided maximal quotient ring [7, Proposición 2.3.10] are examples of FTF rings. Moreover, FTF ring and QF-3 ring are related concepts (for the notion of QF-3 ring, see, e.g., [24]). For instance, any left perfect left FTF ring is QF-3 (see [3, Proposición 2.4.1]) and, in fact, a perfect ring is FTF if and only if it is QF-3 (see [7, Corollary 2.11]). More recently, FTF rings have been considered to characterize the rings for which the category of flat right $R$-modules is abelian [1].

Although the notion of left FTF ring is given in terms of torsion theories it can be restated without reference to the torsion theoretic framework (see Proposition 1.2). In fact, we prove in Theorem 1.8 that a ring $R$ is a two-sided FTF ring if and only if every direct product of copies of $E(R)$ and every direct product of copies of $E(R)$ is flat (here, $E(R)$ and $E(R)$ are, respectively, the injective hulls of $R$ and $R$).

In this note, we will prove that, for a Frobenius ring extension $R \subseteq S$ in the sense of F. Kasch [14], the ring $R$ is left FTF if and only if $S$ is a left FTF. In fact, this result is valid (see Corollary 2.6) for more general
ring extensions including quasi-Frobenius extensions in the sense of B. J. Müller [21]. As an application of the theory of FTF rings developed in [3, 4, 6, 7], in conjunction with the new results expounded here, we will prove the following theorem.

**Theorem.** Let $R \subseteq S$ be a quasi-Frobenius ring extension. The following statements hold.

1. $R$ is almost coherent with left (or right) flat-dominant dimension $\geq 1$ if and only if $S$ is almost coherent with left (or right) flat-dominant dimension $\geq 1$.
2. $R$ has a semi-primary QF-3 two-sided maximal quotient ring if and only if $S$ has a semi-primary QF-3 two-sided maximal quotient ring.
3. If $R$ is almost coherent with left (or right) flat-dominant dimension $\geq 2$, then $S$ is almost coherent with left (or right) flat-dominant dimension $\geq 2$.
4. If $R$ has a QF two-sided maximal quotient ring, then $S$ has a QF two-sided maximal quotient ring.

If, in addition, either $R_S$ is a generator of the category $R$-Mod or $S_R$ is a generator of the category $\text{Mod}_R$ (e.g., if $R \subseteq S$ is a split extension), then the converse statements of (1) and (2) hold.

The definitions of almost coherent ring and of flat-dominant dimension can be found in 1.5 and 1.15, respectively.

1. **FTF RINGS**

All rings are associative with unity and all modules are unital. The injective hull of a module $M$ will be denoted by $E(M)$. The category of all left $R$-modules over a ring $R$ is denoted by $R$-Mod. The reader is assumed to be familiar with the notion of (hereditary) torsion theory. An introduction to the basic properties of hereditary torsion theories on $R$-Mod can be found in [22].

**Definition 1.1.** The ring $R$ is said to be a left FTF ring if there is a hereditary torsion theory $\gamma$ on $R$-Mod such that a left $R$-module $M$ is $\gamma$-torsionfree if and only if $M$ embeds in a flat left $R$-module.

For right FTF rings we will use the notation $\gamma'$; i.e., if $R$ is a right FTF ring, then $\gamma'$ will denote the hereditary torsion theory on $\text{Mod}_R$ such that the $\gamma'$-torsionfree modules are precisely the right $R$-modules that are isomorphic to a submodule of a flat right $R$-module. For a left and right FTF ring $R$ we say simply that $R$ is FTF, and this convention is valid for
any one-sided concept (e.g., a noetherian ring is a left and right noetherian ring).

It is possible to avoid torsion theories in the definition of left FTF ring, as the following characterization shows.

**Proposition 1.2.** A ring $R$ is left FTF if and only if the following two conditions are satisfied.

1. The injective hull of every flat left $R$-module is flat.
2. The direct product of any family of injective and flat left $R$-modules is a flat left $R$-module.

**Proof.** A proof can be found in [3, Proposici´on 2.1.2].

**Remark 1.3.** As a consequence of Proposition 1.2 we note that if $M$ is a $\gamma$-torsionfree left $R$-module over a left FTF ring, then its injective hull $E(\gamma M)$ is a flat $R$-module.

Let $\tau$ be a hereditary torsion theory on $R$-Mod and $M$ a left $R$-module. By $\tau(M)$ we will denote the largest $\tau$-torsion submodule of $M$. A submodule $N$ of $M$ is $\tau$-dense if $M/N$ is $\tau$-torsion. We say that $M$ is $\tau$-finitely generated if it contains a finitely generated $\tau$-dense submodule. If $M$ is finitely generated, then $M$ is said to be $\tau$-finitely presented if $M$ has a finite free presentation with $\tau$-finitely generated kernel. The ring $R$ is $\tau$-coherent [12, Theorem 3.4] if every finitely generated left ideal is $\tau$-finitely presented. We shall recall the localization functor $Q_\tau(-)$ (see [2] or [22, Chap. IX] for further details). Let $\mathcal{I}(\tau)$ be the filter of $\tau$-dense left ideals of $R$. For each left $R$-module $M$ define

$$Q_\tau(M) = \lim_{\to \mathcal{I}(\tau)} \text{Hom}_R(I, M/\tau(M)).$$

It is possible to endow $Q_\tau(R)$ with a structure of ring in such a way that $Q_\tau(M)$ becomes a left $Q_\tau(R)$-module. Moreover, there is a canonical ring homomorphism $R \to Q_\tau(R)$ and, thus, $Q_\tau(M)$ can be considered as a left $R$-module too. This construction leads to the localization functor $Q_\tau(-): R$-Mod $\to Q_\tau(R)$-Mod.

Let $M, N$ be left $R$-modules. The module $N$ is said to be $M$-torsionless if it embeds in a direct product of copies of $M$. When $M = R$, we say simply that $N$ is torsionless. By $E(\mu R)$ we will denote the injective hull of $R$ as a left $R$-module and by $\lambda$ the Lambek torsion theory on $R$-Mod. The $\lambda$-torsionfree left $R$-modules are precisely the $E(\mu R)$-torsionless left $R$-modules. We use the notation $\lambda'$ for the Lambek torsion theory on Mod-$R$.

If a ring $R$ is left FTF, then clearly the torsion theory $\gamma$ is smaller than the Lambek torsion theory. Moreover, we know that, even for commutative FTF rings, $\gamma \neq \lambda$ in general (see [7, Example 3.5]). However, we will show
in Theorem 1.8 that (two-sided) FTF rings can be characterized exclusively by properties of the Lambek torsion theories $\lambda$ and $\lambda'$. The research on Lambek torsion theories developed by M. Hoshino will be useful for this purpose. A direct consequence of [9, Theorem A] is that $P/\lambda(P)$ is torsionless for every finitely presented left $R$-module $P$ if and only if $Q/\lambda'(Q)$ is torsionless for every finitely presented right $R$-module $Q$. This gives the notion of absolutely pure ring relative to the Lambek torsion theory introduced in [11]. This property will be important in our characterization of FTF rings.

**Definition 1.4.** A ring $R$ will be called *almost absolutely pure* if $P/\lambda(P)$ is torsionless for every finitely presented left $R$-module $P$.

**Definition 1.5.** We will say that $R$ is *left almost coherent* if $R$ is $\lambda$-coherent. Analogously, we have the notion of right almost coherent ring.

A module $M$ is said to be $\pi$-flat if every direct product of copies of $M$ is a flat module.

**Proposition 1.6.** The following conditions are equivalent:

1. $E(\mathbb{R}R)$ is $\pi$-flat.
2. $P/\lambda(P)$ embeds in a free left $R$-module for every finitely presented left $R$-module $P$.
3. $R$ is almost absolutely pure and right almost coherent.

**Proof.** First, we will use [16, Théorème 1.2] to prove that (1) and (2) are equivalent. Assume that $E(\mathbb{R}R)$ is $\pi$-flat and let $P$ be a finitely presented left $R$-module. Then $P/\lambda(P)$ embeds in a direct product $E$ of copies of $E(\mathbb{R}R)$. Let $f: P/\lambda(P) \to E$ be such an embedding and let $p: P \to P/\lambda(P)$ be the canonical mapping. Since $E$ is flat, there are morphisms $u: P \to F$ and $v: F \to E$, where $F$ is a free left $R$-module, such that $f \circ p = v \circ u$. Now it is easy to show that $\text{Ker}(u) \subseteq \text{Ker}(p) = \lambda(P)$. But this inclusion entails in fact the equality $\text{Ker}(u) = \lambda(P)$, whence $P/\lambda(P)$ embeds in $F$.

Conversely, let $E$ be any direct product of copies of $E(\mathbb{R}R)$. We have to show that $E$ is flat. Let $f: P \to E$ be a morphism, where $P$ is a finitely presented left $R$-module. By hypothesis, there is an embedding $g: P/\lambda(P) \to F$, where $F$ is a free left $R$-module. Let $h: P/\lambda(P) \to E$ be such that $h \circ p = f$, where $p: P \to P/\lambda(P)$ is the canonical map. There is $v: F \to E$ such that $v \circ g = h$. Put $u = g \circ p$ and observe that, therefore, $v \circ u = f$. This implies that $E$ is flat.

Next, we will prove that (1) and (2) do imply (3). Assume that $R$ satisfies (1) and (2) and consider $\kappa$ to be the torsion theory on Mod-$R$ such that a right $R$-module $M$ is $\kappa$-torsion if and only if $M \otimes_R E(\mathbb{R}R) = 0$. By [12, Corollary 3.5], $R$ is $\kappa$-coherent. Now, it is easy to show that $\kappa \leq \lambda'$ and,
from this, that \( R \) is right almost coherent. Moreover, \( R \) is clearly almost absolutely pure.

Finally, assume that \( R \) is almost absolutely pure and right almost coherent. We use the notation \( M^* = \text{Hom}_R(M,R) \) for any (left or right) \( R \)-module \( M \). Given a finitely presented left \( R \)-module \( P \), there is a canonical monomorphism \( f: P/\lambda(P) \rightarrow P^{**} \). By [12, Theorem 3.4], \( P^* \) is a \( \lambda' \)-finitely generated right \( R \)-module. Thus, there is a finitely generated submodule \( Q \) of \( P^* \) such that \( P^*/Q \) is \( \lambda' \)-torsion. Hence, the canonical map \( P^{**} \rightarrow Q^* \) is a monomorphism. But \( Q^* \) embeds in a free left \( R \)-module. Finally, we can compose all these monomorphisms of left \( R \)-modules to obtain an injective map from \( P/\lambda(P) \) into a free left \( R \)-module.

Remark 1.7. The equivalence \( (3) \Leftrightarrow (1) \) in Proposition 1.6 has been proved by different methods in [10, Proposition 3.5].

Theorem 1.8. The following conditions are equivalent for a ring \( R \).

1. \( R \) is FTF.
2. \( R \) is almost absolutely pure and almost coherent.
3. \( E_{(R,R)} \) and \( E(R_R) \) are \( \pi \)-flat modules.
4. If \( P \) is any finitely presented left (resp. right) \( R \)-module, then \( P/\lambda(P) \) (resp. \( P/\lambda'(P) \)) embeds in a free \( R \)-module.
5. \( R \) is almost coherent and \( E_{(R,R)} \) is flat.

Proof. The statements (2), (3), and (4) are equivalent by Proposition 1.6. Moreover, (1) clearly implies (3). Now, (1) can be deduced from (2) and (3) by [7, Proposition 2.2]. The statement (5) is evidently a consequence of the first four. Finally, assume (5). By [7, Proposition 2.2(i)], \( R \) is a right FTF ring. In particular, \( E(R_R) \) is flat. By the right-handed version of [7, Proposition 2.2(i)], \( R \) is a left FTF ring. Therefore, (1) can be proved from (5).

Remark 1.9. The equivalence \( (2) \Leftrightarrow (5) \) in Theorem 1.8 has been proved in [11] by different methods.

Remark 1.10. The equivalence between (1) and (5) in Theorem 1.8 can be seen as an analogue to the fact that a perfect ring \( R \) is QF-3 if and only if \( E_{(R,R)} \) is projective (see [23]). In fact, this can be obtained as a corollary of the theorem taking into account that the notions of FTF and QF-3 ring coincide over perfect rings [7, Corollary 2.11].

Remark 1.11. Let \( R \) be an FTF ring. By Theorem 1.8 we know that \( R \) is almost coherent. It follows from the proof of [7, Proposition 2.2] (or [3, Lema 2.2.8]) that a right \( R \)-module \( M \) is \( \gamma' \)-torsion if and only if \( M \otimes_R E(R_R) = 0 \). Of course, a left \( R \)-module \( M \) is \( \gamma \)-torsion if and only if \( E(R_R) \otimes_R M = 0 \).
If $R$ is a left FTF ring, then we can compute the associated ring of quotients $Q_\gamma(R)$, which is also a left FTF ring [7, Theorem 3.2].

**Proposition 1.12.** Let $R$ be an FTF ring. There is a ring isomorphism $\theta: Q_\gamma(R) \to Q_\gamma(R)$ which extends the identity map on $R$.

**Proof.** Let us adopt the following notation: $Q = Q_\gamma(R)$, $Q' = Q_\gamma(R)$, $E = E(R)$. Observe that $Q \otimes_R E$ is a flat left $Q$-module. By [7, Theorem 3.2], $Q \otimes_R E$ is $\gamma$-torisonfree. Since $E$ is flat and injective, it follows that $Q/R \otimes_R E$ is isomorphic to a direct summand of $Q \otimes_R E$ and, hence, $Q/R \otimes_R E$ is $\gamma$-torisonfree. But it is clear that $Q/R \otimes_R E$ is also $\gamma$-torison. Hence, $Q/R \otimes_R E = 0$. By Remark 1.11, $Q/R$ is a $\gamma'$-torison right $R$-module. This implies that there is a morphism of right $R$-modules $\theta: Q \to Q'$ such that $\theta(r) = r$ for every $r \in R$. It is easy to see that $\theta$ is a ring homomorphism. In fact, given $q, p \in Q$, there is a $\gamma'$-dense right ideal $I$ of $R$ such that $pI \subseteq R$. For each $r \in I$ we have

$$(\theta(qp) - \theta(q)\theta(p))r = \theta(qpr) - \theta(q)\theta(pr) = \theta(q)pr - \theta(q)pr = 0.$$ 

Therefore, $\theta(qp) - \theta(q)\theta(p)$ is annihilated by $I$ and, since $Q'$ is $\gamma'$-torisonfree, we get $\theta(qp) - \theta(q)\theta(p) = 0$. Analogously, it can be proved that there is a ring homomorphism $\theta': Q' \to Q$ extending the identity map on $R$. Therefore, $\theta' \circ \theta$ is a ring endomorphism on $Q$ which extends the identity on $R$. This entails that $\theta' \circ \theta$ is the identity map on $Q$. Analogously, $\theta \circ \theta' = 1_Q$. 

**Definition 1.13.** An FTF ring will be said to be a maximal FTF ring if $R = Q_\gamma(R) = Q_\gamma(R)$.

**Corollary 1.14.** If $R$ is an FTF ring then $Q_\gamma(R)$ is a maximal FTF ring.

**Proof.** By [7, Theorem 3.2], $Q_\gamma(R)$ is left FTF and $Q_\gamma(R)$ is right FTF. By Proposition 1.12, $Q_\gamma(R) = Q_\gamma(R)$, whence it is FTF. To prove that $Q = Q_\gamma(R)$ is a maximal FTF ring, we have to show that $E(qQ)/Q$ embeds in a flat left $Q$-module. By [7, Lemma 3.1], $E(qQ)/Q = E(R)/Q$ and this left $R$-module is $\gamma$-torisonfree. Therefore, it embeds in a flat left $R$-module. By [7, Theorem 3.2], $E(qQ)/Q$ embeds in a flat left $Q$-module. 

**Definition 1.15.** The ring $R$ has left flat-dominant dimension $\geq n$ if the first $n$ terms of the injective minimal resolution of $_R R$ are flat (see [8]).

A left FTF ring $R$ has left flat-dominant dimension $\geq 1$ and $Q_\gamma(R)$ has left flat-dominant dimension $\geq 2$. If the ring $R$ is noetherian then $R$ has left flat-dominant dimension $\geq n$ if and only if $R$ has right flat-dominant dimension $\geq n$ [8]. If $R$ is noetherian, then $R$ is an FTF ring if and only if $R$ has flat-dominant dimension $\geq 1$ [7, Corollary 2.9]. The following two corollaries are consequences of the foregoing results.
Corollary 1.16. The following conditions are equivalent for a ring $R$.

1. $R$ is an FTF ring.
2. $R$ is almost coherent with left flat-dominant dimension $\geq 1$.
3. $R$ is almost coherent with right flat-dominant dimension $\geq 1$.

Corollary 1.17. The following conditions are equivalent for a ring $R$.

1. $R$ is a maximal FTF ring.
2. $R$ is almost coherent with left flat-dominant dimension $\geq 2$.
3. $R$ is almost coherent with right flat-dominant dimension $\geq 2$.

Finally, we state a result which is a consequence of [18, Theorem 2] in conjunction with Corollary 1.17 and the theory developed in [3, 7]. We say that $R$ is a maximal quotient ring if $R$ coincides with its left maximal quotient ring and with its right maximal quotient ring.

Theorem 1.18. The following conditions are equivalent for a ring $R$.

1. $R$ is a noetherian maximal FTF ring.
2. $R$ is a noetherian ring with flat-dominant dimension $\geq 2$.
3. $R$ is an artinian QF-3 maximal quotient ring.

Proof. (1) $\Rightarrow$ (2). Evident.

(2) $\Rightarrow$ (3). Assume that $R$ is noetherian with flat-dominant dimension $\geq 2$. By [7, Theorem 2.7] and [18, Theorem 2] $R$ is FTF and has a semi-primary QF-3 two-sided maximal quotient ring $Q$ (see also [7, Remark 2.10(2)]). By [4, Theorem 4.6], $Q = Q_\gamma(R)$. Since $R$ has flat-dominant dimension $\geq 2$, $Q = R$. Therefore, $R$ is a noetherian semi-primary QF-3 ring; that is, $R$ is artinian and QF-3.

(3) $\Rightarrow$ (1). This is a consequence of [4, Theorem 4.6] and [7, Theorem 2.7].

2. FROBENIUS EXTENSIONS

Consider $\rho: R \to S$ to be a ring homomorphism. The class of all left $S$-modules that embed in flat left $S$-modules is denoted throughout this section by $\mathcal{F}^S$ and we reserve the notation $\mathcal{F}$ for the class of the submodules of flat left $R$-modules. If the ring $S$ is left FTF, then we denote by $\gamma^S$ the hereditary torsion theory for which $\mathcal{F}^S$ is the class of the $\gamma^S$-torsionfree left $S$-modules. When $R$ is left FTF, the corresponding notation will be $\gamma$.

We will investigate the transfer of the FTF property under the ring homomorphism.
Proposition 2.1. Let $R \subseteq S$ be a ring extension such that $RS$ and $S_R$ are flat. If $S$ is left FTF then $R$ is left FTF.

Proof. We will first prove that $\mathcal{T}$ is the torsionfree class for some torsion theory $\gamma$ on $R$-$\text{Mod}$ and then we will show that $\gamma$ is necessarily hereditary. Since $\mathcal{T}$ is always closed under submodules, we need to check that it is closed under extensions and direct products. Let $R M$ be an extension of modules $R N, R L \in \mathcal{T}$. Since $S_R$ is flat, $S \otimes_R M$ is an extension of $S \otimes_R N$ and $S \otimes_R L$. By [6, Lemma 2.3.(3)], they are $\gamma^S$-torsionfree, whence $S \otimes_R M$ is $\gamma^S$-torsionfree. By [6, Lemma 2.3.(6)], $S \otimes_R M \in \mathcal{T}$. Since $M$ is an $R$-submodule of $S \otimes_R M$, we conclude that $M \in \mathcal{T}$. Analogously, it can be proved that $\mathcal{T}$ is closed under direct products.

We will finish by proving that $\gamma$ is hereditary; that is, the class of all $\gamma$-torsion left $R$-modules is closed under submodules. Consider $N$ to be a submodule of a $\gamma$-torsion $R$-module $M$ and let $G$ be any flat left $R$-module. Observe that $S \otimes_R M$ is $\gamma^S$-torsion, since $\text{Hom}_S(S \otimes_R M, S \otimes_R G) = 0$ for every flat left $S$-module $F$. In particular, $\text{Hom}_S(S \otimes_R M, S \otimes_R G) = 0$, and, since $S \otimes_R N$ embeds in $S \otimes_R M$, it follows that $\text{Hom}_S(S \otimes_R N, S \otimes_R G) = 0$. By adjunction, $\text{Hom}_R(N, S \otimes_R G) = 0$ and, since $G$ is flat, there is an embedding $\text{Hom}_R(N, G)$ into $\text{Hom}_R(N, S \otimes_R G)$ coming from the canonical monomorphism $G \rightarrow S \otimes_R G$. Therefore, $\text{Hom}_R(N, G) = 0$.

The following consequence of Proposition 2.1 is related to the first sentence of [15, Proposition 4].

Corollary 2.2. Let $R \subseteq S$ be a ring extension such that $RS$ and $S_R$ are projective $R$-modules. Assume that both $R$ and $S$ are perfect rings. If $S$ is a QF-3 ring then $R$ is a QF-3 ring.

Proof. By [7, Corollary 2.11] the notions of FTF and QF-3 ring coincide over perfect rings. The corollary follows from Proposition 2.1.

If $R$ is a left FTF ring, then the ring homomorphism $\rho: R \rightarrow S$ induces a torsion theory $\tilde{\gamma}$ in $R$-$\text{Mod}$ with torsion class $\mathcal{T}(\tilde{\gamma})$ consisting of those left $S$-modules that are $\tilde{\gamma}$-torsion as left $R$-modules. Following [17] we will say that $\gamma$ is $S$-good if $S \otimes_R T$ is $\gamma$-torsion for every $\gamma$-torsion left $R$-module $T$. Several characterizations of $S$-good torsion theories which will be useful here were given in [17, Theorem 2.5].

Proposition 2.3. Let $\rho: R \rightarrow S$ be a ring homomorphism. Assume that $RS$ is a finitely generated projective module and that $\text{Hom}_R(RS, R)$ is a flat left $S$-module. If $R$ is a left FTF ring, then $S$ is a left FTF ring, $\gamma$ is $S$-good, and $\gamma^S = \tilde{\gamma}$. 

Proof. First, we will prove that \( \gamma \) is \( S \)-good. Since every module is the sum of its cyclic submodules, it is enough to prove that \( S \otimes_R T \) is \( \gamma \)-torsion for every \( \gamma \)-torsion cyclic left \( R \)-module \( T \). By \([4, \text{Proposition 4.5.}(1)]\), we can assume that \( T \cong R/I \), where \( I \) is a finitely generated left ideal of \( R \). Since \( \text{Hom}_R(R/I, R) \) is flat as a left \( R \)-module, we have \( \text{Hom}_R(S \otimes_R R/I, R) \cong \text{Hom}_R(R/I, \text{Hom}_R(RS, R)) = 0 \). By \([4, \text{Proposition 4.5.}(4)]\), \( S \otimes_R R/I \) is \( \gamma \)-torsion.

Let \( \bar{\gamma} \) denote the hereditary torsion theory induced by \( \gamma \) in \( S \text{-Mod} \). We will prove that \( \mathcal{F} / \bar{\gamma} = \mathcal{F}_S \). By \([17, \text{Theorem 2.5}]\), a left \( S \)-module \( M \) is \( \bar{\gamma} \)-torsionfree if and only if \( _RM \) is \( \gamma \)-torsionfree. Thus, the inclusion \( \mathcal{F} \subseteq \mathcal{F}_S \) is given by \([6, \text{Lemma 2.3.}(5)]\). Now, let \( M \) be a \( \bar{\gamma} \)-torsionfree left \( S \)-module. Then \( _RM \) embeds in a flat left \( R \)-module, say \( F \). By \([16]\), \( F \) is a direct limit of finitely generated free left \( R \)-modules. Now, \( \text{Hom}_R(RS, F) \) is a flat left \( S \)-module. Finally, \( \mathcal{F}_S \subseteq \mathcal{F} / \bar{\gamma} \).

Proposition 2.3 has a corollary which is related to the second part of \([15, \text{Proposition 4}]\). We need the following result from \([3]\), which is also available in \([1, \text{Proposition 2}]\).

**Proposition 2.4** \([3, \text{Proposición 2.4.1}]\). If \( R \) is a left perfect and left \( \text{FTF} \) ring, then \( R \) is a semi-primary \( \text{QF-3} \) ring.

**Corollary 2.5.** Let \( \varphi: R \rightarrow S \) be a ring homomorphism such that \( _RS \) is finitely generated and projective and that \( \text{Hom}_R(RS, R) \) is a projective left \( S \)-module. Assume that \( R \) is a perfect ring and \( S \) is a left perfect ring. If \( R \) is a \( \text{QF-3} \) ring then \( S \) is a semi-primary \( \text{QF-3} \) ring.

**Proof.** If \( R \) is \( \text{QF-3} \), then it is \( \text{FTF} \). By Proposition 2.3, \( S \) is a left \( \text{FTF} \) ring. By Proposition 2.4, \( S \) is a semi-primary \( \text{QF-3} \) ring.

Recall from \([14]\) that a ring extension \( R \subseteq S \) is said to be a Frobenius extension if \( _RS \) is finitely generated and projective and \( _SS_R \cong \text{Hom}_R(RS, R) \). It is well-known that these conditions are equivalent to the corresponding properties on the opposite sides. Somewhat more generally we have the notion of left quasi-Frobenius extension \([21]\). The ring extension \( R \subseteq S \) is said to be left quasi-Frobenius if \( _RS \) is finitely generated and projective and \( _SS_R \) is isomorphic to direct summand of a finite direct sum of copies of \( \text{Hom}_R(RS, R) \). Similarly, we have the concept of right quasi-Frobenius extension. A left and right quasi-Frobenius extension will be referred just as a quasi-Frobenius extension. A straightforward duality argument shows that if \( R \subseteq S \) is a quasi-Frobenius extension then the \( S - R \)-bimodule \( \text{Hom}_R(RS, R) \) is isomorphic to a direct summand of a finite direct sum of copies of \( SS_R \). In particular, it is projective as a left \( S \)-module and as
a right $R$-module. Of course, a symmetric property can be stated for the $R - S$-bimodule $\text{Hom}_R(S, R)$.

**Corollary 2.6.** Let $R \subseteq S$ be a ring extension. Assume that $R S$ is finitely generated and projective, $S \gamma$ is flat, and $\text{Hom}_R(R S, R)$ is a flat left $S$-module (e.g., $R \subseteq S$ is a quasi-Frobenius extension). Then $R$ is left FTF if and only if $S$ is left FTF. In such a case, $\gamma$ is $S$-good, $\gamma^S = \bar{\gamma}$, and we have the natural isomorphism $R Q_S(N) \cong Q_S(R N)$ for every left $S$-module $N$.

**Proof.** By Propositions 2.1 and 2.3, we have only to check the natural isomorphism, which is given by [17, Theorem 2.7].

A module is said to be $\tau$-artinian ($\tau$ is a hereditary torsion theory) if it has the descending chain condition on $\tau$-closed submodules (see [19] for details).

**Theorem 2.7.** Let $R \subseteq S$ be a quasi-Frobenius extension. The following statements hold.

1. $S$ is almost coherent with left (or right) flat-dominant dimension $\geq 1$ if and only if $R$ is almost coherent with left (or right) flat-dominant dimension $\geq 1$.

2. $S$ has a semi-primary QF-3 two-sided maximal quotient ring if and only if $R$ has a semi-primary QF-3 two-sided maximal quotient ring.

**Proof.**

(1) This is a consequence of Corollary 2.6 and Corollary 1.16.

(2) By [5, Theorem] and Corollary 2.6 we can assume that $R$ is left FTF and then show that $R$ is $\gamma$-artinian if and only if $S$ is $\bar{\gamma}$-artinian. Moreover, $\gamma$ is $S$-good. Let $I$ be a left ideal of $S$. By [17, Theorem 2.5], $S/I$ is $\bar{\gamma}$-torsionfree if and only if $R(S/I)$ is $\gamma$-torsionfree. So, the lattice of $\bar{\gamma}$-closed left ideals of $S$ is a sublattice of the lattice of all $\gamma$-closed left $R$-submodules of $R S$. But this last lattice is artinian because $R S$ is finitely generated and $R R$ is $\gamma$-artinian. Therefore, $S$ is $\bar{\gamma}$-artinian. Conversely, assume that $S$ is $\bar{\gamma}$-artinian, and consider $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$, a descending chain of $\gamma$-closed left ideals of $R$. We have a descending chain $SI_1 \supseteq SI_2 \supseteq \cdots \supseteq SI_n \supseteq \cdots$ of left ideals of $S$. Thus, there is a positive integer $n_0$ such that $SI_n$ is $\bar{\gamma}$-dense in $SI_{n_0}$ for every $n \geq n_0$. The flatness of $S_R$ gives the isomorphism $S \otimes_R I_{n_0}/I_n \cong SI_{n_0}/SI_n$ for every $n \geq n_0$. Therefore, $S \otimes_R I_{n_0}/I_n$ is $\gamma$-torsion, as $\gamma$ is $S$-good. Now, $R \subseteq S$ is a two-sided quasi-Frobenius extension and, in particular, $R S_R$ is isomorphic as an $R - R$-bimodule to a direct summand of $\text{Hom}_R(R S, R)^m$ for some positive integer $m$. Therefore, we have a monomorphism of left $R$-modules $S \otimes_R I_{n_0}/I_n \rightarrow (\text{Hom}_R(R S, R) \otimes_R I_{n_0}/I_n)^m$. Recall that $R S$ is finitely generated and projective, which implies the isomorphism of left $R$-modules $\text{Hom}_R(R S, R) \otimes_R I_{n_0}/I_n \cong \text{Hom}_R(R S, I_{n_0}/I_n)$. Now, $I_{n_0}/I_n$ is $\gamma$-torsionfree, which implies, by [17, Theorem 2.5], that $\text{Hom}_R(R S, I_{n_0}/I_n)$ is
\( \gamma \)-torsionfree. Hence, \( S \otimes_R I_{n_0}/I_n = 0 \), as it is a \( \gamma \)-torsion left \( R \)-module that embeds in the \( \gamma \)-torsionfree left \( R \)-module \( \text{Hom}_R(\gamma_S, I_{n_0}/I_n)^m \). To prove that \( I_{n_0}/I_n = 0 \), we know from Remark 1.3 that the injective hull \( E(I_{n_0}/I_n) \) of the \( \gamma \)-torsionfree left \( R \)-module \( I_{n_0}/I_n \) is flat. Thus, in the commutative square diagram

\[
\begin{array}{ccc}
R \otimes_R I_{n_0}/I_n & \longrightarrow & R \otimes_R E(I_{n_0}/I_n) \\
\downarrow & & \downarrow \\
S \otimes_R I_{n_0}/I_n & \longrightarrow & S \otimes_R E(I_{n_0}/I_n)
\end{array}
\]

the right down arrow is an injective \( R \)-module homomorphism. This implies that the down left arrow is a monomorphism, too. Hence, \( I_{n_0}/I_n \cong R \otimes_R I_{n_0}/I_n = 0 \), which finishes the proof.

Remark 2.8. It follows from its proof that the statement (2) in Theorem 2.7 can be rephrased in torsion-theoretical terms as follows: \( R \) is a \( \gamma \)-artinian left FTF ring if and only if \( S \) is a \( \tilde{\gamma} \)-artinian left FTF ring.

Lemma 2.9. Let \( R \subseteq S \) be a quasi-Frobenius ring extension. If \( R \) is a left FTF ring, then there is a natural isomorphism \( Q_{\gamma}(S \otimes_R M) \cong S \otimes_R Q_{\gamma}(M) \) for every left \( R \)-module \( M \).

Proof. Consider a left \( R \)-module \( M \). Using [17, Theorem 2.5], it is easy to check that the canonical morphism of left \( S \)-modules \( S \otimes_R M \to S \otimes_R Q_{\gamma}(R) \) has \( \tilde{\gamma} \)-torsion kernel and cokernel. Thus, if we prove that \( S \otimes_R Q_{\gamma}(M) \) is a faithfully \( \tilde{\gamma} \)-injective left \( S \)-module (see [2] for this notion), then \( S \otimes_R Q_{\gamma}(M) \) and \( Q_{\gamma}(S \otimes_R M) \) are naturally isomorphic. Since \( R \subseteq S \) is a two-sided quasi-Frobenius extension, \( S \otimes_R Q_{\gamma}(M) \) is isomorphic to a direct summand of a finite direct sum of copies of the left \( S \)-module \( \text{Hom}_R(\gamma_S, R) \otimes_R Q_{\gamma}(M) \), which is isomorphic to \( \text{Hom}_R(\gamma_S, Q_{\gamma}(M)) \). Thus, it suffices to prove that \( \text{Hom}_R(\gamma_S, Q_{\gamma}(M)) \) is faithfully \( \tilde{\gamma} \)-injective. But this is a consequence of the adjunction isomorphism

\[
\text{Hom}_S(\_, \text{Hom}_R(\gamma_S, Q_{\gamma}(M))) \cong \text{Hom}_R(\_, Q_{\gamma}(M)).
\]

Theorem 2.10. Let \( R \subseteq S \) be a quasi-Frobenius ring extension.

1. If \( R \) is almost coherent with left (or right) flat-dominant dimension \( \geq 2 \) then \( S \) is almost coherent with left (or right) flat-dominant dimension \( \geq 2 \).

2. If \( R \) has a QF two-sided maximal quotient ring then \( S \) has a QF two-sided maximal quotient ring.
If, in addition, either $RS$ is a generator of the category $R$-$\text{Mod}$ or $SR$ is a generator of the category $\text{Mod}$-$R$, then the converse statements of (1) and (2) hold.

**Proof.** (1) By Corollary 1.17, $R$ is a maximal FTF ring. By Corollary 2.6, $S$ is an FTF ring. By Lemma 2.9,

$$Q_{\gamma}(S) \cong S \otimes R Q_{\gamma}(R) = S \otimes R R \cong S$$

and, hence, $S$ is a maximal FTF ring.

(2) By [7, Proposition 3.6], $R$ is a $\gamma$-artinian left FTF ring such that $\gamma$ is perfect (for the notion of perfect torsion theory see, e.g., [22, p. 231]). By Remark 2.8, we have that $S$ is a $\bar{\gamma}$-artinian left FTF ring. We have to prove that $\bar{\gamma}$ is perfect. Since $\bar{\gamma}$ is of finite type, it is enough, in view of [2, Theorem 4.3], to check that $Q_{\bar{\gamma}}(-)$ is an exact functor. From Corollary 2.6 we know that the functor $Q_{\bar{\gamma}}(-)$ is isomorphic to the composition $R(-) \circ Q_{\gamma}(-)$, where $R(-)$ denotes the restriction of scalars functor. Thus, $R(-) \circ Q_{\gamma}(-)$ is an exact functor, as $\gamma$ is perfect. Since $R(-)$ is a faithful functor, we obtain that $Q_{\bar{\gamma}}(-)$ is exact and $\bar{\gamma}$ is a perfect torsion theory.

Now, assume that $RS$ is a generator of the category $R$-$\text{Mod}$. It follows from the definition of quasi-Frobenius extension that this is equivalent to require that $S_R$ is faithfully flat. Assume that $S$ is almost coherent with left (or right) flat-dominant dimension $\geq 2$, then $Q_{\gamma}(S) = S$ and $R$ is FTF. By Lemma 2.9 again we get $S \otimes R \cong S \otimes R Q_{\gamma}(R)$, which implies, as $S_R$ is faithfully flat, that $R = Q_{\gamma}(R)$. Therefore, $R$ is a maximal FTF ring and we can appeal to Corollary 1.17 to obtain the converse of (1). To see the converse of statement (2) we can argue as in its proof to reduce the problem to prove the exactness of the localization functor $Q_{\gamma}(-)$ from the exactness of $Q_{\bar{\gamma}}(-)$. But this is deduced from the isomorphism $Q_{\bar{\gamma}}(-) \cong S \otimes R Q_{\gamma}(-)$ given in Lemma 2.9, as $S \otimes_R -$ is a faithful functor now. Finally, since statements (1) and (2) and their converse are left–right symmetric, we can replace the added hypothesis “$RS$ is a generator” by the condition “$SR$ is a generator.”

**Remark 2.11.** If the inclusion $R \subseteq S$ makes $R$ into a direct summand of $RS$, then, obviously, $RS$ becomes a generator. Thus, the converse of statement (1) or (2) in Theorem 2.10 holds for split two-sided quasi-Frobenius extensions. Here, we use the notion of split extension in the sense of [13, Definition 2.4.3]; i.e., there is an $R - R$-bimodule map $E: S \to R$ such that $E(1) = 1$.

**Remark 2.12.** The condition “$RS$ is a generator” is not merely technical. In fact, we cannot drop it to obtain the converse of (1) and (2) in Theorem 2.10 even in the case of Frobenius extensions, as the following example shows.
Example 2.13. We will use the presentation given in [13, Example 2.3.1] of an example of K. Morita [20]. Let $S$ be the $4 \times 4$ matrix ring with entries in a commutative field $K$. Consider the $K$-subalgebra $R$ of $S$ with $K$-basis consisting of

$$e_1 = e_{11} + e_{44}, \quad e_2 = e_{22} + e_{33}, \quad e_{21}, e_{31}, e_{42}, e_{43},$$

where the matrices $e_{ij}$ are the obvious ones. Then $R \subseteq S$ is a Frobenius extension but neither $SR$ nor $R_S$ is a generator. Now, $S$ is a simple artinian ring, whence, by Corollary 2.6, $R$ is an FTF ring. Moreover, $\gamma = \lambda$ and, thus, $Q_{\gamma}(R)$ is just the maximal (left, for example) quotient ring $Q_{\text{max}}(R)$ of $R$ (see [7, Proposition 2.6]). Some straightforward computations show that $Q_{\text{max}}(R) = R + Ke_{22} + Ke_{33} + Ke_{23} + Ke_{32}$ and, thus, $R$ is not a maximal FTF ring. Therefore, the converse to Theorem 2.10(1) does not hold in general. Moreover, $Q_{\text{max}}(R)$ is not a QF ring, which gives a counter-example to the converse of Theorem 2.10(2).

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REFERENCES