Continuity of Operators Intertwining with Convolution Operators

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Communicated by D. Sarason

Received October 18, 2001; accepted January 30, 2002

Let $G$ be a locally compact abelian group, let $\mu$ be a bounded complex-valued Borel measure on $G$, and let $T_\mu$ be the corresponding convolution operator on $L^1(G)$. Let $X$ be a Banach space and let $S$ be a continuous linear operator on $X$. Then we show that every linear operator $F: X \to L^1(G)$ such that $FS = T_\mu F$ is continuous if and only if the pair $(S, T_\mu)$ has no critical eigenvalue.

Key Words: automatic continuity; intertwining operators; multipliers.

1. INTRODUCTION

Johnson asked in [5, Note 8(iii)] whether every linear operator $\Phi: L^1(\mathbb{R}) \to L^1(\mathbb{R})$ with the property that $\Phi(\mu*f) = \mu*\Phi(f)$ for each $f \in L^1(\mathbb{R})$ is automatically continuous in the case when $\mu$ is a fixed bounded complex-valued Borel measure on $\mathbb{R}$ whose Fourier–Stieltjes transform is not constant on any open interval. This question may also be found in [2, Unsolved problem 13.5, p. 408]. The same problem has been very recently suggested again by Laursen and Neumann [6]. In fact, these authors asked [6, Open problem 6.3.3, p. 639] the following generalization of Johnson’s question. Suppose that $\mu$ and $\nu$ are bounded complex-valued Borel measures on a locally compact abelian group $G$ such that the corresponding pair $(T_\nu, T_\mu)$ of convolution operators on $L^1(G)$ has no critical eigenvalue. Is every linear operator $\Phi: L^1(G) \to L^1(G)$ for which $T_\nu \Phi = \Phi T_\mu$ automatically continuous?

The aim of this paper is to prove the affirmative solution of this problem.

1Supported by D.G.I.C.Y.T. Grant PB98-1358.
2. OPERATORS INTERTWINING WITH CONVOLUTION OPERATORS

Throughout this section, $G$ stands for a locally compact abelian group, $\hat{G}$ denotes its dual group, and $\hat{f}$ and $\hat{\mu}$ denote the Fourier transforms of $f$ and $\mu$ for all $f \in L^1(G)$ and $\mu \in M(G)$, respectively.

Let $X$ and $Y$ be Banach spaces and let $S$ and $T$ be continuous linear operators on $X$ and $Y$, respectively. A standard requirement in order to obtain a discontinuous linear operator $F : X \to Y$ intertwining with $S$ and $T$ is the existence of a critical eigenvalue of the pair $(S, T)$. We recall that a complex number $\lambda$ is said to be a critical eigenvalue of $(S, T)$ if $\lambda$ is an eigenvalue of $T$ and $(\lambda I_X - S)(X)$ is of infinite codimension in $X$. It is important to note here that if $(S, T)$ has a critical eigenvalue, then there exists a discontinuous linear operator $\Phi : X \to Y$ such that $\Phi S = T \Phi$ (see [7, Lemma 3.2]).

A key notion to study the continuity of a linear map $\Phi$ from a Banach space $X$ into a Banach space $Y$ is that of the separating space $\mathcal{E}(\Phi)$ of $\Phi$ which is defined as follows:

$$\mathcal{E}(\Phi) = \{ y \in Y : \text{there exists } (x_n) \to 0 \text{ in } X \text{ with } (\Phi(x_n)) \to y \}.$$  

The separating space measures the closability of $\Phi$ and the closed graph theorem shows that $\Phi$ is continuous if and only if $\mathcal{E}(\Phi) = \{0\}$. For a thorough discussion of the separating space we refer the reader to [7].

**Theorem 2.1.** Let $G$ be a locally compact abelian group, let $\mu$ be a bounded complex-valued Borel measure on $G$, and let $T_\mu$ be the corresponding convolution operator $f \mapsto \mu * f$ on $L^1(G)$. Let $X$ be a Banach space and let $S$ be a continuous linear operator on $X$. Suppose that the pair $(S, T_\mu)$ has no critical eigenvalue. Then every linear operator $\Phi : X \to L^1(G)$ such that $\Phi S = T_\mu \Phi$ is continuous.

**Proof.** To obtain a contradiction, suppose that $\Phi$ is discontinuous and therefore that $\mathcal{E}(\Phi) \neq \{0\}$.

We first prove that the set

$$\{ \hat{\mu}(\gamma) : \gamma \in \hat{G} \text{ and } \hat{f}(\gamma) \neq 0, \text{for some } f \in \mathcal{E}(\Phi) \}$$

is finite. Conversely, suppose that it is infinite. Let $(\gamma_n)$ be a sequence in $\hat{G}$ such that $\hat{\mu}(\gamma_m) \neq \hat{\mu}(\gamma_n)$ for $n \neq m$ and such that for every $n \in \mathbb{N}$ there exists $f_n \in \mathcal{E}(\Phi)$ such that $\hat{f}_n(\gamma_n) \neq 0$. To shorten notation, we write $\lambda_n = \hat{\mu}(\gamma_n)$ for each $n \in \mathbb{N}$. Let us denote by $I_X$ and $I_G$ the identity operators on $X$ and $L^1(G)$, respectively. Since $(\lambda_n I_G - T_\mu) \Phi = \Phi(\lambda_n I_X - S)$ for each $n \in \mathbb{N}$, the
From Lemma 3.3 we now deduce that

\[(\lambda_1 I_G - T_\mu) \cdots (\lambda_n I_G - T_\mu)(\Xi(\Phi)) = (\lambda_1 I_G - T_\mu) \cdots (\lambda_n I_G - T_\mu)(\Xi(\Phi)).\]

We now observe that the Fourier transform of every function of the latter set vanishes at \(\gamma_{n+1}\). Indeed, we have

\[
[(\lambda_1 I_G - T_\mu) \cdots (\lambda_n I_G - T_\mu)(f)](\gamma_{n+1}) = (\hat{\lambda}_1 - \hat{\mu}(\gamma_{n+1})) \cdots (\hat{\lambda}_n - \hat{\mu}(\gamma_{n+1})) \hat{f}(\gamma_{n+1}) = 0
\]

for each \(f \in L^1(G)\) and finally the continuity of the Fourier transform at \(\gamma_{n+1}\) gives the desired conclusion. Consequently, the Fourier transform of every function of the set \((\lambda_1 I_G - T_\mu) \cdots (\lambda_n I_G - T_\mu)(\Xi(\Phi))\) vanishes at \(\gamma_{n+1}\). We thus get

\[
(\hat{\lambda}_1 - \hat{\mu}(\gamma_{n+1})) \cdots (\hat{\lambda}_n - \hat{\mu}(\gamma_{n+1})) \hat{f}(\gamma_{n+1}) = 0
\]

for each \(f \in \Xi(\Phi)\). Hence \((\hat{\lambda}_1 - \hat{\mu}(\gamma_{n+1})) \cdots (\hat{\lambda}_n - \hat{\mu}(\gamma_{n+1})) \hat{f}(\gamma_{n+1}) = 0\). Since \((\hat{\lambda}_1 - \hat{\mu}(\gamma_{n+1})) \cdots (\hat{\lambda}_n - \hat{\mu}(\gamma_{n+1})) \neq 0\), it follows that \(\hat{f}(\gamma_{n+1}) = 0\), which contradicts the choice of \(f_{n+1}\).

Consequently, there exists \(\alpha_1, \ldots, \alpha_N \in \mathbb{C}\) such that

\[
\{\hat{\mu}(\gamma) : \gamma \in \hat{G} \text{ and } \hat{f}(\gamma) \neq 0, \text{ for some } f \in \Xi(\Phi)\} = \{\alpha_1, \ldots, \alpha_N\}.
\]

This clearly implies

\[
(\alpha_1 I_G - T_\mu) \cdots (\alpha_N I_G - T_\mu)(\Xi(\Phi)) = \{0\}.
\]

By removing some of the scalars from the set \(\{\alpha_1, \ldots, \alpha_N\}\), if necessary, we may assume that \(\alpha_1, \ldots, \alpha_N\) are eigenvalues of \(T_\mu\). On the other hand, since \((S, T_\mu)\) has no critical eigenvalue, it may be concluded that \(\alpha_i I_X - S\) has finite-codimensional range for each \(i = 1, \ldots, N\).

We now write \(M = (\alpha_1 I_X - S) \cdots (\alpha_N I_X - S)(X)\). From what has previously been proved, it follows that the codimension of \(M\) in \(X\) is finite. From [7, Lemma 3.3] we now deduce that \(M\) is closed in \(X\). We shall denote by \(R\) the continuous linear map from \(X\) onto \(M\) given by 

\[
R(x) = (\alpha_1 I_X - S) \cdots (\alpha_N I_X - S)(x) \text{ for each } x \in X
\]

and we shall denote by \(\Psi\) the restriction of \(\Phi\) to \(M\). Since \(\Psi R = (\alpha_1 I_G - T_\mu) \cdots (\alpha_N I_G - T_\mu)\Phi\) and \(\alpha_1 I_G - T_\mu) \cdots (\alpha_N I_G - T_\mu)(\Xi(\Phi)) = \{0\}\), Sinclair [7, Lemma 1.3(i)] shows that \(\Psi R\) is continuous.

Our next goal is to show that \(\Psi\) is continuous. Let \((y_n)\) be a sequence in \(M\) with \(\lim y_n = 0\). By the open mapping theorem, \(R\) is open, and so there exists a sequence \((x_n)\) in \(X\) such that \(\lim x_n = 0\) and \(R(x_n) = y_n\) for each \(n \in \mathbb{N}\).
Since $\Psi R$ is continuous, we have $\lim \Psi(y_n) = \lim \Psi R(x_n) = 0$. Thus $\Psi$ is continuous.

We are now in a position to prove that $\Phi$ is continuous. This just follows from the continuity of $\Phi$ on $M$ and the finite-codimensionality of $M$. This contradicts our assumption on $\Phi$.

Remark 2.1. It should be noted that any convolution operator $T_\mu$ on $L^1(G)$ is a $L^1(G)$-multiplier operator. A careful analysis of the preceding proof shows that Theorem 2.1 holds true if we replace $L^1(G)$ by a linear subspace $Y$ of $M(G)$ which is endowed with a Banach space topology that makes Fourier transform on $Y$ continuous and if we replace $T_\mu$ by any $Y$-multiplier operator on $Y$, i.e. a linear operator $T : Y \to Y$ with the property that there exists a function $a_T$ on $\hat{G}$ such that $\hat{Tf}(\gamma) = a_T(\gamma) \hat{f}(\gamma)$ for all $f \in Y$ and $\gamma \in \hat{G}$.

Taking into account that the eigenvalues of the convolution operator $T_\mu$ are just the complex numbers $\lambda$ such that $\mu$ equals $\lambda$ on some open subset of $\hat{G}$, Theorem 2.1 immediately gives the following.

Corollary 2.1. Let $G$ be a locally compact abelian group, let $\mu$ be a bounded complex-valued Borel measure on $G$, and let $T_\mu$ be the corresponding convolution operator $f \mapsto \mu * f$ on $L^1(G)$. Let $X$ be a Banach space and let $S$ be a continuous linear operator on $X$. Suppose that $\mu$ is not constant on any open subset of $\hat{G}$. Then every linear operator $\Phi : X \to L^1(G)$ such that $\Phi S = T_\mu \Phi$ is continuous.

It is worth pointing out that the preceding results are closely related to the problem of characterizing the topologies on function algebras and function spaces that make a fixed operator continuous [3,4,8,9,10]. The following result illustrates this fact.

Corollary 2.2. Let $G$ be a locally compact abelian group, let $\mu$ be a bounded complex-valued Borel measure on $G$, and let $T_\mu$ be the corresponding convolution operator $f \mapsto \mu * f$ on $L^1(G)$. Then the following assertions are equivalent.

i. Every complete norm $|\cdot|$ on $L^1(G)$ with the property that the convolution operator $T_\mu$ from $(L^1(G),|\cdot|)$ into itself is continuous is automatically equivalent to $||\cdot||_1$.

ii. The codimension of $\{\lambda f - \mu * f : f \in L^1(G)\}$ in $L^1(G)$ is finite whenever $\lambda \in \mathbb{C}$ is such that $\mu$ equals $\lambda$ on some open subset of $G$.

Proof. We first suppose that the second assertion holds. If $|\cdot|$ is any complete norm on $L^1(G)$ that makes $T_\mu$ continuous, then we apply
Theorem 2.1 with $X = (L^1(G), | \cdot |)$, $S = T_\mu$, and being $\Phi$ the identity operator from $(L^1(G), | \cdot |)$ onto $(L^1(G), \| \cdot \|_1)$. We thus obtain that $\Phi$ is continuous and the open mapping theorem now shows that $\Phi$ is a homeomorphism and therefore that $| \cdot |$ is equivalent to $\| \cdot \|_1$.

We now assume that the second assertion fails to be true. Then the codimension of $M = \{ \lambda f - \mu \ast f : f \in L^1(G) \}$ in $L^1(G)$ is infinite for some eigenvalue $\lambda$ of $T_\mu$, which implies that there exists a discontinuous linear functional $\phi$ on $L^1(G)$ such that $\phi(M) = \{0\}$. Thus $\phi(\mu \ast f) = \lambda \phi(f)$ for each $f \in L^1(G)$. Let $u \in L^1(G) \setminus \{0\}$ such that $\mu \ast u = \lambda u$. Since the map $f \mapsto 2f - \phi(f)u$ is a linear bijection from $L^1(G)$ onto itself, it follows that $| f | = \| 2f - \phi(f)u \|_1$ is a complete norm on $L^1(G)$ that is not equivalent to $\| \cdot \|_1$. On the other hand, for every $f \in L^1(G)$ we have

$$| \mu \ast f | = \| 2\mu \ast f - \phi(\mu \ast f)u \|_1 = \| 2\mu \ast f - \lambda \phi(f)u \|_1 = \| \mu \ast (2f - \phi(f)u) \|_1 \leq \| T_\mu \| \| 2f - \phi(f)u \|_1 = \| T_\mu \| \| f \|, $$

where $\| T_\mu \|$ stands for the operator norm of $T_\mu : (L^1(G), \| \cdot \|_1) \to (L^1(G), \| \cdot \|_1)$. This shows that $| \cdot |$ makes $T_\mu$ continuous and therefore that the first assertion fails to be true. □

3. OPERATORS INTERTWINING WITH MULTIPLIERS

In this section we show that $L^1(G)$ may be replaced by an arbitrary Banach space $Y$ and that the convolution operator $T_\mu$ on $L^1(G)$ may be replaced by any multiplier on $Y$.

In the sequel, for any Banach space $X$, $X^*$ stands for the dual space of $X$ and $\mathcal{E}_{X^*}$ for the set of extreme points of the closed unit ball of $X^*$. A continuous linear operator $T$ on a Banach space $X$ is said to be a multiplier if there exists a function $a_T$ on $\mathcal{E}_{X^*}$ such that $f(T(x)) = a_T(f)x$ for all $x \in X$ and $f \in \mathcal{E}_{X^*}$. We note that $a_T$ is uniquely determined and we call it the multiplier function of $T$. For a deep discussion of multipliers on Banach spaces we refer the reader to [1].

**Theorem 3.1.** Let $X$ and $Y$ be Banach spaces and let $S$ and $T$ be continuous linear operators on $X$ and $Y$, respectively. Suppose that $T$ is a multiplier and that the pair $(S, T)$ has no critical eigenvalue. Then every linear operator $\Phi : X \to Y$ such that $\Phi S = T\Phi$ is continuous.

**Proof.** The set

$$\{ a_T(f) : f \in \mathcal{E}_{Y^*} \text{ and } f(y) \neq 0, \text{ for some } y \in \mathcal{E}(\Phi) \}$$
can be checked to be finite by the same method as in the proof of Theorem 2.1 with the Fourier transform at a character $g$ replaced by the action of a functional $f \in \mathcal{E}_Y$. This gives $a_1, \ldots, a_N \in \mathbb{C}$ such that
\[
 f((a_1 I_Y - T) \cdots (a_N I_Y - T)y) = 0
\]
for all $f \in \mathcal{E}_Y$ and $y \in \mathfrak{Z}(\Phi)$, where $I_Y$ stands for the identity operator on $Y$. Thus $(a_1 I_Y - T) \cdots (a_N I_Y - T)(\mathfrak{Z}(\Phi)) = \{0\}$. The rest of the proof runs just as in the proof of Theorem 2.1. 

Analysis similar to that in the preceding section leads to the following results.

**Corollary 3.1.** Let $X$ and $Y$ be Banach spaces and let $S$ and $T$ be continuous linear operators on $X$ and $Y$, respectively. Suppose that $T$ is a multiplier and that its multiplier function $\alpha_T$ is not constant on any open subset of $\mathcal{E}_Y$. Then every linear operator $\Phi : X \to Y$ such that $\Phi S = T \Phi$ is continuous.

**Corollary 3.2.** Let $(X, \| \cdot \|)$ be a Banach space and let $T$ be a multiplier on $X$. Then the following assertions are equivalent.

i. Every complete norm $\| \cdot \|$ on $X$ with the property that the operator $T$ from $(X, \| \cdot \|)$ into itself is continuous is automatically equivalent to $\| \cdot \|$.  

ii. The codimension of $\{ \lambda x - T(x) : x \in X \}$ in $X$ is finite whenever $\lambda$ is an eigenvalue of $T$.

**Remark 3.1.** It is known that for any locally compact Hausdorff space $\Omega$ and for any $a \in C_0(\Omega)$ the multiplication operator $x \mapsto ax$ on $C_0(\Omega)$ is a multiplier. Thus the preceding result can be thought of as an approach in Banach space context of the results in [3, 4, 8, 9, 10].

**REFERENCES**


