Periodic trajectories in Gödel type space–times

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1. Introduction

Aim of this paper is to prove the existence of periodic trajectories in a special class of Lorentzian manifolds, the so-called Gödel type space–times.

**Definition 1.1.** A semi-Riemannian manifold \((\mathcal{M}, \langle \cdot, \cdot \rangle_{L})\) is a Lorentzian manifold of Gödel type if there exists a smooth connected finite dimensional Riemannian manifold \((\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)\) such that \(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2\) and \(\langle \cdot, \cdot \rangle_{L}\) is such that

\[
\langle \cdot, \cdot \rangle_L = \langle \cdot, \cdot \rangle_R + A(x) \, dy^2 + 2B(x) \, dy \, dt - C(x) \, dt^2,
\]

where \(x \in \mathcal{M}_0\), the variables \((y, t)\) are the natural coordinates of \(\mathbb{R}^2\) and \(A, B, C\) are \(C^1\) scalar fields on \(\mathcal{M}_0\) such that

\[
\mathcal{H}(x) = B^2(x) + A(x)C(x) > 0 \quad \text{for all } x \in \mathcal{M}_0.
\]

(Of course, we are making some natural identifications with the product structure; hence, it is \(T_z \mathcal{M} \equiv T_x \mathcal{M}_0 \times \mathbb{R}^2\) for any \(z = (x, y, t) \in \mathcal{M}\).)
Remark 1.2. The assumption (1.2) implies that the metric defined in (1.1) is Lorentzian. In fact, the number \( \nu \) of the negative eigenvalues of \( \langle \cdot, \cdot \rangle_L \) is independent of \( z \in \mathcal{M} \) and equal to 1 (for more details on semi-Riemannian manifolds, cf. [2,16]). Notice that, under our definition, a Gödel type space–time may even not be time-orientable: take \( \mathcal{M}_0 \) equal to a circle \( S^1 \) and put, for each \( x = e^{i\theta} \in S^1 \), \( A(x) = C(x) = \cos \theta, \ B(x) = \sin \theta \). It is not difficult to check that the lightcones rotate \( \pi \) after a round to \( S^1 \), and, thus, this Gödel type space–time is not time-orientable.

Let us point out that Definition 1.1 has been introduced in order to generalize the classical Gödel Universe, which is an exact solution of the Einstein’s field equations in which the matter takes the form of a rotating pressure–free perfect fluid obtained by Kurt Gödel in 1949 (cf. [12]). Anyway, such a definition covers also other models of Lorentzian manifolds interesting from a physical point of view, for example the Gödel–Synge space–times (cf. [19]), some types of Kerr–Schild manifolds and other examples already introduced in [8] (see also [9] and references therein). From a Physics point of view, the importance of this type of space–times justifies the study of their geometrical properties and, in particular, of their geodesics. On the other hand, from a purely mathematical viewpoint, it is worth mentioning the following technical point. Variational methods have been systematically used in order to study some properties of many space–times, as geodesic connectedness or the existence of periodic trajectories (see, for example, [4,14]). In particular, more accurate results have been obtained if, in addition, the space–time is (standard) static or stationary, i.e. it is a product manifold \( \mathcal{M}_0 \times \mathbb{R} \) endowed with a Lorentzian metric which is spacelike on the vectors tangent to the slices \( \mathcal{M}_0 \times \{t_0\}, t_0 \in \mathbb{R} \), and is independent of the time variable \( t \) on \( \mathbb{R} \), so that the vector field \( \partial_t \) is timelike and Killing (in the static case there are no cross terms between the space part \( \mathcal{M}_0 \) and the time axis \( \mathbb{R} \), which may exist in the stationary case). Indeed, in this case the problem can be reduced to a “Riemannian” one. Furthermore, extensions of these results can be obtained when one considers a semi-Riemannian manifold of index \( \nu \) and a distribution of \( \nu \) independent vector fields \( K_1, \ldots, K_\nu \) which commute and span a maximal negative definite subspace at each point (cf. [11]). Gödel type space–times admit two Killing vector fields, \( \partial_y \) and \( \partial_t \), but none of them is necessarily timelike; in fact, their causal characters may even change on the manifold. In spite of this, it was shown by some of the authors in [8] that a reduction to a Riemannian problem is still possible, and we will take advantage of this fact.

Recall that if \( (\mathcal{M}, \langle \cdot, \cdot \rangle_L) \) is a manifold of Gödel type, or more in general a semi-Riemannian manifold, a smooth curve \( z : [a,b] \to \mathcal{M} \ (b > a) \) is a geodesic in \( \mathcal{M} \) if

\[
D_s \dot{z}(s) = 0 \quad \text{for all} \ s \in [a,b],
\]

where \( \dot{z} \) is the tangent field along \( z \) and \( D_s \dot{z} \) is the covariant derivative of \( \dot{z} \) along \( z \) induced by the Levi–Civita connection of \( \langle \cdot, \cdot \rangle_L \). Furthermore, if \( z : [a,b] \to \mathcal{M} \) is a geodesic, there exists a constant \( E_z \), named energy of \( z \), such that

\[
E_z = \langle \dot{z}(s), \dot{z}(s) \rangle_L \quad \text{for all} \ s \in [a,b].
\]
Hence, \( z \) is said timelike, respectively, lightlike, spacelike, if its energy \( E_z \) is negative, respectively null, positive. This classification is called causal character of geodesics and comes from General Relativity (cf. [16]).

Particular geodesics in Gödel type manifolds are the \((Y, T)\)-periodic trajectories which we define as follows.

**Definition 1.3.** Let \( Y, T \in \mathbb{R} \) be fixed. A geodesic \( z : [a, b] \to \mathcal{M}, z(s) = (x(s), y(s), t(s)) \), is a \((Y, T)\)-periodic trajectory in \( \mathcal{M} \) if the following conditions hold

\[
\begin{align*}
x(b) &= x(a), & \dot{x}(b) &= \dot{x}(a), \\
y(b) &= y(a) + Y, & \dot{y}(b) &= \dot{y}(a), \\
t(b) &= t(a) + T, & \dot{t}(b) &= \dot{t}(a).
\end{align*}
\]

In particular, a closed geodesic is a \((0, 0)\)-periodic trajectory.

As the metric (1.1) is independent of the variables \( y \) and \( t \), we choose the normalization \( y(a) = t(a) = 0 \) in order to avoid trivial multiplicity results. Moreover, without loss of generality, in what follows we assume \([a, b] = [0, 1]\).

**Remark 1.4.** When it is \( C > 0 \), the physical interpretation of \((0, T)\)-periodic trajectories in Gödel type space–times is equal to the physical interpretation of \( T \)-periodic trajectories in standard stationary space–times: each observer travelling through the integral curves of \( \partial_y \), may interpret that each slice with constant variable \( t \) is his “space” (at least if \( A > 0 \)), and each \((0, T)\)-periodic trajectory is truly periodic for him in his space (analogously, this happens when \( A < 0 \) for \((Y, 0)\)-periodic trajectories). Moreover, for each \((Y, T) \neq (0, 0)\) a new couple of variables \((u, v) \equiv (u(y, t), v(y, t))\) can be defined by means of the linear relations \( y = Yv - Tu, t = Tv + Yu \) and a \((Y, T)\)-periodic trajectory would be also periodic in \( u \) and would have periodic derivative in \( v \) with \( v(1) - v(0) = 1 \). Thus, if \( \partial_y = Y\partial_y + T\partial_t \) is timelike, then \((Y, T)\)-periodic trajectories can be seen as \( 1 \)-periodic trajectories of a stationary space–time. Other interpretations are also possible for other kinds of reference frames (in the sense of [20]) making natural for the trajectories the double control of the advanced steps \( Y \) and \( T \).

**Remark 1.5.** Let \( Y, T \in \mathbb{R} \). A \((Y, T)\)-periodic trajectory \( \tilde{z} : [0, 1] \to \mathcal{M}, \tilde{z}(s) = (\tilde{x}(s), \tilde{y}(s), \tilde{t}(s)) \), will be called trivial if there exists \( \tilde{x} \in \mathcal{M}_0 \) such that \( \tilde{x}(s) \equiv \tilde{x} \) is a constant curve. It is not difficult to check from the geodesic equations that a curve \( \tilde{z}(s) \) is a trivial \((Y, T)\)-periodic trajectory, if and only if \( \tilde{x}(s) \) is a constant curve \( \tilde{x}(s) \equiv \tilde{x} \), where \( \tilde{x} \) is a critical point of \( Y^2A(x) + 2YTB(x) - T^2C(x) \), and \( \tilde{y}, \tilde{t} \) are straight lines of slope \( Y, T \), respectively (see also Remark 3.4). Clearly, for any couple \((Y, T)\), trivial \((Y, T)\)-periodic trajectories always occur when \( \mathcal{M}_0 \) is compact.

In this paper a variational approach allows to prove that, under suitable assumptions on the coefficients \( A, B, C \), for any non-zero couple \((Y, T) \in \mathbb{R}^2\) there exist infinitely
many non-trivial spacelike \((Y,T)\)-periodic trajectories in \(\mathcal{M}\) with higher and higher energy.

Previous results on periodic trajectories are stated in manifolds where just one of the periods, say \(T\), can be taken into account. Results in Lorentzian manifolds of splitting type were obtained with variational methods in [4] (static case, i.e., an orthogonal splitting metric independent of time), [5] (static case with boundary), [6] (orthogonal splitting metrics depending on time) and [1] (stationary case). Moreover, other results for non-orthogonal splitting metrics depending on time have been obtained with non-variational methods in [21] (see also references therein).

Recall that all the previous manifolds (but those ones with boundary) are globally hyperbolic and the results obtained with non-variational methods are applicable just to causal (i.e., timelike or lightlike) trajectories.

Let us point out that a Gödel type space–time with \(A;C\equiv 0\) is a stationary manifold and, in particular, if \(B\equiv 0\) the space–time is static. Then, some results for the static case can be obtained under our approach. Nevertheless, the previous restrictions will not be imposed and also the study of spacelike geodesics will be covered.

Here, we use a variational principle which is essentially the same introduced in [8] in order to study the geodesical connectedness in Gödel type space–times. However, with respect to the case of geodesics joining two points, two more problems arise in the study of periodic trajectories: the existence of trivial solutions (hence, it is interesting to look for non-trivial ones) and the lack of compactness if \(\mathcal{M}_0\) is just complete (then, we need some controls at infinity).

We will use the following notations:

\[
a(x) = \int_0^1 \frac{A(x)}{\mathcal{H}(x)} \, ds, \quad b(x) = \int_0^1 \frac{B(x)}{\mathcal{H}(x)} \, ds, \quad c(x) = \int_0^1 \frac{C(x)}{\mathcal{H}(x)} \, ds, \tag{1.4}
\]

\[
\mathcal{L}(x) = b^2(x) + a(x)c(x) \tag{1.5}
\]

for any continuous curve \(x: [0,1] \to \mathcal{M}_0\); moreover, \(A^1(\mathcal{M}_0)\) will denote the space of closed \(H^1\)-curves in \(\mathcal{M}_0\) (see Section 2 for the precise definition).

We can state the main theorems.

**Theorem 1.6.** Let \((\mathcal{M}, \langle \cdot, \cdot \rangle_L)\) be a Lorentzian manifold of Gödel type. Let us assume that

- \((\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)\) is a compact Riemannian manifold such that its fundamental group \(\pi_1(\mathcal{M}_0)\) is finite or it has infinitely many conjugacy classes;
- for all \(x \in A^1(\mathcal{M}_0)\) it is \(|\mathcal{L}(x)| > 0\);
- there exist some positive constants \(k_1, k_2, k_3\) such that for all \(x \in A^1(\mathcal{M}_0)\) it is

\[
\left| \frac{a(x)}{\mathcal{L}(x)} \right| \leq k_1, \quad \left| \frac{b(x)}{\mathcal{L}(x)} \right| \leq k_2, \quad \left| \frac{c(x)}{\mathcal{L}(x)} \right| \leq k_3.
\]

Then, for any non-zero couple \((Y,T) \in \mathbb{R}^2\) there exist infinitely many non-trivial spacelike distinct \((Y,T)\)-periodic trajectories in \(\mathcal{M}\) whose energies diverge positively.
It is possible to extend Theorem 1.6 to Gödel type space–times \( \mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2 \) when the Riemannian part \( \mathcal{M}_0 \) is just complete. To this aim we add some assumptions on the metric (see (H_4)) and, as in [5] or [14], we introduce a suitable function \( U \) which is strictly convex at infinity (see (H_5) and (H_6)).

Theorem 1.7. Let \((\mathcal{M}, \langle \cdot, \cdot \rangle_L)\) be a Gödel type space–time such that
\( (\mathcal{M}_0, \langle \cdot, \cdot \rangle_R) \) is complete, not contractible in itself and its fundamental group \( \pi_1(\mathcal{M}_0) \) is finite or it has infinitely many conjugacy classes.

Let us assume that (H_2), (H_3) hold and, moreover,
\( \text{(H_4)} \) there exist some positive constants \( K_1, K_2, K_3 \) such that for all \( x \in \mathcal{M}_0 \) the coefficients \( A, B \) and \( C \) satisfy
\[
\left| \frac{A(x)}{\mathcal{H}(x)} \right| \leq K_1, \quad \left| \frac{B(x)}{\mathcal{H}(x)} \right| \leq K_2, \quad \left| \frac{C(x)}{\mathcal{H}(x)} \right| \leq K_3,
\]
\( \text{(H_5)} \) there exist \( x_0 \in \mathcal{M}_0, \ U \in C^2(\mathcal{M}_0, \mathbb{R}_+) \) and some positive constants \( R, \rho, \lambda, \) such that
\[
x \in \mathcal{M}_0, \quad d(x, x_0) \geq R \Rightarrow H_R^U(x)[\xi, \xi] \geq \lambda \langle \xi, \xi \rangle_R \quad \text{for all } \xi \in T_x \mathcal{M}_0,
\]
\( \text{(H_6)} \) taken \( x_0 \) and \( U \) as in (H_5), there results
\[
\liminf_{d(x, x_0) \to +\infty} \langle \nabla U(x), \nabla A(x) \rangle_R \geq 0,
\]
\[
\lim_{d(x, x_0) \to +\infty} \langle \nabla U(x), \nabla B(x) \rangle_R = 0,
\]
\[
\limsup_{d(x, x_0) \to +\infty} \langle \nabla U(x), \nabla C(x) \rangle_R \leq 0,
\]
where \( d(\cdot, \cdot) \) is the distance in \( \mathcal{M}_0 \) and \( H_R^U(x)[\xi, \xi] \) denotes the Hessian of the function \( U \) at \( x \) in the direction \( \xi \) both induced on \( \mathcal{M}_0 \) by its Riemannian structure \( \langle \cdot, \cdot \rangle_R \).
Then, for any non-zero couple \( (Y, T) \in \mathbb{R}^2 \) there exist infinitely many non-trivial space-like distinct \( (Y, T) \)-periodic trajectories in \( \mathcal{M} \) whose energies diverge positively.

Remark 1.8. When the hypothesis (H_1) (respectively (H'_1)) on \( \pi_1(\mathcal{M}_0) \) does not hold in Theorem 1.6 (respectively in Theorem 1.7) the existence of at least one non-trivial \((Y, T)\)-periodic trajectory can also be stated (see Remarks 3.7, 4.3).

Let us point out that the hypotheses in Theorems 1.6 and 1.7 can be simplified if \( Y = 0 \) or \( T = 0 \). Indeed, if \( Y = 0 \) (respectively, \( T = 0 \)) assumptions (H_3) and (H_4) become
\[
\text{(H'_3)} \quad \left| \frac{C(x)}{\mathcal{H}(x)} \right| \leq k_3 \quad \text{(resp. } \left| \frac{A(x)}{\mathcal{H}(x)} \right| \leq k_1 \text{) for all } x \in \mathcal{A}^1(\mathcal{M}_0),
\]
\[
\text{(H'_4)} \quad \left| \frac{C(x)}{\mathcal{H}(x)} \right| \leq K_3 \quad \text{(resp. } \left| \frac{A(x)}{\mathcal{H}(x)} \right| \leq K_1 \text{) for all } x \in \mathcal{M}_0.
\]
In particular, if we look for \((0,0)\)-periodic trajectories, i.e., closed geodesics in \(\mathcal{M}\), also the hypotheses \((H'_3)\) and \((H'_4)\) can be avoided. In fact, it can be proved that if \((H_2)\) holds, then \(z=(x,y,t)\) is a closed geodesic in \(\mathcal{M}\), if and only if \(x\) is a closed geodesic in the Riemannian manifold \(\mathcal{M}_0\) while \(y, t\) are constant (cf. Remark 2.2; the hypothesis \((H_2)\) is unavoidable, see Section 5(C)).

Anyway, Theorems 1.6 and 1.7 do not imply the existence of infinitely many \((0,0)\)-periodic trajectories, and so, in order to look for closed geodesics in a Gödel type space–time, it is enough to study the existence of closed geodesics in a Riemannian manifold; hence, all the known results in this field can be applied (see, e.g., [13] for classical results and [7] and its references for some more recent ones).

**Remark 1.9.** In [8, Remark 1.4] simple conditions on \(A,B,C\) which imply \((H_2),(H_3)\) are stated. However, recall that if \(\mathcal{M}_0\) is compact and \((H_2)\) is replaced by the stronger condition \((H'_2)\) there exists \(v>0\) such that \(|\mathcal{L}(x)| \geq v\) for all \(x \in A^1(\mathcal{M}_0)\), then \((H'_3)\) holds.

On the other hand, it is also easy to give conditions on \(A,B,C\) in order to obtain timelike \((Y,T)\)-periodic trajectories, even though they must be more restrictive in order to avoid a too bad behavior of lightcones. For example, assume that \(\mathcal{M}_0\) is compact, \(A,C \geq 0\) and \(|B| \geq \varepsilon > 0\); then, for any fixed \(Y \neq 0\) with sign \(Y = -\text{sign} B\) the number of \((Y,T)\)-periodic timelike trajectories goes to infinity when \(T \to \infty\). In particular, this always happens in the warped case \(A \equiv C \equiv 0\), studied in Section 5(B).

In Section 5, we will give some examples where Theorems 1.6 and 1.7 can be applied, and discuss their hypotheses further. More precisely, we apply these theorems: first, to a simple example where the causal character of \(\partial_y, \partial_t\) changes and, second, to the warped case. We also discuss how results for static space–times can be obtained and give a new result, Corollary 5.1. Finally, the classical Gödel space–time is also studied. We point out that previous theorems do not apply to this manifold because it does not satisfy their assumptions. In fact, it does not have non-trivial \((Y,T)\)-periodic trajectories for \(Y \neq 0\) and it contains \((0,0)\)-periodic trajectories with non-constant projection onto the \((y,t)\) plane.

For some other physical examples of Gödel type manifolds where Theorems 1.6 and 1.7 hold, see, e.g., the introduction in [8].

### 2. Variational principle

Let \(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2\) be a Gödel type space–time equipped with the Lorentzian metric (1.1). Fix \((Y,T) \in \mathbb{R}^2\).

Our goal is to state a suitable variational principle which reduces the research of \((Y,T)\)-periodic trajectories in \(\mathcal{M}\) to the study of critical points of a functional depending only on the Riemannian component.

 Taken \(I = [0,1]\), let \(H^1(I,\mathcal{M}_0)\) be the set of curves \(x : I \to \mathcal{M}_0\) such that for any local chart \((U,\varphi)\) of \(\mathcal{M}_0\), with \(U \cap x(I) \neq \emptyset\), the curve \(\varphi \circ x\) belongs to the Sobolev
space $H^1(x^{-1}(U), \mathbb{R}^n)$, $n = \dim \mathcal{M}_0$. This implies that $H^1(I, \mathcal{M}_0)$ is equipped with a structure of infinite dimensional manifold modelled on the Hilbert space $H^1(I, \mathbb{R}^n)$ and for any $x \in H^1(I, \mathcal{M}_0)$ the tangent space to $H^1(I, \mathcal{M}_0)$ at $x$ admits the following identification

$$T_xH^1 \equiv \{ \xi \in H^1(I, T_{\mathcal{M}_0}) : \xi(s) \in T_{x(s)}\mathcal{M}_0 \text{ for all } s \in I \},$$

with $T_{\mathcal{M}_0}$ tangent bundle of $\mathcal{M}_0$. Let us remark that, by the Nash Imbedding Theorem (cf. [15]), $\mathcal{M}_0$ can be smoothly isometrically imbedded in some Euclidean space $\mathbb{R}^N$; hence, $H^1(I, \mathcal{M}_0)$ is a submanifold of the Hilbert space $H^1(I, \mathbb{R}^N)$ (for more details, see [17]).

For simplicity, we will denote the Riemannian metric on $\mathcal{M}_0$ with $\langle \cdot, \cdot \rangle$.

Define

$$A^1(\mathcal{M}_0) = \{ x \in H^1(I, \mathcal{M}_0) : x(0) = x(1) \}.$$ 

It is known that $A^1(\mathcal{M}_0)$ is a Riemannian submanifold of $H^1(I, \mathcal{M}_0)$ whose tangent space in $x \in A^1(\mathcal{M}_0)$ is given by

$$T_xA^1(\mathcal{M}_0) = \{ \xi \in T_xH^1 : \xi(0) = \xi(1) \}.$$ 

Moreover, $A^1(\mathcal{M}_0)$ is complete if $\mathcal{M}_0$ is complete (i.e., if it is a complete submanifold of the Euclidean space $\mathbb{R}^N$).

Since we are interested in $(Y, T)$-periodic trajectories $z = (x, y, t)$ such that $y(0) = t(0) = 0$, let us define

$$H_Y = \{ y \in H^1(I, \mathbb{R}) : y(0) = 0, \ y(1) = Y \},$$

$$H_T = \{ t \in H^1(I, \mathbb{R}) : t(0) = 0, \ t(1) = T \}.$$ 

Assumed

$$Y_* : s \in I \mapsto Y_s \in \mathbb{R}, \quad T_* : s \in I \mapsto T_s \in \mathbb{R},$$

it is

$$H_Y = H_0^1 + Y_*, \quad H_T = H_0^1 + T_*,$$

where

$$H_0^1 = \{ u \in H^1(I, \mathbb{R}) : u(0) = u(1) = 0 \}.$$ 

Then, $H_Y$, $H_T$ are closed affine submanifolds of the Hilbert space $H^1(I, \mathbb{R})$ and $H_0^1$ is the tangent space at every one of their points.

Classical variational tools imply that $\tilde{z}$ is a $(Y, T)$-periodic trajectory in $\mathcal{M}$ if and only if $\tilde{z}$ is a critical point of the action functional

$$F(z) = \frac{1}{2} \int_0^1 \langle \dot{z}, \dot{z} \rangle_L \ ds$$

$$= \frac{1}{2} \int_0^1 (\langle \dot{x}, \dot{x} \rangle + A(x)\dot{y}^2 + 2B(x)\dot{y}\dot{t} - C(x)t^2) \ ds,$$

$z = (x, y, t)$, on the Hilbert manifold $Z_{Y,T} \equiv A^1(\mathcal{M}_0) \times H_Y \times H_T$. 

It is easy to prove that $F$ is a $C^1$ functional on $Z_{Y,T}$ and

$$F'(z)[\zeta] = \int_0^1 \langle \dot{x}, \dot{\zeta} \rangle \, ds + \frac{1}{2} \int_0^1 \left( \langle \nabla A(x), \zeta \rangle \dot{y}^2 + 2A(x) \dot{y}\dot{\eta} \right) \, ds$$

$$+ \int_0^1 \left( \langle \nabla B(x), \zeta \rangle \dot{y} + B(x) \dot{\eta} + B(x) \dot{\eta} \dot{\tau} \right) \, ds$$

$$- \frac{1}{2} \int_0^1 \left( \langle \nabla C(x), \zeta \rangle \dot{\tau}^2 + 2C(x) \dot{\eta} \dot{\tau} \right) \, ds$$

for any $z = (x, y, t) \in Z_{Y,T}$ and $\zeta = (\xi, \eta, \tau) \in T_z Z_{Y,T} \equiv T_x A^1(\mathcal{M}_0) \times H_0^1 \times H_0^1$.

Since the action functional $F$ is unbounded both from above and from below, a new functional on $A^1(\mathcal{M}_0)$ is defined in such a way that it is bounded from below and its critical points are related to those ones of $F$.

This variational approach has been introduced in [8] in order to study geodesics joining two given points in Gödel type manifolds. Anyway, for completeness, we outline the main ideas of the proof.

First of all, let us define

$$\phi_y : A^1(\mathcal{M}_0) \to H_Y \quad \text{and} \quad \phi_t : A^1(\mathcal{M}_0) \to H_T$$

such that, if $x \in A^1(\mathcal{M}_0)$, for all $s \in I$ it is

$$\phi_y(x)(s) = \frac{Yb(x) - Tc(x)}{\mathcal{L}(x)} \int_0^s \frac{B(x(\sigma))}{\mathcal{H}(x(\sigma))} \, d\sigma$$

$$+ Ya(x) + Tb(x) \int_0^s \frac{C(x(\sigma))}{\mathcal{H}(x(\sigma))} \, d\sigma,$$

$$\phi_t(x)(s) = - \frac{Yb(x) - Tc(x)}{\mathcal{L}(x)} \int_0^s \frac{A(x(\sigma))}{\mathcal{H}(x(\sigma))} \, d\sigma$$

$$+ Ya(x) + Tb(x) \int_0^s \frac{B(x(\sigma))}{\mathcal{H}(x(\sigma))} \, d\sigma.$$ 

Clearly, $\phi_y$ and $\phi_t$ are two $C^1$ maps on $A^1(\mathcal{M}_0)$.

**Proposition 2.1.** Let $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ be a Gödel type manifold such that $(H_2)$ holds. Then, the following statements are equivalent:

(i) $z \in Z_{Y,T}$ is a critical point of $F$,

(ii) $z = (x, y, t)$ is such that $x \in A^1(\mathcal{M}_0)$ is a critical point of the functional

$$J(x) = \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds + \frac{Y^2a(x) + 2YTb(x) - T^2c(x)}{2\mathcal{L}(x)}$$

(2.2)
and
\[ y = \phi_y(x), \quad t = \phi_t(x), \]
where \(a, b, c\) and \(\mathcal{L}\) are as in (1.4) and (1.5) and \(\phi_y, \phi_t\) are as above. Moreover, if (i) or (ii) holds, it is
\[ F(x, \phi_y(x), \phi_t(x)) = J(x). \]

**Proof.** Since \(F\) is a \(C^1\) functional on \(Z_{Y,T}\), let us define the partial derivatives of \(F\) in \(z = (x,y,t)\) as follows:
\[ F_x(z)[\xi] = F'(z)[(\xi, 0, 0)] \quad \text{for all } \xi \in T_xA^1(\mathcal{M}_0), \]
\[ F_y(z)[\eta] = F'(z)[(0, \eta, 0)] \quad \text{for all } \eta \in H^1_0, \]
\[ F_t(z)[\tau] = F'(z)[(0, 0, \tau)] \quad \text{for all } \tau \in H^1_0. \]

Clearly, all the critical points of \(F\) in \(Z_{Y,T}\) belong to \(N = \{ z \in Z_{Y,T} : F_y(z) = F_t(z) = 0 \} \) (2.3)

Arguing as in the proof of [8, Lemma 2.3], it follows that \(N\) is a submanifold of \(Z_{Y,T}\), since it is the graph of the \(C^1\) map \(\Phi : A^1(\mathcal{M}_0) \to H_T \times H_T\) defined as \(\Phi(x) = (\phi_y(x), \phi_t(x))\) for every \(x \in A^1(\mathcal{M}_0)\).

Assume \(J\) is equal to the restriction of \(F\) to \(N\), i.e.,
\[ J(x) = F(x, \Phi(x)) \quad \text{for all } x \in A^1(\mathcal{M}_0). \]

Then, it is easy to see that \(z \in Z_{Y,T}\) is a critical point of \(F\), if and only if \(z = (x,y,t) \in N\), i.e., \((y,t) = \Phi(x)\), and \(J'(x) = 0\), as,
\begin{align*}
J'(x)[\xi] &= F'_x(x)[\xi] = F'(z)[(\xi, 0, 0)] \\
&= \int_0^1 \langle \dot{x}, \dot{\xi} \rangle ds + \frac{1}{2} \int_0^1 \langle \nabla A(x), \xi \rangle \dot{y}^2 ds \\
&\quad + \int_0^1 \langle \nabla B(x), \xi \rangle \dot{y} \dot{t} ds - \frac{1}{2} \int_0^1 \langle \nabla C(x), \xi \rangle \dot{t}^2 ds \quad (2.4)
\end{align*}

for all \(\xi \in T_xA^1(\mathcal{M}_0)\). \(\square\)

**Remark 2.2.** Let \(\mathcal{M}\) be a Gödel type space–time such that \((H_2)\) holds. If we look for closed geodesics in \(\mathcal{M}\), i.e., geodesics in \(\mathcal{M}\) which satisfy the boundary conditions (1.3) with \(Y = T = 0\), then, by Proposition 2.1 and the definitions of \(\phi_y, \phi_t\), a curve \(\tilde{z} = (\tilde{x}, \tilde{y}, \tilde{t}) \in Z_{0,0}\) is a critical point of \(F\), if and only if \(\tilde{y} = \tilde{t} = 0\) while \(\tilde{x}\) is a critical point of the functional
\[ f(x) = \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle ds \quad \text{on } A^1(\mathcal{M}_0). \]
So, when \((H_2)\) holds, a curve \(\tilde{z} = (\tilde{x}, \tilde{y}, \tilde{t})\) in \(\mathcal{M}\) is a closed geodesic, if and only if \(\tilde{x}\) is a closed geodesic in the Riemannian manifold \(\mathcal{M}_0\) and \(\tilde{y}, \tilde{t}\) are constant.

Now, let us recall the main definitions and results of the Ljusternik–Schnirelman theory on manifolds (for more details, see, e.g., [18,22]).

**Definition 2.3.** Let \(X\) be a topological space. Given \(A \subseteq X\), \(\text{cat}_X(A)\) is the Ljusternik–Schnirelman category of \(A\) in \(X\), that is the least number of closed and contractible subsets of \(X\) covering \(A\). Otherwise, it is \(\text{cat}_X(A) = +\infty\) if it is not possible to cover \(A\) with a finite number of such sets.

We assume \(\text{cat}(X) = \text{cat}_X(X)\).

**Definition 2.4.** Let \(\Lambda\) be a Riemannian manifold modelled on a Hilbert space. A \(C^1\) functional \(g: \Lambda \to \mathbb{R}\) satisfies the Palais–Smale condition, briefly \((PS)\), if every sequence \((x_n)_{n \in \mathbb{N}}\) in \(\Lambda\), such that

\[
\sup_{n \in \mathbb{N}} |g(x_n)| < +\infty \quad \text{and} \quad \lim_{n \to +\infty} g'(x_n) = 0,
\]

has a convergent subsequence (here, \(g'(x_n)\) goes to 0 in the norm induced on the cotangent bundle by the Riemannian metric on \(\Lambda\)).

**Theorem 2.5** (Ljusternik–Schnirelman theorem). Let \(\Lambda\) be a complete Riemannian manifold and let \(g: \Lambda \to \mathbb{R}\) be a \(C^1\) functional which satisfies \((PS)\). If \(k \in \mathbb{N}\), \(k \geq 1\), let us define

\[
\Gamma_k = \{ A \subseteq \Lambda: \text{cat}_\Lambda(A) \geq k \}, \tag{2.5}
\]

\[
c_k = \inf_{A \in \Gamma_k} \sup_{x \in A} g(x). \tag{2.6}
\]

If \(\Gamma_k \neq \emptyset\) and \(c_k \in \mathbb{R}\), then \(c_k\) is a critical value of \(g\).

If \(g\) is a functional on a manifold \(\Lambda\), for any \(c \in \mathbb{R}\) we set

\[
g_c^\leq = \{ x \in \Lambda : g(x) \leq c \},
\]

\[
g_c^\geq = \{ x \in \Lambda : g(x) > c \}.
\]

**Remark 2.6.** Let \(\Lambda\) and \(g\) be as in Theorem 2.5. If \(g\) is bounded from below, then for any \(c \in \mathbb{R}\) it is

\[
\text{cat}_\Lambda(g_c^\leq) < +\infty. \tag{2.7}
\]

As a consequence of the Ljusternik–Schnirelman theorem, we can prove the following propositions.
Proposition 2.7. Let $g$ be a $C^1$ functional on a smooth Riemannian manifold $A$ such that $\text{cat}(A) = +\infty$. Assume that (2.7) holds for all $c \in \mathbb{R}$. If $L \in \mathbb{R}$ is such that $gL \neq \emptyset$, then there exists $k_L \in \mathbb{N}$ such that $c_{k_L} \geq L$, where $c_{k_L}$ is as in (2.6).

Proof. Let $L \in \mathbb{R}$ be such that $gL \neq \emptyset$. By the definition (2.6) it is enough to prove that there exists $k_L \in \mathbb{N}$ such that $B \cap gL \neq \emptyset$ for all $B \in \Gamma_{k_L}$, where $\Gamma_{k_L}$ is defined as in (2.5).

In fact, otherwise, for any $k \in \mathbb{N}$ there exists $B_k \in \Gamma_k$ such that $B_k \cap gL = \emptyset$, then

$$B_k \subset g^L, \quad \text{cat}_A(B_k) \geq k \Rightarrow \text{cat}_A(g^L) \geq k$$

in contradiction with (2.7). \qed

Corollary 2.8. Let $A$ be a complete Riemannian manifold which has compact subsets of arbitrarily high category. Let $g : A \rightarrow \mathbb{R}$ be a $C^1$ functional bounded from below but not from above. Assume that $g$ satisfies (PS). Then there exists an increasing diverging sequence of critical levels of $g$ on $A$.

Proof. Let $n \in \mathbb{N}$, $n \geq 1$, be fixed. Since $g$ is not bounded from above, then the “superlevel” $g_n$ is not empty; moreover, by Remark 2.6 the condition (2.7) holds for all $c \in \mathbb{R}$. By Proposition 2.7 and the topological assumptions on $A$ it follows that there exist $n \in \mathbb{N}$ and a compact set $K_n \in \Gamma_{k_n}$ such that

$$n \leq c_{k_n} \leq \sup_{x \in K_n} g(x) < +\infty.$$ 

So, Theorem 2.5 implies $c_{k_n}$ is a critical level of $g$ and $\lim_{n \rightarrow +\infty} c_{k_n} = +\infty$. \qed

The following result, due to Fadell and Husseini (cf. [10]), allows to evaluate the Lusternik–Schnirelman category of $\Lambda^1(X)$, space of loops of an open subset of $\mathbb{R}^N$.

Anyway, standard arguments allow to apply such a result even when $X = \mathcal{M}_0$ is a manifold.

Proposition 2.9. Let $X$ be an open subset of $\mathbb{R}^N$ which is connected, not contractible in itself and such that its fundamental group $\pi_1(X)$ does not have the property that it is an infinite group with finitely many conjugacy classes. Then $\text{cat}(\Lambda^1(X)) = +\infty$. Moreover, $\Lambda^1(X)$ possesses compact subsets of arbitrarily high category.

3. The compact case

Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ be a manifold of G"odel type equipped with the Lorentzian metric (1.1) such that $\mathcal{M}_0$ is at least complete and the assumptions (H$_2$), (H$_3$) hold. Fix a non-zero couple $(Y,T) \in \mathbb{R}^2$.

Since Proposition 2.1 applies, then $(Y,T)$-periodic trajectories in $\mathcal{M}$ can be searched looking for critical points of the $C^1$ functional $J : \Lambda^1(\mathcal{M}_0) \rightarrow \mathbb{R}$ defined in (2.2).

In order to apply Corollary 2.8 the following lemmas are useful.
**Lemma 3.1.** The functional $J$ is bounded from below on $\Lambda^1(\mathcal{M}_0)$.

**Proof.** The proof is an obvious consequence of the hypothesis (H$_3$):

$$J(x) \geq \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds - \frac{1}{2} (k_1 Y^2 + 2k_2 |YT| + k_3 T^2).$$

\[ \square \]

**Remark 3.2.** Taken $k(Y,T) = k_1 Y^2 + 2k_2 |YT| + k_3 T^2$, the proof of Lemma 3.1 implies

$$\int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds \leq 2J(x) + k(Y,T).$$

Let us remark that trivial periodic trajectories in $\mathcal{M}$ correspond to constant critical points of $J$ in $\Lambda^1(\mathcal{M}_0)$; hence, we need more information about $J$ on constant curves.

Define $h_{Y,T} : \mathcal{M}_0 \to \mathbb{R}$ such that

$$h_{Y,T}(x) = Y^2 A(x) + 2YTB(x) - T^2 C(x).$$

Clearly, $h_{Y,T}$ is a $C^1$ map. Taken $k \in \mathcal{M}_0$, by (1.4) and (1.5), on the corresponding constant map in $\Lambda^1(\mathcal{M}_0)$ there results

$$\mathcal{L}(k) = \frac{1}{\mathcal{H}(k)}, \quad \frac{a(k)}{\mathcal{L}(k)} = A(k), \quad \frac{b(k)}{\mathcal{L}(k)} = B(k), \quad \frac{c(k)}{\mathcal{L}(k)} = C(k),$$

hence,

$$J(k) = \frac{1}{2} h_{Y,T}(k).$$

**Lemma 3.3.** Taken $k \in \mathcal{M}_0$, the following statements are equivalent:

(i) $k$ is a trivial critical point of $J$ on $\Lambda^1(\mathcal{M}_0)$;

(ii) $k$ is a critical point of $h_{Y,T}$ on $\mathcal{M}_0$.

**Proof.** By (3.2) it follows

$$\phi_y(k) = Y_*, \quad \phi_t(k) = T_*,$$

where $\phi_y$ and $\phi_t$ are as in Section 2 while $Y_*, T_*$ are defined in (2.1). So, (2.4) gives $J'(k) = 0$ if and only if

$$\int_0^1 \langle \nabla h_{Y,T}(k), \zeta \rangle \, ds = 0 \quad \text{for all } \zeta \in T_k \Lambda^1(\mathcal{M}_0),$$

i.e., $\nabla h_{Y,T}(k) \equiv 0$. \[ \square \]

**Remark 3.4.** If (H$_2$) holds, by Lemma 3.3 it follows that $z : [0,1] \to \mathcal{M}, z = (x,y,t)$, is a trivial periodic trajectory in $\mathcal{M}$ if and only if $x \equiv k$ is a critical point of $h_{Y,T}$ while $y$ and $t$ are straight lines of slope $Y$ and $T$, respectively (see the definitions in (3.4)). Let us point out that such a result does not need the assumption (H$_2$) and can also be proved just using the geodesic equations.
**Corollary 3.5.** If \( x \in A^1(\mathcal{M}_0) \) is a critical point of \( J \) such that \( 2J(x) > k(Y, T) \), then \( x \) is not constant.

**Proof.** It is easy to see that
\[
|A(x)| \leq k_1, \quad |B(x)| \leq k_2, \quad |C(x)| \leq k_3 \quad \text{for all} \quad x \in \mathcal{M}_0
\]
(trivial estimates if \( \mathcal{M}_0 \) is compact, otherwise coming from (3.2) and (H3)) which imply
\[
\sup_{x \in \mathcal{M}_0} |h_{Y,T}(x)| \leq k(Y, T).
\]
So, the result follows by (3.3) and Lemma 3.3. \( \square \)

**Lemma 3.6.** Let \( \mathcal{M}_0 \) be compact. Then the functional \( J \) satisfies (PS) in \( A^1(\mathcal{M}_0) \).

**Proof.** Let \( (x_n)_{n \in \mathbb{N}} \subset A^1(\mathcal{M}_0) \) be such that
\[
(J(x_n))_{n \in \mathbb{N}} \quad \text{is bounded} \quad \text{and} \quad \lim_{n \to \infty} J'(x_n) = 0. \quad (3.5)
\]
Clearly, by (3.1) it follows that
\[
\left( \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \, ds \right)_{n \in \mathbb{N}} \quad \text{is bounded},
\]
then, since \( \mathcal{M}_0 \) is compact, the sequence \( (x_n)_{n \in \mathbb{N}} \) is bounded in \( H^1(I, \mathbb{R}^N) \) and there exists \( x \in H^1(I, \mathbb{R}^N) \) such that
\[
x_n \rightharpoonup x \quad \text{weakly in} \quad H^1(I, \mathbb{R}^N), \quad x_n \to x \quad \text{uniformly in} \quad I \quad (3.6)
\]
(up to subsequences). Clearly, \( x \in A^1(\mathcal{M}_0) \). By [3, Lemma 2.1] there exists two bounded sequences \( (\xi_n)_{n \in \mathbb{N}} \) and \( (v_n)_{n \in \mathbb{N}} \) in \( H^1(I, \mathbb{R}^N) \), \( \xi_n \in T_x A^1(\mathcal{M}_0) \), such that
\[
x_n - x = \xi_n + v_n \quad \text{for all} \quad n \in \mathbb{N},
\]
\[
\xi_n \rightharpoonup 0 \quad \text{weakly} \quad \text{and} \quad v_n \to 0 \quad \text{strongly in} \quad H^1(I, \mathbb{R}^N). \quad (3.7)
\]
Our aim is to verify that \( \xi_n \to 0 \) strongly in \( H^1(I, \mathbb{R}^N) \).
In fact, taken \( y_n = \phi_y(x_n), \quad t_n = \phi_t(x_n) \) and \( z_n = (x_n, y_n, t_n) \in N \) (see (2.3)), by (2.4) and (3.5) it is
\[
J'(x_n)[\xi_n] = F'(z_n)[(\xi_n, 0, 0)] = o(1),
\]
i.e.,
\[
o(1) = \int_0^1 \langle \dot{x}_n, \dot{\xi}_n \rangle \, ds + \frac{1}{2} \int_0^1 \langle \nabla A(x_n), \dot{\xi}_n \rangle \dot{y}_n^2 \, ds
\]
\[
+ \int_0^1 \langle \nabla B(x_n), \dot{\xi}_n \rangle \dot{y}_n \, ds - \frac{1}{2} \int_0^1 \langle \nabla C(x_n), \dot{\xi}_n \rangle \dot{t}_n^2 \, ds. \quad (3.8)
\]
Let us remark that, since the Palais–Smale sequence \((x_n)_{n \in \mathbb{N}}\) is bounded and \((H_3)\) holds, the definitions of \(\phi_j\) and \(\phi_t\) imply that
\[
\left( \int_0^1 \dot{y}_n^2 \, ds \right)_{n \in \mathbb{N}} \quad \text{and} \quad \left( \int_0^1 t_n^2 \, ds \right)_{n \in \mathbb{N}}
\]
are bounded, so, (3.6) and (3.7) give
\[
\int_0^1 \langle \nabla A(x_n), \dot{\xi}_n \rangle \dot{y}_n^2 \, ds = o(1),
\]
\[
\int_0^1 \langle \nabla B(x_n), \dot{\xi}_n \rangle \dot{y}_n \dot{t}_n \, ds = o(1),
\]
\[
\int_0^1 \langle \nabla C(x_n), \dot{\xi}_n \rangle \dot{t}_n^2 \, ds = o(1).
\]
Finally, since \(x_n = x + \dot{\xi}_n + v_n\), by (3.8) and
\[
\int_0^1 \langle \dot{x}, \dot{\xi}_n \rangle \, ds = o(1), \quad \int_0^1 \langle \dot{v}_n, \dot{\xi}_n \rangle \, ds = o(1)
\]
it follows that
\[
\int_0^1 \langle \dot{\xi}_n, \dot{\xi}_n \rangle \, ds = o(1). \quad \square
\]

**Proof of Theorem 1.6.** From the hypothesis \((H_1)\) it follows that Proposition 2.9 applies; moreover, since Lemmas 3.1 and 3.6 hold, by Corollary 2.8 there exists an increasing diverging sequence of critical levels of \(J\) on \(A^1(\mathcal{H}_0)\). Clearly, by Corollary 3.5, infinitely many of the corresponding critical points, should not be constant; hence, by Proposition 2.1, infinitely many non-trivial \((Y, T)\)-periodic trajectories exist. \(\square\)

**Remark 3.7.** If \(\mathcal{H}_0\) is compact but it does not satisfy \((H_1)\), the other hypotheses of Theorem 1.6 imply the existence of a critical point of \(J\) for curves at any conjugacy class \(\mathcal{C}\) of the fundamental group of \(\mathcal{H}_0\). More precisely, let \(A^1_{\mathcal{C}}\) be the subset of \(A^1(\mathcal{H}_0)\) containing the curves which lie in the conjugacy class \(\mathcal{C}\). It can be proved that \(A^1_{\mathcal{C}}\) is an open and closed subset of \(A^1(\mathcal{H}_0)\); thus, \(J\) attains a minimum point in \(A^1_{\mathcal{C}}\) which is also a critical point in \(A^1(\mathcal{H}_0)\). If \(\mathcal{C}\) is the trivial conjugacy class this result is not interesting since \(J\) attains its minimum in the minimum points of \(h_{Y,T}\), and then the corresponding trajectories are trivial. But if \((H_1)\) does not hold, then at least there is a non-trivial conjugacy class and, so, a non-trivial \((Y, T)\)-periodic trajectory is found.
4. The complete case

Let \( \mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2 \) be a Gödel type manifold equipped with the Lorentzian metric (1.1) such that \( \mathcal{M}_0 \) is complete and the assumptions \((H_2)\) and \((H_3)\) hold.

Fixed a non-zero couple \((Y, T) \in \mathbb{R}^2\), by Proposition 2.1 the \((Y, T)\)-periodic trajectories in \( \mathcal{M} \) are critical points of the functional \( J : \mathcal{A}^1(\mathcal{M}_0) \to \mathbb{R} \) defined in (2.2). Unluckily, since \( \mathcal{M}_0 \) is complete but in general unbounded, \( J \) could not satisfy the Palais–Smale condition. In fact, a sequence \((x_n)_{n \in \mathbb{N}} \) in \( \mathcal{M}_0 \) may exist such that \( d(x_n, \bar{x}) \to +\infty \) and \( h_{Y, T}'(x_n) \to 0 \) as \( n \to +\infty \) (for a certain \( \bar{x} \in \mathcal{M}_0 \)) while in \( \mathcal{A}^1(\mathcal{M}_0) \) it is a sequence which satisfies the conditions (3.5) and has no converging subsequences.

In order to overcome such a problem, the existence of a “convex at infinity” map allows to introduce a family of penalized functionals \((F_\varepsilon)_{\varepsilon > 0}\) in such a way that every \( J_\varepsilon \), associated to \( F_\varepsilon \), satisfies \((PS)\).

Thus, let \( U \) be such that the hypothesis \((H_5)\) holds. It can be proved that

\[
\langle \nabla U(x), \nabla U(x) \rangle^{1/2} \geq \lambda \, d(x, x_0) - c_1, \tag{4.1}
\]

\[
U(x) \geq \frac{\lambda}{2} \, d^2(x, x_0) - c_2 \, d(x, x_0) - c_3 \tag{4.2}
\]

for suitable positive constants \( c_1, c_2, c_3 \) and any \( x \in \mathcal{M}_0 \), where \( \lambda \) and \( x_0 \) are as in \((H_5)\) (cf. [5, Lemma 2.2]).

Fixed \( \varepsilon > 0 \), let \( \psi_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+ \) be a \( C^2 \) “cut-function” such that

\[
\psi_\varepsilon(s) = \begin{cases} 
0 & \text{if } 0 \leq s \leq \frac{1}{\varepsilon}, \\
\sum_{n=3}^{+\infty} \frac{\lambda^n}{n!} \left( s - \frac{1}{\varepsilon} \right)^n & \text{if } s > \frac{1}{\varepsilon}.
\end{cases}
\]

It is easy to prove that there exist some positive constants \( a, b \) such that

\[
\psi_\varepsilon'(s) \geq \lambda \psi_\varepsilon(s) \geq as - b \quad \text{for all } s \in \mathbb{R}_+, \tag{4.3}
\]

\[
\psi_\varepsilon(s) \leq \psi_\varepsilon'(s) \quad \text{for all } s \in \mathbb{R}_+. \tag{4.4}
\]

For any \( x \in \mathcal{M}_0 \), define

\[
U_\varepsilon(x) = \psi_\varepsilon(U(x)).
\]

Then, the action functional \( F \) can be penalized by means of \( U_\varepsilon(x) \); more precisely, define

\[
F_\varepsilon(z) = F(z) + \int_0^1 U_\varepsilon(x) \, ds, \quad z = (x, y, t) \in Z_{Y, T}.
\]

Clearly, \( F_\varepsilon \) is of class \( C^1 \); moreover, since the penalization term does not depend on the variables \((y, t)\), for all \( z = (x, y, t) \in Z_{Y, T} \) and \( \eta, \tau \in H_0^1 \) it is

\[
F_\varepsilon'(z)[(0, \eta, 0)] = F'(z)[(0, \eta, 0)], \quad F_\varepsilon'(z)[(0, 0, \tau)] = F'(z)[(0, 0, \tau)].
\]
Then, arguing as in Proposition 2.1, taken \( \phi_y \) and \( \phi_t \) as in Section 2, there results that \( z = (x, y, t) \) is a critical point of \( F \), if and only if \( x \) is a critical point of

\[
J(x) = J(x) + \int_0^1 U(x) \, ds \quad \text{in} \; A^1(\mathcal{M}_0),
\]

while \( y = \phi_y(x) \), \( t = \phi_t(x) \). Moreover, \( J(x) = F(z) \) and

\[
J'(x)[\xi] = F'(z)[(\xi, 0, 0)]
\]

\[
= F'(z)[(\xi, 0, 0)] + \int_0^1 \psi'_\xi(U(x))(\nabla U(x), \xi) \, ds
\]

for all \( \xi \in T_xA^1(\mathcal{M}_0) \).

Let us point out that, since \( \psi \) is positive, (3.1) implies

\[
\int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds \leq 2J(x) + k(Y, T) \quad \text{for all} \; x \in A^1(\mathcal{M}_0);
\]

hence, \( J \) is bounded from below.

**Lemma 4.1.** For all \( \varepsilon > 0 \) the functional \( J_\varepsilon \) satisfies (PS) in \( A^1(\mathcal{M}_0) \).

**Proof.** Fixed \( \varepsilon > 0 \), let \( (x_n)_{n \in \mathbb{N}} \subset A^1(\mathcal{M}_0) \) be such that

\[
(J_\varepsilon(x_n))_{n \in \mathbb{N}} \text{ is bounded and} \lim_{n \to \infty} J'_\varepsilon(x_n) = 0.
\]

By (4.7) and (4.8) it follows that

\[
\left( \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \, ds \right)_{n \in \mathbb{N}} \text{ is bounded.}
\]

If it is

\[
\sup\{d(x_n(s), x_0): s \in I, \; n \in \mathbb{N}\} = + \infty
\]

(taken \( x_0 \) as in (H3)), then

\[
\lim_{n \to \infty} \left( \inf_{s \in I} d(x_n(s), x_0) \right) = + \infty
\]

which implies, by (4.2), that

\[
\lim_{n \to \infty} \left( \inf_{s \in I} U(x_n(s)) \right) = + \infty.
\]

Clearly, (4.3) and this last formula gives

\[
\lim_{n \to \infty} \int_0^1 U_\varepsilon(x_n) \, ds = + \infty.
\]
On the other hand, by (3.1) and (4.5) it follows
\[ \int_0^1 U(x_n) \, ds \leq J(x_n) + \frac{k(Y, T)}{2}, \]
then, by (4.8),
\[ \left( \int_0^1 U(x_n) \, ds \right)_{n \in \mathbb{N}} \]
is bounded
in contradiction with (4.10).
Thus, \( \sup \{ d(x_n(s), x_0) : s \in I, n \in \mathbb{N} \} < + \infty \); hence, by (4.9), \( (x_n)_{n \in \mathbb{N}} \) is bounded in \( H^1(I, \mathbb{R}^N) \), so (3.6), (3.7) hold.
It is easy to verify that (4.6), \( (x_n)_{n \in \mathbb{N}} \) bounded and \( \xi_n \rightharpoonup 0 \) weakly in \( H^1(I, \mathbb{R}^N) \) (see (3.7)) imply
\[ \int_0^1 \psi'(U(x_n))(\nabla U(x_n), \xi_n) \, ds = o(1), \]
then, arguing as in the last part of the proof of Lemma 3.6, the conclusion follows. \( \square \)

In order to avoid the penalization argument, the following proposition needs.

**Proposition 4.2.** Let \( \mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2 \) be a G\ö del type manifold such that \( \mathcal{M}_0 \) is complete and the assumptions (H2)–(H6) hold. Taken any couple of positive constants \( m, M \) with
\[ \frac{k(Y, T)}{2} < m < M \] (4.11)
there exists \( \varepsilon_0 = \varepsilon_0(m, M) > 0 \) such that for every \( \varepsilon \in \mathcal{X}_0, \varepsilon_0[ \) if \( x_\varepsilon \in \mathcal{A}^1(\mathcal{M}_0) \) satisfies
\[ J'_\varepsilon(x_\varepsilon) = 0, \quad m \leq J_\varepsilon(x_\varepsilon) \leq M, \] (4.12)
then
\[ J(x_\varepsilon) = J'_\varepsilon(x_\varepsilon), \quad J'(x_\varepsilon) = 0. \] (4.13)

**Proof.** Obviously, if there exists \( \varepsilon_0 > 0 \) such that (4.12) implies
\[ \sup_{s \in I} U(x_\varepsilon(s)) < \frac{1}{\varepsilon_0}, \] (4.14)
with \( \varepsilon \leq \varepsilon_0 \), then \( \psi'_\varepsilon(U(x_\varepsilon(s))) = \psi'(U(x_\varepsilon(s))) = 0 \) for all \( s \in I \) and (4.13) holds.
Thus, in order to prove (4.14), let us argue by contradiction and assume that for each \( n \in \mathbb{N} \) there exist \( \varepsilon_n > 0 \) and \( x_n \in \mathcal{A}^1(\mathcal{M}_0) \) such that \( \varepsilon_n \rightharpoonup 0 \) and
\[ J'_{\varepsilon_n}(x_n) = 0, \quad m \leq J_{\varepsilon_n}(x_n) \leq M, \] (4.15)
\[ \sup_{s \in I} U(x_n(s)) \geq \frac{1}{\varepsilon_n}. \] (4.16)
It is easy to prove that, since $U$ is continuous, (4.16) implies
\[
\sup \{ d(x_n(s), x_0); s \in I, \ n \in \mathbb{N} \} = + \infty,
\] (4.17)
moreover, by (4.7) and (4.15) it follows that
\[
\left( \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \ ds \right)_{n \in \mathbb{N}}
\] is bounded. (4.18)

Clearly, (4.17) and (4.18) give
\[
\lim_{n \to +\infty} \left( \inf_{s \in I} d(x_n(s), x_0) \right) = + \infty;
\] (4.19)
hence, by (4.2) it is
\[
\lim_{n \to +\infty} \left( \inf_{s \in I} U(x_n(s)) \right) = + \infty.
\]

Fixed $n \in \mathbb{N}$, assume $u_n(s) = U(x_n(s))$, $y_n = \phi_1(x_n)$, $t_n = \phi_2(x_n)$.
As $J_n'(x_n) = 0$, by (4.6) and (2.4) classical arguments imply that $x_n$ is a $C^2$ map such that $\dot{x}_n(0) = \dot{x}_n(1)$ and
\[
D^R_s \dot{x}_n = \frac{1}{2} \dot{y}_n^2 \nabla A(x_n) + \dot{y}_n \dot{t}_n \nabla B(x_n) - \frac{1}{2} \dot{t}_n^2 \nabla C(x_n) + \psi_n'(U(x_n)) \nabla U(x_n),
\]
where $D^R_s \dot{x}_n$ is the covariant derivative of $\dot{x}_n$ along $x_n$ induced by the Levi–Civita connection of $\langle \cdot, \cdot \rangle$ (e.g., cf. [14]).
Then $u_n \in C^2(I, \mathbb{R})$ and $u_n(0) = u_n(1)$, $\dot{u}_n(0) = \dot{u}_n(1)$, where for all $s \in I$ it is
\[
\dot{u}_n(s) = \langle \nabla U(x_n(s)), \dot{x}_n(s) \rangle.
\]

From the above equation and
\[
\ddot{u}_n(s) = H^U_R(x_n(s))[\dot{x}_n(s), \dot{x}_n(s)] + \langle \nabla U(x_n(s)), D^R_s \dot{x}_n(s) \rangle, \quad s \in I,
\]
it follows
\[
0 = \int_0^1 \ddot{u}_n \ ds = \int_0^1 \left( H^U_R(x_n)[\dot{x}_n, \dot{x}_n] + \langle \nabla U(x_n), D^R_s \dot{x}_n \rangle \right) \ ds
\]
\[
= \int_0^1 H^U_R(x_n)[\dot{x}_n, \dot{x}_n] \ ds + \frac{1}{2} \int_0^1 \langle \nabla U(x_n), \nabla A(x_n) \rangle \dot{y}_n^2 \ ds
\]
\[
+ \int_0^1 \langle \nabla U(x_n), \nabla B(x_n) \rangle \dot{y}_n \dot{t}_n \ ds - \frac{1}{2} \int_0^1 \langle \nabla U(x_n), \nabla C(x_n) \rangle \dot{t}_n^2 \ ds
\]
\[
+ \int_0^1 \psi_n'(U(x_n)) \langle \nabla U(x_n), \nabla U(x_n) \rangle \ ds.
\]
Let us remark that (H4) and the definitions of $\phi_y$ and $\phi_t$ imply that
\[ \left( \int_0^1 \dot{y}_n^2 \, ds \right)_{n \in \mathbb{N}} \quad \text{and} \quad \left( \int_0^1 \dot{t}_n^2 \, ds \right)_{n \in \mathbb{N}} \]
are bounded, then by (H6) and (4.19) if $n \to +\infty$ we obtain
\[ \int_0^1 \langle \nabla U(x_n), \nabla A(x_n) \rangle \dot{y}_n^2 \, ds \geq o(1), \]
\[ \int_0^1 \langle \nabla U(x_n), \nabla B(x_n) \rangle \dot{y}_n \dot{t}_n \, ds = o(1), \]
\[ \int_0^1 \langle \nabla U(x_n), \nabla C(x_n) \rangle \dot{t}_n^2 \, ds \leq o(1), \]
which gives
\[ 0 \geq \int_0^1 H_R^U(x_n)[\dot{x}_n, \dot{x}_n] \, ds + \int_0^1 \psi'_a(U(x_n)) \langle \nabla U(x_n), \nabla U(x_n) \rangle \, ds + o(1). \]
Furthermore, if $n$ is large enough, for all $s \in I$ by (4.1) and (4.19) it follows
\[ \langle \nabla U(x_n(s)), \nabla U(x_n(s)) \rangle \geq 2, \]
while (H5) implies
\[ H_R^U(x_n(s))[\dot{x}_n(s), \dot{x}_n(s)] \geq \lambda \langle \dot{x}_n(s), \dot{x}_n(s) \rangle. \]
Thus, if $n$ is large enough, it is
\[ 0 \geq \lambda \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \, ds + 2 \int_0^1 \psi'_a(U(x_n)) \, ds + o(1). \quad (4.20) \]
On the other hand, by the definition of $J_c$, (H3) and (4.15) imply
\[ m \leq J_c(x_n) \leq \frac{1}{2} \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \, ds + \frac{k(Y,T)}{2} + \int_0^1 \psi'_a(U(x_n)) \, ds \]
(with $k(Y,T)$ defined in Remark 3.2), then (4.20) becomes
\[ 0 \geq \lambda (2m - k(Y,T)) + 2 \int_0^1 (\psi'_a(U(x_n)) - \lambda \psi'_a(U(x_n))) \, ds + o(1) \]
and, by (4.3), this last inequality is in contradiction with (4.11) as $n$ goes to infinity. \qed

**Proof of Theorem 1.7.** Defined
\[ f(x) = \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds \]
for all $c \in \mathbb{R}$ the inequality (3.1) implies that $J^c \subset f^{2c+k(Y,T)}$. 
Since \( \text{cat}_{A^1(\mathcal{M}_0)}(f^{2c+k(Y,T)}) < +\infty \) (for more details, see [7, Lemma 4.1]), there results
\[
\text{cat}_{A^1(\mathcal{M}_0)}(J^c) < +\infty.
\]
(4.21)

Fixed
\[
m > \frac{k(Y,T)}{2},
\]
(4.22)
by Proposition 2.9, (4.21) and \( J \) unbounded from above, Proposition 2.7 implies that there exists \( \tilde{k} \in \mathbb{N}, \tilde{k} = \tilde{k}(m) \), such that
\[
\inf_{A \in \Gamma_{\tilde{k}}} \sup_{x \in A} J(x) \geq m,
\]
(4.23)
where \( \Gamma_{\tilde{k}} \) is defined in (2.5).
On the other hand, for all \( \varepsilon \in ]0,1] \) by (4.4) and (4.5) it is
\[
J(x) \leq J_\varepsilon(x) \leq J_1(x) \quad \text{for all } x \in A^1(\mathcal{M}_0);
\]

hence, if we assume
\[
c_{\varepsilon,k} = \inf_{A \in \Gamma_{\tilde{k}}} \sup_{x \in A} J_\varepsilon(x)
\]
for any \( \varepsilon > 0, k \in \mathbb{N}, \) (4.23) implies that
\[
m \leq c_{\varepsilon,\tilde{k}} \leq c_{1,\tilde{k}}.
\]
(4.24)
Furthermore, by Proposition 2.9 there exists a compact set \( \tilde{K} \) in \( A^1(\mathcal{M}_0) \) such that
\[
\text{cat}_{A^1(\mathcal{M}_0)}(\tilde{K}) \geq \tilde{k},
\]
so (4.24) implies
\[
m \leq c_{\varepsilon,\tilde{k}} \leq M \quad \text{where } M = \max_{x \in \tilde{K}} J_1(x), \varepsilon \in ]0,1].
\]
(4.25)

By Lemma 4.1 and Theorem 2.5 it follows that \( c_{\varepsilon,\tilde{k}} \) is a critical level of \( J_\varepsilon \); moreover, Proposition 4.2 and (4.22), (4.25) imply that \( c_{\varepsilon,\tilde{k}} \) is a critical level of \( J \), too, if \( \varepsilon \leq \varepsilon_0 \) for a certain \( \varepsilon_0 \in ]0,1] \).
Let us point out that any corresponding critical point \( x_{\tilde{k}} \) of \( J \) is not trivial as a consequence of (4.22), (4.25) and Corollary 3.5.

Thus, constructing two sequences \((m_i)_{i \in \mathbb{N}}\) and \((M_i)_{i \in \mathbb{N}},\) where
\[
m_0 = m, \quad M_0 = M, \quad m_i < M_i < m_{i+1} \quad \text{for all } i \in \mathbb{N},
\]
such that at any couple \( m_i, M_i \) the previous arguments apply, we can find a sequence of critical points of \( J \) whose critical levels diverge positively; whence, by Proposition 2.1, infinitely many non-trivial \((Y,T)\)-periodic trajectories exist on \( \mathcal{M}. \)

\begin{remark}
Let \( \mathcal{M}_0 \) be complete and such that \((H_2)-(H_6)\) are satisfied. Arguing as in the compact case (see Remark 3.7), if \((H_1')\) does not hold but \( \mathcal{M}_0 \) is complete and not simply connected, then at least one non-trivial \((Y,T)\)-periodic trajectory exists on \( \mathcal{M}. \)
\end{remark}
5. Examples

5.1. Killing vector fields with non-constant causal character

Let $\mathcal{M}_0$ be a compact Riemannian manifold (or a manifold satisfying the hypotheses $(H'_1)$ and $(H_5)$ of Theorem 1.7) and choose any differentiable mapping $\theta: \mathcal{M}_0 \to \mathbb{R}$ whose range contains the interval $[0, \pi]$, i.e., $[0, \pi] \subseteq \theta(\mathcal{M}_0)$. Put

$$A(x) \equiv C(x) \equiv \sin(\theta(x)), \quad B(x) \equiv \varphi(\theta(x)),$$

where $\varphi: \mathbb{R} \to \mathbb{R}$ is a function. Notice that the Killing vector fields $\partial_y, \partial_t$ do not have a constant causal character on all $\mathcal{M}$. If $m := \inf(\varphi) > 0$ then, clearly, there exists $\nu > 0$ such that $L(x) \geq \nu$ for all $x \in A^1(\mathcal{M}_0)$. So, by Remark 1.9, Theorem 1.6 is applicable (recall also that if $\mathcal{M}_0$ is just complete, the hypothesis $(H.4)$ of Theorem 1.7 will also hold).

5.2. Warped and static cases

Consider a Gödel type manifold such that all the coefficients $A, B, C$ are equal up to a multiplicative constant. That is, there exist a $C^1$ real function $\beta$ on $\mathcal{M}_0$ and some real constants $\alpha_1, \alpha_2, \alpha_3$, such that the coefficients $A, B, C$ satisfy

$$A(x) = \alpha_1 \beta(x), \quad B(x) = \alpha_2 \beta(x), \quad C(x) = \alpha_3 \beta(x)$$

and $\alpha_1^2 + \alpha_2 \alpha_3 > 0$, $\beta(x) \neq 0$ for all $x \in \mathcal{M}_0$ ($\beta$ will be assumed positive, without loss of generality). This manifold is a warped space–time (see, for example, [16, Chapter 7]), with base $\mathcal{M}_0$, warping function $\sqrt{\beta}$ and fiber $(\mathbb{R}^2, g_F \equiv \alpha_1 \, dy^2 + 2 \alpha_2 \, dy \, dt - \alpha_3 \, dt^2)$. Recall that the fiber is isometric to $\mathbb{L}^2$ (in fact, no loss of generality would occur if $\alpha_1 = \alpha_3 = 1$, $\alpha_2 = 0$ are assumed). The projection of any geodesic on the $(y, t)$-plane is a reparametrization of a straight line (it can be checked directly from the expressions for $\phi_y$ and $\phi_t$, see below). For any $(Y, T)$-periodic trajectory, the sign of $g_F((Y, T), (Y, T)) = Y^2 \alpha_1 + 2YT \alpha_2 - T^2 \alpha_3 \equiv K(Y, T)$ determines if this straight line in the fiber is timelike, spacelike or lightlike.

Recall that, now, it is

$$\mathcal{L}(x) = \frac{1}{x_2^2 + x_1 x_3} \left( \int_0^1 \frac{1}{\beta(x)} \, ds \right)^2 > 0$$

for all $x \in A^1(\mathcal{M}_0)$. So, the condition $(H_2)$ always holds. Moreover, if $\varepsilon < \beta < M$, for some $\varepsilon, M > 0$, then $(H_3)$ and $(H_4)$ also hold. In particular, if $\mathcal{M}_0$ is compact then infinitely many $(Y, T)$-periodic trajectories exist for any $(Y, T) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Notice, also, that an existence result for timelike geodesic is obtained whenever $YT \to -\infty$.

In this case, the variational setting is similar to that one which is found looking for periodic trajectories in a static Lorentzian manifold. In fact, by Proposition 2.1, the functional $J$ on $A^1(\mathcal{M}_0)$ becomes

$$J(x) = \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds + \frac{K(Y, T)}{2} \left( \int_0^1 \frac{1}{\beta(x)} \, ds \right)^{-1}.$$

(5.1)
So, according to the choice of the periods \((Y, T)\), different situations occur: if \(K(Y, T) = 0\), then \(J\) is just the energy functional of the closed geodesics in the Riemannian manifold \(\mathcal{M}_0\); if \(K(Y, T) < 0\), then \(J\) is essentially equal to the functional introduced in the study of geodesics in static Lorentzian manifolds (cf. [4]); if \(K(Y, T) > 0\), \(J\) is positive hence the study of its critical points is easier.

From a more general viewpoint, consider any complete semi-Riemannian manifold \((F, g_F)\), a positive function \(\beta\) on \(\mathcal{M}_0\) and the warped product \((\mathcal{M}_0 \times F, g)\) where, under natural identifications, \(g = \langle \cdot, \cdot \rangle_R + \beta g_F\). Let \(\pi_0 : \mathcal{M}_0 \times F \rightarrow \mathcal{M}_0\), \(\pi_F : \mathcal{M}_0 \times F \rightarrow F\) be the natural projections. It is well-known that, for any geodesic \(\gamma\) of the warped product, the projection \(\pi_F \circ \gamma\) is a pregeodesic (i.e., a geodesic up to a reparametrization) of \((F, g_F)\). Thus, choose a geodesic \(\hat{\gamma} \equiv \hat{\gamma}(t)\) of \((F, g_F)\). We will say that the geodesic \(\gamma\) of \((\mathcal{M}_0 \times F, g)\) is \(T\)-periodic along \(\hat{\gamma}\) if \(\pi_F \circ \gamma(s) = \hat{\gamma}(t(s))\) for a reparametrization \(t = t(s)\) such that

\[
\begin{align*}
    t(0) &= 0, & t(1) &= T, & \dot{t}(0) &= \dot{t}(1).
\end{align*}
\]

If \(\hat{\gamma}\) is lightlike then \(\pi_0 \circ \gamma\) is a geodesic in \(\mathcal{M}_0\) and the reparametrization \(t(s)\) must be any affine function. As a consequence, fixed a lightlike \(\hat{\gamma}\) and choosing \(\pi_0 \circ \gamma\) as a closed geodesic, then a \(T\)-periodic trajectory along \(\hat{\gamma}\) is obtained for every \(T \in \mathbb{R}\) (giving rounds to \(\pi_0 \circ \gamma\), infinitely many such trajectories are also found, which are geometrically distinct in the space–time, if \(T \neq 0\)). When \(\hat{\gamma}\) is timelike (respectively, spacelike) then the problem is equivalent to study the functional (5.1) for \(K(Y, T) < 0\) (respectively, \(K(Y, T) > 0\)). In particular, results analogous to those of Theorems 1.6 and 1.7 (with \(A \equiv C \equiv 0, B \equiv \beta\)) can be stated in this case.

Static space–times can be regarded as warped space–times with \((F, g_F) = (\mathbb{R}, -dr^2)\) and the standard choice of \(\hat{\gamma}\) is \(\hat{\gamma}(t) = t\). In particular, we obtain the existence of infinitely many \(T\)-periodic trajectories either when \(\mathcal{M}_0\) is compact or when it is not contractible, with \(0 < \inf \beta < \sup \beta < +\infty\) and a function \(U\) as in Theorem 1.7 exists.

Nevertheless, this result can be strenghtened if \(T\)-periodic trajectories in a static space–time \((\mathcal{M}_0 \times \mathbb{R}, g = \langle \cdot, \cdot \rangle_R - \beta \, dt^2)\) are regarded as \((0, T)\)-periodic trajectories in a Gödel type space–time \(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2\) with \(A \equiv 1, B \equiv 0, C \equiv \beta\). In this case the conditions \((H'_1)\) and \((H'_2)\) are automatically satisfied if \(\beta = \beta(x)\) is bounded and far from zero. So, when \(\mathcal{M}_0\) is not compact, the only relevant hypotheses of Theorem 1.7 are \((H'_1)\), \((H_5)\) and the last one of \((H_6)\). More precisely:

**Corollary 5.1.** Let \((\mathcal{M}_0 \times \mathbb{R}, g, g = \langle \cdot, \cdot \rangle_R - \beta \, dt^2)\) be a static space–time such that \(0 < \inf \beta < \sup \beta < +\infty\) and

\((H'_1)\) \((\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)\) is complete, not contractible in itself and its fundamental group \(\pi_1(\mathcal{M}_0)\) is finite or it has infinitely many conjugacy classes;

\((H_5)\) there exist \(x_0 \in \mathcal{M}_0\), \(U \in C^2(\mathcal{M}_0, \mathbb{R}_+)\) and some positive constants \(R, \rho, \lambda, \) such that

\[
x \in \mathcal{M}_0, \quad d(x, x_0) \geq R \Rightarrow H^U_R(x)[\xi, \xi] \geq \lambda \langle \xi, \xi \rangle_R \quad \text{for all } \xi \in T_x \mathcal{M}_0;
\]

\((H'_6)\) taken \(x_0\) and \(U\) as in \((H_5)\), there results

\[
\limsup_{d(x, x_0) \to +\infty} \langle \nabla U(x), \nabla \beta(x) \rangle_R \leq 0.
\]
Then, for any $T \in \mathbb{R}$ there exist infinitely many non-trivial spacelike distinct $T$-periodic trajectories in $\mathcal{M}_0 \times \mathbb{R}$ whose energies diverge positively.

Clearly, the hypotheses (H$_5$), (H’$_6$) can be dropped in the compact case.

5.3. Classical Gödel space–time (CGS)

Now, our goal is to study periodic trajectories in CGS. We must emphasize that previous results cannot be applied to this space–time, especially because their coefficients $A, B, C$ do violate some of the hypotheses of our theorems and, in particular, among them, the essential condition (H$_2$). The following facts were shown in [8], where Gödel type space–times were studied in order to determine their geodesic connectedness: (a) even when violations of the hypotheses of variational theorems less severe than those in CGS occur, variational methods may be non-applicable and geodesically disconnected–space–time may appear (in fact, an explicit counterexample was given which satisfied (H$_2$) but not (H$_3$); none of these conditions are satisfied by CGS); (b) nevertheless, CGS is geodesically connected, because of the symmetries and special characteristics of the metric.

Our study of periodic trajectories in CGS is motivated not only by the importance of this classical space–time but also because it provides an example of the role of some of our hypotheses.

We will use the study of geodesics in CGS carried out in [8], and summarize the consequences for $(Y, T)$-periodic trajectories.

First of all, recall that CGS is $\mathbb{R}^4$ equipped with a Gödel type metric where $\mathcal{M}_0 = \mathbb{R}^2$ is the standard Euclidean space and

$$
\langle \cdot, \cdot \rangle = dx^2 + dx^2 - \frac{1}{2}e^{-2x} dy^2 - 2e^{-x} dy dt - dt^2,
$$

with $x = (x_1, x_2) \in \mathbb{R}^2$ and $x > 0$ given constant ($\omega = x/\sqrt{2}$ is the magnitude of the vorticity of the flow). This metric is a semi-Riemannian product, and we will drop the irrelevant factor $(\mathbb{R}, dx^2)$. After a change of coordinates obtained by putting $x = \sqrt{2}x e^{-x}$, CGS can be written as $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ with

$$(\mathcal{M}_0, \langle \cdot, \cdot \rangle)_R = (\mathbb{R}_+, (dx/(2x))^2),$$

$$
\langle \cdot, \cdot \rangle_L = \frac{1}{(2x)^2} dx^2 - \frac{1}{(2x)^2} dy^2 - 2\frac{\sqrt{2}}{2x} dy dt - dt^2, \quad (5.2)
$$

where $(x, y, t) \in \mathbb{R}_+ \times \mathbb{R}^2$.

It is easy to see that none of the conditions (H$_1$)–(H$_4$) of Theorem 1.7 is satisfied by CGS. Geodesics of CGS can be divided into the following three classes:

1. Trivial trajectories with $Y = 0$. Recall that if $Y = 0$ then $K(Y, T)$ is independent of $x$; thus, a trivial trajectory can be found (recall that this is true even if (H$_2$) does
not hold, see Remark 1.5). These \((0, T)\)-periodic trajectories are obtained by putting \((x(s), y(s)) \equiv (x_0, y_0) \) (constant), \(t(s) \equiv sT\).

2. Geodesics whose projection on the \((x, y)\) half-plane \(\mathbb{R}_+ \times \mathbb{R}\) is a straight line or a piece of circumference (this piece of circumference can be extended to a circumference of the plane \(\mathbb{R}^2 \supseteq \mathbb{R}_+ \times \mathbb{R}\)). If this projection is a straight line parallel to the \(y\)-axis, a trivial \((Y, T)\)-periodic trajectory with \(Y \neq 0\) is obtained. Otherwise, these geodesics are not \((Y, T)\)-periodic trajectories, even though they are complete.

3. Geodesics whose projection on the \((x, y)\) half-plane \(\mathbb{R}_+ \times \mathbb{R}\) is a complete circumference. All these geodesics are (non-trivial) \((Y, T)\)-periodic trajectories with \(Y = 0\), and they exist for all the values of \(T \in \mathbb{R}\). In particular, there exist closed geodesics (i.e., \((0, 0)\)-periodic trajectories) such that their projections on \(\mathcal{M}_0\) are not closed geodesics (this happens because of the violation of the hypothesis \((H_2)\), see Remark 2.2).

References

