A Mehler–Heine-type formula for Hermite–Sobolev orthogonal polynomials

Laura Castaño-García\textsuperscript{a}, Juan J. Moreno-Balcázar\textsuperscript{b,c,*}

\textsuperscript{a}Departamento de Matemáticas, I.E.S. Seritium, Jerez de la Frontera, Cádiz, Spain
\textsuperscript{b}Departamento de Estadística y Matemática Aplicada, Universidad de Almería, La Cañada de San Urbano s/n, 04120 Almeria, Spain
\textsuperscript{c}Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, Spain

Received 29 October 2001; received in revised form 10 April 2002

Abstract

We consider a Sobolev inner product such as

\[ (f, g)_S = \int f(x)g(x) \, d\mu_0(x) + \lambda \int f'(x)g'(x) \, d\mu_1(x), \quad \lambda > 0, \tag{1} \]

with \((\mu_0, \mu_1)\) being a symmetrically coherent pair of measures with unbounded support. Denote by \(Q_n\) the orthogonal polynomials with respect to (1) and they are so-called Hermite–Sobolev orthogonal polynomials. We give a Mehler–Heine-type formula for \(Q_n\) when \(\mu_1\) is the measure corresponding to Hermite weight on \(\mathbb{R}\), that is, \(d\mu_1 = e^{-x^2} \, dx\) and as a consequence an asymptotic property of both the zeros and critical points of \(Q_n\) is obtained, illustrated by numerical examples. Some remarks and numerical experiments are carried out for \(d\mu_0 = e^{-x^2} \, dx\). An upper bound for \(|Q_n|\) on \(\mathbb{R}\) is also provided in both cases.

\(\text{MSC: Primary 42C05; Secondary 33C25}\

\Keywords{Sobolev orthogonal polynomials; Asymptotics; Mehler–Heine-type formulas}

\textsuperscript{*} This research was partially supported by Spanish Project of MCYT (BMF 2001-3878-C02-02), Junta de Andalucía (FQM 0229) and European Project INTAS 2000-272.

\textsuperscript{*} Corresponding author. Departamento de Estadística y Matemática Aplicada, Universidad de Almería, La Cañada de San Urbano s/n, 04120 Almeria, Spain Tel.: +34-950-015661; Fax: +34-950-015167.

\textit{E-mail address:} balcazar@ual.es (J.J. Moreno-Balcázar).

0377-0427/02/$ - see front matter \(\copyright\) 2002 Elsevier Science B.V. All rights reserved.

PII: S 0377-0427(02)00552-6
1. Introduction

We consider the Sobolev inner product

\[ (f, g)_S = \int f(x)g(x) \, d\mu_0(x) + \lambda \int f'(x)g'(x) \, d\mu_1(x), \quad \lambda > 0, \tag{2} \]

where \( \mu_i, i = 1, 2 \) are positive Borel measures with support \( I_i \subseteq \mathbb{R} \), respectively. Denote for \( Q_n(x) = 2^n x^n + \cdots \), those polynomials that are orthogonal with respect to (2). The Sobolev orthogonal polynomials were introduced in [4] in connection with the least-squares simultaneous approximation of a function and its derivatives. In the early 1990s, Iserles et al. introduced in [3] the fruitful concept of coherent pair of measures and, for symmetric measures, symmetrically coherent pair. Later, Meijer gave in [6] a complete classification of all coherent pairs and symmetrically coherent pairs. In particular, it was established that at least one of the measures in each coherent pair has to be classic (i.e., Jacobi, Laguerre o Hermite). Thus, if one of the measures corresponds to Hermite weight function, \( e^{-x^2} \, dx \) on \( \mathbb{R} \), there are only two possibilities (see [6]):

(a) Case I. \( d\mu_0 = (x^2 + a^2)e^{-x^2} \, dx, \, d\mu_1 = e^{-x^2} \, dx, \, a \in \mathbb{R} \).

(b) Case II. \( d\mu_0 = e^{-x^2} \, dx, \, d\mu_1 = \frac{e^{-x^2}}{x^2 + a^2} \, dx, \, a \in \mathbb{R} \setminus \{0\} \).

Let \( (\mu_0, \mu_1) \) be a pair of measures of Cases I or II, then \( Q_n \) are so-called Hermite–Sobolev orthogonal polynomials. Analytic properties of these polynomials, such as asymptotics for \( Q_n(x) \) in \( \mathbb{C} \setminus \mathbb{R} \) or Plancherel–Rotach-type asymptotics in \( \mathbb{C} \setminus [-\sqrt{2}, \sqrt{2}] \), have been obtained in [2]. Also, in the aforementioned paper, the accumulation sets of the zeros of \( Q_n \) before and after an appropriate scaling of the plane are obtained.

On the other hand, we think that the Mehler–Heine-type formulas for Sobolev orthogonal polynomials are interesting, both analytically and numerically, since they are the natural way to establish a limit relation between these orthogonal polynomials and the well-known Bessel function \( J_k(x) \) defined as (see e.g. [7, p. 15]):

\[ J_k(x) = \sum_{j=0}^{\infty} \frac{(-1)^j (x/2)^{2j+k}}{j! \Gamma(j+k+1)}. \]

In this sense, a Mehler–Heine-type formula for the so-called non-diagonal Laguerre–Sobolev orthogonal polynomials has been obtained in [5].

Here, we look for a Mehler–Heine-type formula for the orthogonal polynomials \( Q_n \). Thus, in the Case I, we give a limit relation between the appropriately scaled Hermite–Sobolev polynomials \( Q_n \) and the elementary trigonometric functions \( \sin(x) \) and \( \cos(x) \) (these function can be expressed, see [7, f.(1.71.2), p. 15], in terms of \( J_{1/2}(x) \) and \( J_{-1/2}(x) \), respectively). This result allows us to know the asymptotic behavior of \( Q_n \) (and of its derivatives) in the neighborhood of 0. As a consequence of this, we obtain an asymptotic property of the zeros and critical points of \( Q_n \), supported by some numerical examples in Section 4. Also, we discuss some problems of Case II and, in Section 4 a conjecture about the small zeros of \( Q_n \) is done in this case, supported by numerical examples. Finally, in Section 3, we give an upper bound for \( |Q_n| \) on \( \mathbb{R} \), analogous to that for Hermite polynomials.
The notation that we use in this work: \( H_n \) denotes the Hermite polynomial orthogonal with respect to the inner product

\[
(f,g) := \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2} \, dx,
\]

with the normalization \( H_n(x) = 2^n x^n + \cdots \) and \( Q_n \) are chosen with the same leading coefficient. Finally, we also denote

\[
\varphi(x) = x + \sqrt{x^2 - 1}, \quad k_n = (H_n, H_n), \quad \tilde{k}_n = (Q_n, Q_n).
\]

On the other hand, in order to obtain Theorem 1, we use the well-known Mehler–Heine-type formula for Hermite polynomials, that is, for \( j \) fixed (see, for example, [1, p. 346] or, [7, p. 193] using the relation between Hermite and Laguerre polynomials) we have:

\[
\lim_{n \to \infty} \frac{(-1)^n \sqrt{n + j} Q_{2n}(x/(2\sqrt{n} + j))}{2^{2n}n!} = \frac{1}{\sqrt{\pi}} \cos(x),
\]

\[
\lim_{n \to \infty} \frac{(-1)^n H_{2n+1}(x/(2\sqrt{n} + j))}{2^{2n+1}n!} = \frac{1}{\sqrt{\pi}} \sin(x),
\]

both uniformly on compact subsets of \( \mathbb{C} \).

2. Mehler–Heine-type formula

First, we consider the Case I and so we have the inner product

\[
(f,g)_{S} = \int_{-\infty}^{\infty} f(x)g(x)(x^2 + a^2)e^{-x^2} \, dx + \lambda \int_{-\infty}^{\infty} f'(x)g'(x)e^{-x^2} \, dx. \quad \lambda > 0, \quad a \in \mathbb{R}.
\]

In this situation we get:

**Theorem 1.** Let \( \Theta(\lambda) = \varphi(1 + 2\lambda)/(\varphi(1 + 2\lambda) - 1) \). The following Mehler–Heine-type formulas for the polynomials \( Q_n(x) = 2^n x^n + \cdots \) orthogonal with respect to (5) hold:

\[
\lim_{n \to \infty} \frac{(-1)^n \sqrt{n} Q_{2n}(x/(2\sqrt{n}))}{2^{2n}n!} = \Theta(\lambda) \frac{\cos(x)}{\sqrt{\pi}},
\]

\[
\lim_{n \to \infty} \frac{(-1)^n Q_{2n+1}(x/(2\sqrt{n}))}{2^{2n+1}n!} = \Theta(\lambda) \frac{\sin(x)}{\sqrt{\pi}},
\]

both uniformly on compact subsets of \( \mathbb{C} \).

**Proof.** The polynomials \( H_n \) and \( Q_n \) satisfy the relation (see, for example, [2, Lemma 2.1]):

\[
H_n = Q_n + a_{n-2} Q_{n-2}, \quad n \geq 0,
\]
where \(a_{n-2} = k_n/(4k_{n-2})\), \(n \geq 2\), and \(a_{-1} = a_{-2} = 0\). Applying (6) in a recursive way, we obtain

\[
Q_m(x) = \sum_{i=0}^{[m/2]} (-1)^i b_i^{(m)} H_{m-2i}(x), \quad m \geq 0,
\]

where

\[
b_i^{(m)} = \prod_{j=1}^{i} a_{m-2j} \quad \text{for } i \geq 1 \text{ and } b_0^{(m)} = 1
\]

and \([m]\) means the greatest integer less than or equal to \(m\). First we consider \(m\) as even; that is, \(m = 2n\). Then, scaling the variable \(x\) in (7) we can write

\[
\frac{(-1)^n \sqrt{n} Q_{2n}(x/(2\sqrt{n}))}{2^{2n}n!} = \sum_{i=0}^{n} \frac{b_{i}^{(2n)} \sqrt{n}}{2^{2i} \prod_{j=0}^{i-1} (n-j)} \frac{(-1)^{n-i} H_{2n-2i}(x/(2\sqrt{n}))}{2^{2n-2i}(n-i)!} := \sum_{i=0}^{n} g_{n,i}(x/(2\sqrt{n})),
\]

where

\[
g_{n,i}(x/(2\sqrt{n})) = (-1)^{n-i} c_i^{(2n)} \sqrt{n} H_{2n-2i}(x/(2\sqrt{n})) \frac{2^{2n-2i}(n-i)!}{2^{2i} \prod_{j=0}^{i-1} (n-j)},
\]

with

\[
c_i^{(2n)} = \frac{b_i^{(2n)}}{2^{2i} \prod_{j=0}^{i-1} (n-j)},
\]

and the assumption \(\prod_{j=0}^{i-1} (n-j) = 1\). On the other hand, in [2, Lemma 2.2] it was established that the sequence \(\{a_n/(2(n+2))\}\) is uniformly bounded by \(r := 1/(1+2\lambda) < 1\) and

\[
\lim_{n \to \infty} \frac{a_n}{2(n+2)} = \frac{1}{\varphi(1+2\lambda)}.
\]

Thus, using the bound for \(\{a_n/(2(n+2))\}\), we obtain, for \(i = 0, \ldots, n\), that \(|c_i^{(2n)}| \leq r^i\). Now, if \(x\) belongs to a compact subset of \(\mathbb{C}\), using (3), we have for \(n\) large enough and \(0 \leq i \leq n\),

\[
\left| \frac{\sqrt{n} H_{2n-2i}(x/(2\sqrt{n}))}{2^{2n-2i}(n-i)!} \right| \leq \mathcal{M},
\]

where \(\mathcal{M}\) is a constant and, therefore,

\[
|g_{n,i}(x/(2\sqrt{n}))| \leq \mathcal{M} r^i.
\]

Then, taking into account (3) and (9), we have, for every fixed non-negative integer \(i\),

\[
\lim_{n \to \infty} g_{n,i}(x/(2\sqrt{n})) = \frac{\cos(x)}{\sqrt{\pi}} \left( \frac{1}{\varphi(1+2\lambda)} \right)^i.
\]
Finally, from (10) to (11) and using Lebesgue’s dominated convergence theorem, we have
\[
\lim_{n \to \infty} \frac{(-1)^n \sqrt{n} Q_{2n}(x/(2\sqrt{n}))}{2^{2n} n!} = \frac{\cos(x)}{\sqrt{\pi}} \sum_{i=0}^{\infty} \left( \frac{1}{\varphi(1+2\lambda)} \right)^i \frac{\varphi(1+2\lambda)}{\sqrt{\pi}} \frac{\varphi(1+2\lambda)-1}{2^{2n} n!},
\]

If \( m \) is odd using relation (4), we can proceed as the even case. □

From this theorem we can obtain additional information about zeros of \( Q_n \). We know that these zeros accumulate in \( \mathbb{R} \) when \( n \to \infty \). Now, we have

**Corollary 1.** Let \( x_{n,i} \) be the zeros of \( Q_n \). Then
\[
\lim_{n \to \infty} 2\sqrt{n} x_{2n,i} = (2i - 1) \frac{\pi}{2}, \quad \lim_{n \to \infty} 2\sqrt{n} x_{2n+1,i} = i\pi, \quad i \in \mathbb{Z}.
\]

**Proof.** Use Theorem 1 and the Theorem of Hurwitz (see, for example, [7, Theorem 1.9.3, p. 22]). □

Since we have uniform convergence in the result obtained in Theorem 1, we can get asymptotic results for the derivatives of \( Q_n \). In particular, we have
\[
\lim_{n \to \infty} \frac{(-1)^n Q'_{2n}(x/(2\sqrt{n}))}{2^{2n+1} n!} = -\Theta(\lambda) \frac{\sin(x)}{\sqrt{\pi}},
\]
\[
\lim_{n \to \infty} \frac{(-1)^n Q'_{2n+1}(x/(2\sqrt{n}))}{2^{2n+2} \sqrt{n} n!} = \Theta(\lambda) \frac{\cos(x)}{\sqrt{\pi}},
\]
both uniformly on compact subsets of \( \mathbb{C} \). Thus, we have asymptotic information about the critical points \( y_{n,i} \) of \( Q_n \), that is,
\[
\lim_{n \to \infty} 2\sqrt{n} y_{2n,i} = i\pi, \quad \lim_{n \to \infty} 2\sqrt{n} y_{2n+1,i} = (2i - 1) \frac{\pi}{2}, \quad i \in \mathbb{Z}.
\]

Now, we turn to Case II, that is, we consider the Sobolev inner product
\[
(f,g)_S = \int f(x)g(x)e^{-x^2} \, dx + \lambda \int f'(x)g'(x) \frac{e^{-x^2}}{x^2 + a^2} \, dx, \quad \lambda > 0, \quad a \in \mathbb{R} \setminus \{0\}
\]
and let \( Q_n \) be the orthogonal polynomials with respect to (12). We get the following result:

**Proposition 1.** It holds,
\[
\lim_{n \to \infty} \frac{(-1)^n \sqrt{n} Q_{2n}(x/(2\sqrt{n}))}{2^{2n} n!} = \lim_{n \to \infty} \frac{(-1)^n Q_{2n+1}(x/(2\sqrt{n}))}{2^{2n+1} n!} = 0,
\]
uniformly on compact subsets of \( \mathbb{C} \).
Proof. We know a relation between these Sobolev orthogonal polynomials and Hermite polynomials (see [2, Lemma 2.5])
\[ R_{n+2}(x) = H_{n+2}(x) + \sigma_n \frac{n+2}{n} H_n(x) = Q_{n+2}(x) + \tilde{a}_n Q_n(x), \quad n \geq 1, \] (13)
where \( \sigma_n \) are non-zero constants. We also know (see [2, Lemma 2.4])
\[ \lim_{n \to \infty} \frac{\sigma_n}{2n} = 1. \]
Thus, using (3)–(4), we can establish that
\[ \lim_{n \to \infty} \frac{(-1)^n \sqrt{n} R_{2n}(x/(2\sqrt{n}))}{2^{2n} n!} = \lim_{n \to \infty} \frac{(-1)^n R_{2n+1}(x/(2\sqrt{n}))}{2^{2n+1} n!} = 0, \] (14)
uniformly on compact subsets of \( \mathbb{C} \).
Therefore, taking into account Lemma 2.6 in [2], that is, the sequence \( \{\tilde{a}_n/(2(n+2))\} \) is uniformly bounded by \((1 + a^2)/(1 + a^2 + 2\lambda)\) and
\[ \lim_{n \to \infty} \frac{\tilde{a}_n}{2(n+2)} = \frac{1}{\varphi(1+2\lambda)}, \]
and using (14), it only remains to proceed as in Theorem 1 in order to obtain the result. \( \Box \)

Remark. The result of Proposition 1 corresponds well with Theorem 2.7 in [2] where it was established that
\[ \lim_{n \to \infty} \frac{Q_n(x)}{H_n(x)} = 0, \]
uniformly on compact subsets of \( \mathbb{C} \setminus \mathbb{R} \). We think that to improve the result of Proposition 1, it would be necessary to obtain an adequate Mehler–Heine-type formula for the polynomials \( R_n(x) = 2^n x^n + \cdots \) which are in some sense very close to the orthogonal polynomials associated with the measure \( d\mu_1 = (e^{-x^2}/(x^2 + a^2)) \, dx \). Obviously, it is not possible to obtain any asymptotic information about the zeros of \( Q_n \) from Proposition 1. In Section 4 a conjecture about the zeros of \( Q_n \) is done, supported by numerical experiments.

3. Upper bound for \( |Q_n| \)

We give an upper bound for \( |Q_n| \) on \( \mathbb{R} \), analogous to that for the Hermite polynomials.

Proposition 2. It holds,
(a) In Case I,
\[ |Q_n(x)| < ke^{x^2/2} 2^{n/2} n! \left( \frac{1 - r^{[n/2]+1}}{1 - r} \right), \quad x \in \mathbb{R}, \]
(b) In Case II,
\[ |Q_n(x)| < ke^{x^2/2} 2^{n/2} n! \left( 1 + \sqrt{5} + \frac{\sqrt{5}}{3} a^2 \right) \left( \frac{1 - s^{[n/2]+1}}{1 - s} \right), \quad x \in \mathbb{R}, \quad a \neq 0, \]
where \( k \approx 1.086435, \ r = 1/(1 + 2\lambda), \ s = (1 + a^2)/(1 + a^2 + 2\lambda), \) \( n!! \) denotes the double-factorial of \( n, \) that is, \( n!! = \prod_{j=0}^{\lfloor n/2 \rfloor - 1} (n - 2j). \)

**Proof.** (a) From (7), we have

\[
\frac{|Q_n(x)|}{2^{n/2} \sqrt{n!}} \leq \sum_{i=0}^{\lfloor n/2 \rfloor} b_i^{(n)} 2^i \sqrt{\frac{(n-2i)!}{n!}} \frac{|H_{n-2i}(x)|}{2^{(n/2)-i} \sqrt{(n-2i)!}}.
\]

Now, using the relation \(|H_n(x)|/(2^{n/2} \sqrt{n!}) < ke^{x^2/2}\) (see [1, f.(22.14.17) p. 346]) we get

\[
\frac{|Q_n(x)|}{2^{n/2} \sqrt{n!}} < ke^{x^2/2} \sum_{i=0}^{\lfloor n/2 \rfloor} b_i^{(n)} 2^i \prod_{j=0}^{i-1} (n-2j) \left( \frac{i-1}{\prod_{j=0}^{i-1} (n-j)} \right) \frac{|H_{n-2i}(x)|}{\prod_{j=0}^{i-1} (n-j)}.
\]

Then, using (see [2, Lemma 2.2])

\[
\frac{b_i^{(n)}}{2^i \prod_{j=0}^{i-1} (n-2j)} = \prod_{j=1}^{i} \frac{a_{n-2j}}{2(n-2j+2)} < \left( \frac{1}{1 + 2\lambda} \right)^i = r^i
\]

we get

\[
\frac{|Q_n(x)|}{2^{n/2} \sqrt{n!}} < ke^{x^2/2} \sqrt{\frac{n!!}{(n-1)!!} \frac{1 - r^{\lfloor n/2 \rfloor + 1}}{1 - r}}.
\]

It only remains to use \( \sqrt{n!!/(n-1)!!} \sqrt{n!} = n!! \)

(b) Indeed, relation (13) can be rewritten as

\[
R_n(x) = Q_n + \tilde{a}_{n-2} Q_{n-2}, \quad n \geq 0,
\]

(15)
where $\tilde{a}_{-2} = \tilde{a}_{-1} = \tilde{a}_0 = 0$ being $R_i(x) = H_i(x)$, $i = 0, 1, 2$, and $\tilde{a}_n = \sigma_n((n+2)/n)(\tilde{k}_n/k_n)$, $n \geq 1$ (see [2, Lemma 2.5]). Thus, applying (15) in a recursive way we obtain

$$Q_n(x) = \sum_{i=0}^{[n/2]} (-1)^i \tilde{b}_i^{(m)} R_{m-2i}, \quad m \geq 0,$$

where $\tilde{b}_i^{(m)} = \prod_{j=1}^{i} \tilde{a}_{m-2j}$, $i \geq 1$ and $\tilde{b}_0^{(m)} = 1$.

Therefore, as in (a), we have

$$|Q_n(x)| \leq \sum_{i=0}^{[n/2]} |\tilde{b}_i^{(n)}| |R_{n-2i}(x)| \frac{2^{i(n/2)-i} \sqrt{(n-2i)!}}{2^{n/2} \sqrt{n!}}.$$ 

On the other hand,

$$R_n(x) = H_n(x) + \sigma_{n-2} \frac{n}{n-2} H_{n-2}, \quad n \geq 3,$$

where (see [2, f.(2.12)])

$$\sigma_n = \frac{k_{n+1}'}{4k_{n-1}} > 0, \quad n \geq 1, \quad \text{with} \quad k_n' = \int_{-\infty}^{\infty} T_n^2(x) \frac{e^{-x^2}}{x^2 + a^2} \, dx,$$

being $T_n$ the orthogonal polynomials with respect to the inner product $(f, g) = \int_{-\infty}^{\infty} f(x)g(x)(e^{-x^2}/(x^2 + a^2)) \, dx$ and with the same leading coefficient as $H_n$.

We know (see [2, f.(2.19)]) that

$$\sigma_{n+1} + \frac{4n(n-1)}{\sigma_{n-1}} = 4(n + a^2) + 2, \quad n \geq 2,$$

then $\sigma_{n+1} < 4(n + a^2) + 2$ for $n \geq 2$. Thus, for $n \geq 5$,

$$\frac{|R_n(x)|}{2^{n/2} \sqrt{n!}} \leq \frac{|H_n(x)|}{2^{n/2} \sqrt{n!}} + \frac{\sigma_{n-2}}{2(n-2)} \sqrt{\frac{n}{n-1} \frac{|H_{n-2}(x)|}{2^{n/2-1} \sqrt{(n-2)!}}} < ke^{x^2/2} \left(1 + \frac{\sigma_{n-2}}{2(n-2)} \sqrt{\frac{n}{n-1}}\right) \leq ke^{x^2/2} \left(1 + \sqrt{5} + \frac{\sqrt{5}}{3}a^2\right).$$

Since $R_4(x) = H_4(x) + 2\sigma_2 H_2(x)$ and $R_5(x) = H_5(x) + 3\sigma_1 H_1(x)$, and straightforward computations show that $\sigma_1 < 2$ and $\sigma_2 < 4$, we have that, for $n \geq 0$, it holds

$$\frac{|R_n(x)|}{2^{n/2} \sqrt{n!}} < ke^{x^2/2} \left(1 + \sqrt{5} + \frac{\sqrt{5}}{3}a^2\right).$$

Now, in order to obtain the result, we can proceed exactly as in (a) taking into account that (see [2, Lemma 2.6]):

$$\frac{\tilde{b}_i^{(n)}}{2^i \prod_{j=0}^{i-1} (n-2j)} = \prod_{j=1}^{i} \frac{\tilde{a}_{n-2j}}{2(n-2j+2)} < \left(\frac{1 + a^2}{1 + a^2 + 2\lambda}\right)^i = s^i. \quad \Box$$
4. Numerical examples and remarks

We illustrate Corollary 1 with two numerical examples. We compare the limit values \((2i - 1)\pi/2\) and \(i\pi\), where \(i = 1, 2, 3, 4\), with the first four positive real zeros of \(Q_{2n}\) and \(Q_{2n+1}\) rescaled by the factor \(2\sqrt{n}\), respectively, for \(n = 25, 50, 75, 100\). Note that \(Q_n\) are symmetric, that is, \(Q_n(-x) = (-1)^i Q_n(x)\). In order to obtain the numerical results, we use relation (6) and the recurrence relation for the coefficients \(a_n\) in (6) given by (see [2, f.(2.3)]):

\[
a_n = \frac{4(n + 1)(n + 2)}{2(2\lambda + 1)n + 1 + 2a^2) - a_{n-2}}, \quad n \geq 2
\]

with

\[
a_0 = \frac{4}{1 + 2a^2}, \quad a_1 = \frac{12}{3 + 2a^2 + 4\lambda}.
\]

First example: \(a = 0\) and \(\lambda = 1\).

<table>
<thead>
<tr>
<th>(2\sqrt{n}x_{2n,i})</th>
<th>(i = 1)</th>
<th>(i = 2)</th>
<th>(i = 3)</th>
<th>(i = 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n = 25)</td>
<td>1.5698692268</td>
<td>4.7111057199</td>
<td>7.8568557403</td>
<td>11.0101937059</td>
</tr>
<tr>
<td>(n = 50)</td>
<td>1.5702245533</td>
<td>4.7110548442</td>
<td>7.8530299109</td>
<td>10.9969170269</td>
</tr>
<tr>
<td>(n = 75)</td>
<td>1.5703920503</td>
<td>4.7113465231</td>
<td>7.8528123556</td>
<td>10.9951312645</td>
</tr>
<tr>
<td>(n = 100)</td>
<td>1.5704845829</td>
<td>4.7115498501</td>
<td>7.8529034979</td>
<td>10.9947380373</td>
</tr>
</tbody>
</table>

Limit value \((2i - 1)\pi/2\) 1.5707963268 4.7123889804 7.8539816340 10.9955742876

Second example: \(a = 16.25\) and \(\lambda = 7.2\).

<table>
<thead>
<tr>
<th>(2\sqrt{n}x_{2n+1,i})</th>
<th>(i = 1)</th>
<th>(i = 2)</th>
<th>(i = 3)</th>
<th>(i = 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n = 25)</td>
<td>3.1091797287</td>
<td>6.2212173042</td>
<td>9.3390062419</td>
<td>12.4655126418</td>
</tr>
<tr>
<td>(n = 50)</td>
<td>3.1249658267</td>
<td>6.2506757089</td>
<td>9.3778760451</td>
<td>12.5073179403</td>
</tr>
<tr>
<td>(n = 75)</td>
<td>3.1304135054</td>
<td>6.2611622556</td>
<td>9.3925818469</td>
<td>12.5250900005</td>
</tr>
<tr>
<td>(n = 100)</td>
<td>3.1331726373</td>
<td>6.2665351221</td>
<td>9.4002774525</td>
<td>12.5345899285</td>
</tr>
</tbody>
</table>

Limit value \(i\pi\) 3.1415926536 6.2831853072 9.4247779608 12.5663706144

<table>
<thead>
<tr>
<th>(2\sqrt{n}x_{2n,i})</th>
<th>(i = 1)</th>
<th>(i = 2)</th>
<th>(i = 3)</th>
<th>(i = 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n = 25)</td>
<td>1.5638248280</td>
<td>4.6929891583</td>
<td>7.8267166702</td>
<td>10.968140574</td>
</tr>
<tr>
<td>(n = 50)</td>
<td>1.5673453033</td>
<td>4.7024188792</td>
<td>7.838425814</td>
<td>10.976782318</td>
</tr>
<tr>
<td>(n = 75)</td>
<td>1.5685056399</td>
<td>4.7056877955</td>
<td>7.8433828210</td>
<td>10.981334737</td>
</tr>
<tr>
<td>(n = 100)</td>
<td>1.5690824506</td>
<td>4.7073436604</td>
<td>7.8458937829</td>
<td>10.9849260131</td>
</tr>
</tbody>
</table>

Limit value \((2i - 1)\pi/2\) 1.5707963268 4.7123889804 7.8539816340 10.9955742876
\[2\sqrt{n}x_{2n+1,i} \quad \begin{array}{cccc}
i = 1 & i = 2 & i = 3 & i = 4 \\n = 25 & 3.0974699203 & 6.1978284839 & 9.3039995611 & 12.4189787398 \\
 = 50 & 3.1192959362 & 6.2393393699 & 9.3608801311 & 12.4846727358 \\
 = 75 & 3.1266791037 & 6.2536944055 & 9.3813825738 & 12.510812193 \\
 = 100 & 3.1303897036 & 6.2609696616 & 9.3919302800 & 12.5234622650 \\
\end{array}\]

Limit value \(i\pi\) \[3.1415926536 \quad 6.2831853072 \quad 9.4247779608 \quad 12.5663706144\]

We also give a numerical example about critical points of \(Q_n\). For example, we take \(a = 16.25\) and \(\lambda = 7.2\).

\[2\sqrt{n}y_{2n,i} \quad \begin{array}{cccc}
i = 1 & i = 2 & i = 3 & i = 4 \\n = 25 & 3.1594830497 & 6.3221561784 & 9.4912516037 & 12.6700874601 \\
 = 50 & 3.1505015656 & 6.3017887843 & 9.4546498604 & 12.6098781249 \\
 = 75 & 3.1475288458 & 6.2954052502 & 9.4439772711 & 12.593593674 \\
 = 100 & 3.1460444775 & 6.2922840149 & 9.4389138295 & 12.5861294535 \\
\end{array}\]

Limit value \(i\pi\) \[3.1415926536 \quad 6.2831853072 \quad 9.4247779608 \quad 12.5663706144\]

\[2\sqrt{n}y_{2n+1,i} \quad \begin{array}{cccc}
i = 1 & i = 2 & i = 3 & i = 4 \\n = 25 & 1.5638124248 & 4.6929519452 & 7.8266546359 & 10.9680271832 \\
 = 50 & 1.5673412691 & 4.7024067765 & 7.8386224094 & 10.976589897 \\
 = 75 & 1.5685036668 & 4.7056818763 & 7.8433729554 & 10.981916256 \\
 = 100 & 1.5690812843 & 4.7073401616 & 7.8458804145 & 10.9849178491 \\
\end{array}\]

Limit value \((2i - 1)\pi/2\) \[1.5707963268 \quad 4.7123889804 \quad 7.8539816340 \quad 10.9955742876\]

Finally, we turn to Case II again. In order to obtain some light about the asymptotic behavior of the small zeros of \(Q_n\) in this case (see Remark after Proposition 1), we have done some numerical experiments. For the computations, we have used relations (13) and (16), and the recurrence relation for \(\tilde{a}_n\) (see [2, f.(2.15)]):

\[\tilde{a}_n = \frac{((n + 2)/n)k_n\sigma_n}{k_n + n^2k_{n-2}(\sigma_{n-2}/(n - 2))^2 + 16\lambda n^2k_{n-3}\sigma_{n-2} - nk_{n-2}(\sigma_{n-2}/(n - 2))\tilde{a}_{n-2}} \quad n \geq 3.\]

We denote by \(x_{n,i}\) and \(t_{n,i}\) the positive zeros of \(Q_n\) and \(T_n\), respectively. Note that, as in the proof of (b) in Proposition 2, \(T_n\) are the orthogonal polynomials associated to the measure \(d\mu = (e^{-x^2}/(x^2 + a^2))dx\), \(a \in \mathbb{R}\setminus\{0\}\), with the same leading coefficient as \(H_n(x)\), \(Q_n(-x) = (-1)^nQ_n(x)\) and \(T_n(-x) = (-1)^nT_n(x)\). We have done several numerical experiments. Here, we show one of them where we have chosen \(a = 1.5\) and \(\lambda = 2\), obtaining then, the following results:
Conjecture. After these experiments, we conjecture that the zeros of $Q_n$ are in some sense close to those of $T_n$.

Acknowledgements

The authors thank the anonymous referees for their very useful corrections and comments which have made the paper more readable and comprehensible.

References