NUMERICAL-RADIUS-ATTAINING POLYNOMIALS

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Abstract

In this paper we discuss a version of James theorem making use of polynomials which attain their numerical radius. In addition, we obtain a characterization of finite-dimensional Banach spaces in terms of such polynomials.

1. Introduction

The celebrated James theorem asserts—for the special case of the unit ball—that a Banach space is reflexive if, and only if, each bounded linear functional attains its norm. A parallel version for numerical radius was proven in [1]. In order to be more precise and introduce the notation, for a Banach space $E$, $\Pi(E)$ denotes the set

$$\Pi(E) := \{(x, x^*) \in E \times E^*: \|x\| = \|x^*\| = x^*(x) = 1\}$$

and the numerical radius of a bounded and linear operator $T : E \rightarrow E$ is the real number

$$v(T) := \sup\{|x^*(Tx)| : (x, x^*) \in \Pi(E)\}.$$

Such an operator $T$ is said to attain its numerical radius provided that

there exists $(x_0, x_0^*) \in \Pi(E)$ such that $|x_0^*(Tx_0)| = v(T)$.

Then, the mentioned version of James’ theorem for numerical radius can be stated as follows: a Banach space is reflexive provided that every rank-one operator attains the numerical radius.

Harris [6] introduced the generalization of the numerical radius for holomorphic functions, which was used by Choi and Kim [4] for polynomials. Given a natural number $n \geq 1$, the numerical radius of an $n$-homogeneous continuous polynomial $P : E \rightarrow E$ is the real number $v(P)$ defined by

$$v(P) := \sup\{|x^*(P(x))| : (x, x^*) \in \Pi(E)\}.$$

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and \( P \) attains its numerical radius when
\[
\text{there exists } (x_0, x_0^*) \in \Pi(E) \text{ such that } |\lambda_0^n(Px_0)| = v(P).
\]

In section 2 we establish a James-type result for numerical-radius-attaining polynomials, and in section 3 we arrive at a characterization of finite-dimensional Banach spaces by means of polynomials attaining their numerical radius.

In the rest of this paper, if \( E \) is a Banach space, \( B_E \) and \( S_E \) will denote the closed unit ball and the unit sphere of \( E \), respectively.

2. A polynomial James theorem for numerical radius

In order to state the version of James’ theorem in terms of polynomials and numerical radius, we shall begin with an example. In the following, we shall denote by \( P^{(n)}(E, E) \) the Banach space of all \( n \)-homogeneous polynomials on \( E \), endowed with its usual sup norm.

**Example 1** There exists a reflexive Banach space \( E \) such that for all \( n \geq 1 \) some polynomial in \( P^{(n)}(\ell_2, \ell_2) \) does not attain its numerical radius.

We can take, for instance, \( E = \ell_2 \). In order to define the polynomial \( P \in P^{(n)}(\ell_2, \ell_2) \) that does not attain the numerical radius, let us fix a sequence \( \alpha \in \ell_\infty \) satisfying
\[
|\alpha(j)| < 1 = \|\alpha\|_\infty \quad \text{for all } j \geq 1.
\]

Then we define the polynomial \( Px : \ell_2 \to \ell_2 \) given by
\[
Px := \sum_{j \geq 1} \alpha(j)x(j)^ne_j \quad (x \in \ell_2),
\]
where \( \{e_j\}_{j \geq 1} \) is the usual basis for \( \ell_2 \). Since
\[
v(P) = \sup \{ |\langle x, Px \rangle| : \|x\|_2 = 1 \}
\]
(\( \langle , \rangle \) denotes the usual inner product on \( \ell_2 \)), we begin by fixing \( x \in S_{\ell_2} \). Then
\[
|\langle x, Px \rangle| = \left| \sum_{j \geq 1} \alpha(j)x(j)^{n+1} \right| \leq \sum_{j \geq 1} |\alpha(j)||x(j)|^2|x(j)|^{n-1}
\leq \sum_{j \geq 1} |x(j)|^2 = 1,
\]
because \( \|x\|_2 = 1 \) and, for all \( j \geq 1 \), \( |\alpha(j)| < 1 \). However,
\[
\sup_{j \geq 1} |\langle e_j, Pe_j \rangle| = \sup_{j \geq 1} |\alpha(j)| = \|\alpha\|_\infty = 1.
\]

Therefore, \( v(P) = 1 \) and \( P \) does not attain its numerical radius.

In order to state the appropriate version of James’s theorem for numerical-radius-attaining polynomials, we introduce the following notation and some technical results (Lemmas 2 and 3). For \( n \geq 1 \), \( x_0 \in E \) and \( x_1^*, \ldots, x_n^* \in E^* \), we write \( P_{x_1^*, \ldots, x_n^*, x_0} \) to stand the \( n \)-homogeneous polynomial on \( E \) defined by
\[
P_{x_1^*, \ldots, x_n^*, x_0}(x) := x_1^*(x) \cdots x_n^*(x)x_0 \quad (x \in E).
\]
In what follows, \( \prod_{k=1}^{m} \cdots \) is interpreted to be 1 whenever \( m < l \).
LEMMA 2 Let \( n \geq 1 \) and let \( \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \) be complex numbers. Then

\[
\prod_{j=1}^{n} \alpha_j - \prod_{j=1}^{n} \beta_j = \sum_{j=1}^{n} (\alpha_j - \beta_j) \prod_{p=1}^{j-1} \alpha_p \prod_{q=j+1}^{n} \beta_q.
\]

Proof. We shall proceed inductively. For \( n = 1 \) the statement is clearly satisfied. Now, if one assumes that it is satisfied for some \( n \geq 1 \), then

\[
\prod_{j=1}^{n+1} \alpha_j - \prod_{j=1}^{n+1} \beta_j = \left( \prod_{j=1}^{n} \alpha_j \right) (\alpha_{n+1} - \beta_{n+1}) + \beta_{n+1} \left( \prod_{j=1}^{n} \alpha_j - \prod_{j=1}^{n} \beta_j \right)
\]

\[
= \left( \prod_{j=1}^{n} \alpha_j \right) (\alpha_{n+1} - \beta_{n+1}) + \sum_{j=1}^{n} (\alpha_j - \beta_j) \prod_{p=1}^{j-1} \alpha_p \prod_{q=j+1}^{n} \beta_q
\]

\[
= \sum_{j=1}^{n+1} (\alpha_j - \beta_j) \prod_{p=1}^{j-1} \alpha_p \prod_{q=j+1}^{n+1} \beta_q,
\]

as required.

A generalization of the following result is proved in [5, Theorem 6]. Here we give an elementary proof.

LEMMA 3 Let \( E \) be a Banach space, \( n \geq 1 \), \( x_1^*, \ldots, x_n^* \in E^* \) and \( x_0 \in E \). Then

\[
v(P_{x_1^* \ldots x_n^*, x_0}) = \sup\{|x^{**}(x_1^*) \cdots x^{**}(x_n^*)x^*(x_0)| : (x^*, x^{**}) \in \Pi(E^*)\}.
\]

Proof. We shall assume, with no loss of generality, that \( \|x_0\| \leq 1 \) and that for all \( j = 1, \ldots, n \), \( \|x_j^*\| \leq 1 \). It is clear that

\[
v(P_{x_1^* \ldots x_n^*, x_0}) \leq \sup\{|y^{**}(x_1^*) \cdots y^{**}(x_n^*)y^*(x_0)| : (y^*, y^{**}) \in \Pi(E^*)\},
\]

so we shall prove the other inequality.

Let \( \epsilon > 0 \) and \( (y_0^*, y_0^{**}) \in \Pi(E) \), and let us fix \( \delta > 0 \) satisfying \( \delta(n\delta + 2(n+1)) < \epsilon \). Goldstine’s theorem provides an element \( z \in S_E \) such that

\[
\text{for all } j = 1, \ldots, n, \quad |(y_0^{**} - z)(x_j^*)| < \delta^2 \quad (3.1)
\]

and

\[
|(y_0^{**} - z)(y_0^*)| < \delta^2.
\]

Hence, we obtain that

\[
\text{Re } y_0^*(z) > \text{Re } y_0^{**}(y_0^*) - |(y_0^{**} - z)(y_0^*)| > 1 - \delta^2.
\]

Then, the Bishop–Phelps–Bollobás theorem [3] gives \( (y, y^*) \in \Pi(E) \) such that

\[
\|z - y\| < 2\delta \quad \text{and} \quad \|y_0^* - y^*\| < 2\delta. \quad (3.2)
\]
Hence
\[
|y_0^{**}(x_1^*) \cdots y_0^{**}(x_n^*)y_0^*(x_0)| - |x_1^*(y) \cdots x_n^*(y)y^*(x_0)| \\
= |y_0^*(x_0)| \left( \prod_{j=1}^n |y_0^{**}(x_j^*)| \right) - \left( \prod_{j=1}^n |x_j^*(z)| \right) \\
+ |y_0^*(x_0)| \left( \prod_{j=1}^n |x_j^*(z)| - n |x_j^*(y)| \right) \\
+ |y_0^*(x_0)| \left( |y_0^*(x_0)| - |y^*(x_0)| \right) \prod_{j=1}^n |x_j^*(y)| \\
\leq \sum_{j=1}^n |(y_0^{**} - z)(x_j^*)| + n |x_j^*(z - y)| \\
+ \|y_0^* - y^\star\| \quad \text{(by (3.1) and (3.2))} \\
\leq n \delta^2 + 2n \delta + 2 \delta < \epsilon.
\]

Thus,
\[
v(P) \geq |x_1^*(y) \cdots x_n^*(y)y^*(x_0)| \geq |y_0^{**}(x_1^*) \cdots y_0^{**}(x_n^*)y_0^*(x_0)| - \epsilon.
\]

Finally, since this inequality holds for every \( \epsilon > 0 \) and \((y_0^*, y_0^{**}) \in \Pi(E)\), it follows that
\[
v(P) \geq \sup_{(y^*, y^{**}) \in \Pi(E^*)} |y^{**}(x_1^*) \cdots y^{**}(x_n^*)y^*(x_0)|.
\]

Let us observe that, if we define \( \hat{\Pi}(E) \) to be the set
\[
\hat{\Pi}(E) := \{(x, x^*) \in S_E \times S_{E^*} : |x^*(x)| = 1\},
\]
then it is easy to check that for a continuous homogeneous polynomial \( P \) on \( E \),
\[
v(P) = \sup_{(x, x^*) \in \hat{\Pi}(E)} \Re x^*(Px)
\]
and that \( P \) attains its numerical radius if, and only if, the above supremum is a maximum.

In view of [1, Theorem 1] and Example 1, the next result seems to be the James theorem we are looking for.
THEOREM 4 Let $E$ be a Banach space, $x_0 \in E \setminus \{0\}$, $n \geq 0$ and $x_1^*, \ldots, x_n^* \in E^* \setminus \{0\}$ such that for all $x^* \in E^*$, $P_{x_1^* \cdots x_n^*} x_0$ attains its supremum on $B$.

Then $E$ is reflexive.

Proof. If one defines

$$B := \{x_1^*(z) \cdots x_n^*(z)z : (z, z^*) \in \hat{E}(E)\},$$

then it is clear that the polynomial $P_{x_1^* \cdots x_n^*} x_0$ attains its numerical radius if, and only if, $\Re x^*$ attains its supremum on $B$. Hence, in view of James’ theorem [7], the fact that for all $x^* \in E^*$ the polynomial $P_{x_1^* \cdots x_n^*} x_0$ attains its numerical radius can be equivalently stated by saying that the set $\mathcal{C}(B)$ (the closed convex hull of $B$) is weakly compact. Finally, let us show that this condition implies the reflexivity of $E$. Indeed, if it were not the case, the Bishop–Phelps theorem [2] provides $(z_0^*, z_0^{**}) \in \Pi(E^*)$ such that

$$z_0^{**} \not\in E \cup \bigcup_{j=1}^n \ker x_j^*$$

(3.1)

and

$$\|z_0^{**} - x_0\| < 1.$$  

(3.2)

It follows from condition (3.1) that

$$z_0^{**}(x_1^*) \cdots z_0^{**}(x_n^*) \neq 0$$

(3.3)

and in view of (3.2) we have that

$$|z_0^{**}(x_0^*) - z_0^{**}(x_0)| < 1$$

and thus $(z_0^{**}(x_0^*) = 1)$,

$$z_0^*(x_0) \neq 0.$$  

(3.4)

Therefore, by (3.1), (3.3) and (3.4) we have that

$$z_0^{**}(x_1^*) \cdots z_0^{**}(x_n^*)z_0^*(x_0)z_0^{**} \not\in E.$$

Since $\mathcal{C}(B)$ is weakly compact, it follows that $\mathcal{C}(B) = \mathcal{C}(B) \subset E$ ($\mathcal{C}(B)$ is the weak-$*$ closed convex hull of $B$) and by using the Hahn–Banach theorem there exists $y^* \in E^*$ such that

$$\sup_B \Re y^* < \Re z_0^{**}(x_1^*) \cdots z_0^{**}(x_n^*)z_0^*(y^*)z_0^*(x_0)$$

or, in other words,

$$\nu(P_{x_1^* \cdots x_n^*} y^*, x_0) < \Re z_0^{**}(x_1^*) \cdots z_0^{**}(x_n^*)z_0^*(y^*)z_0^*(x_0).$$

But thanks to Lemma 3,

$$\Re z_0^{**}(x_1^*) \cdots z_0^{**}(x_n^*)z_0^{**}(y^*)z_0^*(x_0) \leq \nu(P_{x_1^* \cdots x_n^*} y^*, x_0),$$

which is impossible. Consequently, $E$ is reflexive.

This theorem generalizes the corresponding result for operators ($n = 0$). In addition, the proof we have given here is more direct that the one appearing in [1].
3. A characterization of finite-dimensional Banach spaces in terms of numerical-radius-attaining polynomials

Now we prove that the converse of Theorem 4 does not hold. At the same time, we shall obtain a characterization of finite-dimensional Banach spaces by means of numerical-radius-attaining polynomials.

Given a Banach space \( E \), \( x \in E \), \( x^* \in E^* \) and \( n \geq 1 \), we write \( P^n_{x^*,x} \) for the continuous \( n \)-homogeneous polynomial on \( E \) defined by \( P^n_{x^*,x}(e) := x^*(e)^n x \) \( (e \in E) \).

**Lemma 5** Let \( E \) be a Banach space, \( x^*_0 \in S_{E^*} \), \( z_0 \in S_E \) and \( n \geq 1 \) such that the numerical radius of the continuous \( n \)-homogeneous polynomial \( P^n_{x^*_0,z_0} \) is 1 and \( z_0 \) is not contained in any proper line segment in \( S_E \). Then \( P^n_{x^*_0,z_0} \) attains its numerical radius if, and only if, \( |x^*_0(z_0)| = 1 \).

**Proof.** One direction can be checked immediately. For the other one, suppose that there exists \((x, x^*) \in \Pi(E)\) such that

\[
1 = v(P^n_{x^*_0,z_0}) = |x^*_0(x)|^n |x^*(z_0)|.
\]

Therefore, \( |x^*_0(x)| = |x^*(z_0)| = 1 \). Then, if \( \lambda \in \mathbb{K} \) with \( |\lambda| = 1 \) and \( x^*(\lambda z_0) = 1 \), we have that

\[
\|\lambda z_0 + x\| \geq |x^*(\lambda z_0 + x)| = 2.
\]

By using the assumption, \( x = \lambda z_0 \) and thus we obtain \( |x^*_0(z_0)| = |x^*_0(x)| = 1 \).

**Lemma 6** Let \( E \) be a Banach space, \( n \geq 1 \), \( x_0 \in S_E \) and \( z_0 \in B_E \). Assume that \( \{x_j\}_{j \geq 1} \) is a sequence in \( B_E \) satisfying

(i) \( \lim_{j \geq 1} \|x_j + z_0\| = 2 \),

(ii) \( z_0 \) is not in any proper line segment of \( S_E \),

(iii) \( \{x_j\}_{j \geq 1} \) converges to \( x_0 \) in the weak topology, and

(iv) \( z_0 \not\in \mathbb{K} x_0 \).

Then there exists \( x^*_0 \in S_{E^*} \) such that the continuous \( n \)-homogeneous polynomial \( P^n_{x^*_0,z_0} \) does not attain its numerical radius.

**Proof.** We choose \( x^*_0 \in S_{E^*} \) with \( x^*_0(x_0) = 1 \) and we shall prove that the polynomial \( P^n_{x^*_0,z_0} \) does not attain the numerical radius. It follows from (iii) that

\[
\lim_{j \geq 1} x^*_0(x_j) = x^*_0(x_0) = 1. \tag{6.1}
\]

By using (i) and the Bishop–Phelps–Bollobás theorem [3], we can find a sequence \( \{(y_j, y^*_j)\}_{j \geq 1} \) in \( \Pi(E) \) such that

\[
\lim_{j \geq 1} \|x_j - y_j\| = 0 \quad \text{and} \quad \lim_{j \geq 1} y^*_j(z_0) = 1, \tag{6.2}
\]

and so \( z_0 \in \hat{S}_E \). Moreover, from (6.1) and (6.2) we have that

\[
\lim_{j \geq 1} x^*_0(y_j) = 1.
\]
so the numerical radius of $P^n_{x_0, z_0}$ is 1, since

$$1 \leq \sup_{j \geq 1} |x_0^*(y)| |y^*_j(z_0)| \leq v(P^n_{x_0, z_0}) \leq 1.$$ 

Conditions (iv) and (ii) give that $|x_0^*(z_0)| < 1$. The preceding lemma shows that $P^n_{x_0, z_0}$ does not attain its numerical radius.

Not only do the above results provide us with a counterexample for the converse of Theorem 4, but they also give us the following general renorming result, previously stated for operators ($n = 1$) on Banach spaces with a Schauder basis [1, Example].

**THEOREM 7** A Banach space is finite-dimensional if, and only if, for each equivalent norm there exists $n \geq 1$ such that every rank-one continuous $n$-homogeneous polynomial attains its numerical radius.

**Proof.** A simple compactness argument gives us one direction. Therefore, we just have to prove that any infinite-dimensional Banach space $E$ admits an equivalent norm for which some (continuous) monomial does not attain its numerical radius. In view of Theorem 4 we can assume $E$ to be reflexive. Otherwise, the original norm satisfies the desired condition.

First of all, let us consider the separable case. If $E$ is separable and infinite-dimensional, we can find a positive number $K > 0$ and a countable biorthogonal system $\{(e_j, e^*_j)\}_{j \geq 1}$ in $S_E \times KBE^*$ such that the space generated by $\{e_j : j \geq 1\}$ is (norm) dense in $E$ and the subset $\{e^*_j : j \geq 1\}$ separates the elements in $E$ (see [9] or [10]). Let $A$ be the subset of $E$ given by

$$A := \left\{ x \in E : \sum_{j \geq 1} \frac{1}{\varepsilon_j} |e^*_j(x)|^2 \leq 1 \right\},$$

where $\{\varepsilon_j\}_{j \geq 1}$ is a fixed sequence in $\ell_1$ satisfying $\varepsilon_1 = 1$, and for all $j \geq 1$, $0 < \varepsilon_{j+1} < \varepsilon_j$. Since $\{\varepsilon_j\}_{j \geq 1} \in \ell_1$ and the set $\{e^*_j : j \geq 1\}$ separates the points in $E$, it holds that

$$\text{for all } x \in A, \quad x = \sum_{j \geq 1} e^*_j(x)e_j.$$ 

It is also clear that the set $A$ is (norm) compact. Now let us consider the subset $B$ given by

$$B := \text{co} \left\{ \frac{1}{2K} E \cup \overline{\text{co}}(e_2, e_2 + e_j : j \geq 3) \cup A \right\},$$

which is the unit ball of an equivalent norm $\| \cdot \|$ on $E$ (co and aco are, respectively, ‘convex hull’ and ‘absolutely convex hull’). Let $Y$ be the space $E$ endowed with the new norm. Then its dual norm is given by

$$\|y^*\| = \max \left\{ \frac{1}{2K} |y^*(e_1)|, \max_{j \geq 3} |y^*(e_2)|, \max_{j \geq 1} |y^*(e_j)|, \max_{a \in A} |y^*(a)| \right\}$$

$$= \max \left\{ \frac{1}{2K} |y^*(e_1)|, \max_{j \geq 3} |y^*(e_2)|, \max_{j \geq 1} |y^*(e_j)|, \left( \sum_{j \geq 1} \varepsilon_j^2 |y^*(e_j)|^2 \right)^{\frac{1}{2}} \right\}.$$
for any $y^* \in Y^*$, where we write $| \cdot |$ for the original norm in $E$. Now we take the elements

$$z_0 = e_1, \quad x_0 = e_2, \quad x_n = e_2 + e_j \quad (j \geq 3).$$

We finish the proof in the separable case by checking that $z_0, x_j \in B_Y, x_0 \in S_Y$ and proving that the four conditions in Lemma 6 are satisfied. We know by the definition of $B$ that $z_0, x_0, x_j \in B_Y$ and, in fact, $\|x_0\| = 1$ because $e_2^*(x_0) = 1$ and $\|e_2^*\| = 1$. Conditions (iii) and (iv) also hold since $\{e_j^*: j \geq 3\}$ generates a weak$^*$ dense subspace of $E^*$, and, since $E$ is reflexive, it is a dense subspace of $E^*$. In order to check condition (i) we just consider the functionals

$$x_j^* := (1 - \varepsilon_j)e_1^* + e_j^* \quad (j \geq 3);$$

then $\|x_j^*\| \leq 1$ and

$$\lim_{j \to \infty} x_j^*(x_j + z_0) = \lim_{j \to \infty} 2 - \varepsilon_j = 2.$$

But $2 - \varepsilon_j \leq \|x_j + z_0\| \leq 2$, so

$$\lim_{j \to \infty} \|x_j + z_0\| = 2.$$

Finally, we check condition (ii). In fact we shall show a stronger condition (ii$'$): $z_0$ is a point of smoothness for the new norm and the unique functional $z_0^* \in S_{Y^*}$ such that $z_0^*(z_0) = 1$ is also a point of smoothness (of the dual norm). Thus, if $z_0^* \in S_{Y^*}$ satisfies $z_0^*(z_0) = z_0^*(e_1) = 1$, then

$$1 = \|z_0^*\|^2 \geq \sum_{j \geq 1} e_j^2|z_0^*(e_j)|^2 = |z_0^*(e_1)|^2 + \sum_{j \geq 2} e_j^2|z_0^*(e_j)|^2,$$

and thus for all $j \geq 2$, $z_0^*(e_j) = 0$. The linear span of $\{e_j: j \geq 1\}$ is dense in $E$, so the previous condition implies that $z_0^* = e_1^*$ and $z_0$ is a point of smoothness. On the other hand, we have that

$$|e_1^*(x)| < 1 \quad \text{for all } x \in \frac{1}{2K}B_E \cup \overline{a_0(x_2, e_2 + e_j)}_{j \geq 3},$$

and so $e_1^*$ only attains its norm at elements in $A$. If $a \in A$ is such an element, then $e_1^*(a) = 1$ and the fact that $\sum_{j \geq 1}(1/e_j^2)|e_j^*(a)|^2 \leq 1, e_1 = 1$, gives that for all $j \geq 2$, $e_j^*(a) = 0$. But the functionals $\{e_j^*: j \geq 1\}$ separate the points of $E$ so $a = e_1 = z_0$, and $z_0^*$ is also smooth.

To conclude we deal with the general case. Assume now that $E$ is reflexive and infinite-dimensional. Then there exists a separable (infinite-dimensional) complemented subspace $E_0$ of $E$ (see [8, Proposition 1]). In view of the proof in the separable case, we can assume that there is an equivalent norm on $E_0$, satisfying the conditions (i), (iii) and (iv) in Lemma 6 and (ii$'$). Hence, there is an element $z_0 \in S_{E_0}$ and a functional $x_0^* \in S_{E_0^*}$ such that the polynomial $P_{x_0^*, z_0}$ does not attain its numerical radius.

By renorming $E$ we can assume that we have the decomposition $E = E_0 \oplus Y$ for some closed subspace $Y$ of $E$. Since the $\ell_2$-sum preserves smoothness, one can directly check that $x_0^* \in E^*$ and $z_0 \in E$ still satisfy the conditions (i), (ii$'$) and (iv). Therefore, by using Lemma 6 again, there is a space isomorphic to $E$ such that the polynomial $P_{x_0^*, z_0}$ does not attain its numerical radius.

We finish by showing that Theorem 4 cannot be derived from [1, Theorem 1].
PROPOSITION 8 If $n \geq 2$, then there exist a Banach space $E$, an element $x_0 \in E$ and $n$ functionals $x_1^*, \ldots, x_n^* \in E^*$ such that the polynomial $P_{x_1^*, \ldots, x_n^*, x_0}$ attains its numerical radius but the rank-one operator $P_{x_1^*, x_0} = x_1^* \otimes x_0$ does not.

Proof. Let $E$ be the sequence space $\ell_1$ and

$$x_1^* = \left\{ \left( 1 - \frac{1}{j} \right) \right\},$$

$$x_2^* = \cdots = x_n^* = e_2^*$$

($\{e_j^*\}_{j \geq 1}$ is the sequence of functionals associated with the usual basis of $\ell_1$) and $x_0 = e_1 \in S_{\ell_1}$. Let us first check that the rank-one operator $x_1^* \otimes x_0$ does not attain the numerical radius. Otherwise, one could find $(z, z^*) \in \Pi(\ell_1)$ satisfying

$$|z^*(x_0)x_1^*(z)| = v(x_1^* \otimes x_0) = 1.$$

Then

$$|x_1^*(z)| = 1$$

and $x_1^*$ is a norm-attaining functional on $\ell_1$, which is not the case.

Now let us show that the polynomial $P_{x_1^*, \ldots, x_n^*, x_0}$ attains its numerical radius. On the one hand, if $(x, x^*) \in \Pi(\ell_1)$,

$$|x^*(P_{x_1^*, \ldots, x_n^*, x_0}(x))| \leq |x_1^*(x) \cdots x_n^*(x)|$$

$$= |x(2)|^{n-1} \sum_{j \geq 2} \left( 1 - \frac{1}{j} \right) |x(j)|$$

$$= |x(2)|^{n-1} \sum_{j \geq 2} \left( 1 - \frac{1}{j} \right) |x(j)|$$

$$= |x(2)|^{n-1} \left( \frac{1}{2} |x(2)| + \sum_{j \geq 3} \left( 1 - \frac{1}{j} \right) |x(j)| \right)$$

$$\leq |x(2)|^{n-1} \left( \frac{1}{2} \left( 1 - \sum_{j \geq 3} |x(j)| \right) + \sum_{j \geq 3} \left( 1 - \frac{1}{j} \right) |x(j)| \right)$$

$$\leq |x(2)|^{n-1} \left( \frac{1}{2} \left( 1 + \sum_{j \geq 3} |x(j)| \right) \right)$$

$$\leq |x(2)|^{n-1} \left( 1 - \frac{|x(2)|}{2} \right)$$

$$\leq \frac{1}{2}.$$  

On the other, if we write $y^*$ for the sequence in $S_{\ell_\infty}$

$$y^*(j) = 1 \quad (j \geq 1),$$
then \((e_2, y^*) \in \Pi(\ell_1)\) and
\[y^*(Px^*_1 \cdots x^*_n, x^*_0) = \frac{1}{2}.
\]
Therefore,
\[v(Px^*_1 \cdots x^*_n, x^*_0) = \frac{1}{2}\]
and \(Px^*_1 \cdots x^*_n, x^*_0\) attains its numerical radius (at \((e_2, y^*)\)).

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