**Extrinsic Killing spinors**

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**Abstract.** Under intrinsic and extrinsic curvature assumptions on a Riemannian spin manifold and its boundary, we show that there is an isomorphism between the restriction to the boundary of parallel spinors and extrinsic Killing spinors of non-negative Killing constant. As a corollary, we prove that a complete Ricci-flat spin manifold with mean-convex boundary isometric to a round sphere, is necessarily a flat disc.

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1. **Introduction**

On a compact $k$-dimensional Riemannian spin manifold $Q$, real Killing spinors are characterized as eigenspinor fields of the Dirac operator associated with the smallest eigenvalues $\pm \sqrt{\frac{k}{4(k-1)} R_0^Q}$, where $R_0^Q$ denotes the infimum over $Q$ of the scalar curvature. In other words, these are eigenspinor fields of the Dirac operator satisfying the limiting-case of the Friedrich inequality:

$$\lambda^2 \geq \frac{k}{4(k-1)} R_0^Q.$$  \hspace{1cm} (1)

Note that Inequality (1) is an immediate consequence of the spinorial Cauchy-Schwarz inequality and it is only interesting in the case where $R_0^Q > 0$. A

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natural question arises: Is there any optimal lower bound for \(\lambda^2\) for manifolds with \(R^Q_0 \leq 0\)?

Recently, in [HMZ1] it has been shown that if \(Q := \partial M\) is the boundary of an \((n + 1)\)-dimensional compact Riemannian spin manifold \(M\) with non-negative scalar curvature \(R^M\) whose mean curvature \(H\) is non-negative, then the first non-negative eigenvalue \(\lambda\) of the Dirac operator of \(\partial M\) satisfies

\[
\lambda \geq \frac{n}{2} H_0, \tag{2}
\]

where \(H_0\) is the infimum of the mean-curvature. Furthermore, using the conformal covariance of the boundary Dirac operator together with an appropriate conformal boundary condition, Inequality (2) was improved by showing that \(H_0\) could be replaced by an extrinsic scalar conformal invariant (see [HMZ2]). Note that if the Einstein tensor of the manifold \(M\) is non-negative, then by the Gauss formula and by the Cauchy-Schwarz inequality it follows

\[
\frac{n}{2} H_0 \geq \frac{n}{4(n - 1)} R_0^{\partial M},
\]

where the last inequality could be strict (for example take \(\partial M\) to be a revolution tori in \(\mathbb{R}^3\)). It then became clear that one can get subtle information on a spin manifold via extrinsic invariants.

In this paper, we define the notion of extrinsic Killing spinors on the boundary \(\partial M\) of an \((n + 1)\)-dimensional compact Riemannian spin manifold \(M\) where, under some curvature assumptions and appropriate boundary conditions, we show that extrinsic Killing spinors could be extended to parallel spinors (see Theorem 3). The idea is to show that, under such conditions, equality in (2) is achieved by such spinors. One of the consequences of this result is to show that if a spin manifold, of dimension at least 3, is complete Ricci-flat with mean-convex boundary isometric to a round sphere, then it is a flat disc (see Corollary 6).

2. Riemannian spin manifolds with boundary

Let \(M\) be an \((n + 1)\)-dimensional Riemannian spin manifold with non-empty boundary \(\partial M\). We will always assume that \(M\) is connected while \(\partial M\) could be disconnected. Denote by \(\langle . , . \rangle\) its scalar product and by \(\nabla\) its corresponding Levi-Civita connection on the tangent bundle \(TM\). We fix a spin structure (and so a corresponding orientation) on the manifold \(M\) and we denote by \(SM\) the associated spinor bundle, which is a complex vector bundle of rank \(2\left[\frac{n+1}{2}\right]\) and by

\[
\gamma : \mathcal{C}\ell(M) \longrightarrow \text{End}_{\mathbb{C}}(SM)
\]

the Clifford multiplication, defined on the Clifford bundle \(\mathcal{C}\ell(M)\). It is well-known (see [LM1]) that on the complex spinor bundle \(SM\), there exist a natural Hermitian metric \(\langle . , . \rangle\) and a compatible spinorial Levi-Civita connection, also denoted by
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V. The (classical) Dirac operator $D$ on the manifold $M$ is the first order elliptic differential operator acting on spinor fields given locally by

$$D = \sum_{i=1}^{n+1} \gamma(e_i) \nabla_{e_i},$$

where $\{e_1, \ldots, e_{n+1}\}$ is a local orthonormal frame of $TM$.

The boundary hypersurface $\partial M$ is also an oriented Riemannian manifold with the orientation and the metric induced from the ambient space. Since its normal bundle is trivial, the Riemannian manifold $\partial M$ is also spin and so we have a corresponding spinor bundle $\mathbb{S}\partial M$, a Clifford multiplication $\gamma^{\partial M}$, a spinorial Levi-Civita connection $\nabla^{\partial M}$ and an intrinsic Dirac operator $D^{\partial M}$. It is not difficult to check (see for example [Ba2, BFGK, HMZ1, HMZ2, Bur, Tr]) that the restriction of the spinor bundle of $M$ to its boundary is related to the intrinsic Hermitian spinor bundle $\mathbb{S}\partial M$ by

$$\mathbb{S} := \mathbb{S}M|_{\partial M} \cong \begin{cases} \mathbb{S}\partial M & \text{if } n \text{ is even} \\ \mathbb{S}\partial M \oplus \mathbb{S}\partial M & \text{if } n \text{ is odd}. \end{cases}$$  \hspace{1cm} (3)

For any spinor field $\psi \in \Gamma(\mathbb{S})$ on the boundary hypersurface $\partial M$ and for any vector field $X \in \Gamma(T\partial M)$, define on the restricted bundle $\mathbb{S}$ the Clifford multiplication $\gamma^\mathbb{S}$ and the connection $\nabla^\mathbb{S}$ by

$$\gamma^\mathbb{S}(X)\psi := \gamma(X)\gamma(N)\psi,$$

$$\nabla^\mathbb{S}_X \psi := \nabla_X \psi - \frac{1}{2} \gamma^\mathbb{S}(AX)\psi = \nabla_X \psi - \frac{1}{2} \gamma(AX)\gamma(N)\psi,$$  \hspace{1cm} (5)

where $N$ denotes a unit normal vector field on $\partial M$. Then, $\gamma^\mathbb{S}$ and $\nabla^\mathbb{S}$ satisfy the same compatibility relations as for $\nabla$ and $\gamma$ with the additional compatibility relation

$$\nabla^\mathbb{S}_X (\gamma(N)\psi) = \gamma(N)\nabla^\mathbb{S}_X \psi.$$  \hspace{1cm} (6)

Taking into account the relation between the Hermitian bundles $\mathbb{S}$ and $\mathbb{S}\partial M$, one can see that

$$\nabla^\mathbb{S} \cong \begin{cases} \nabla^{\partial M} & \text{if } n \text{ is even} \\ \nabla^{\partial M} \oplus \nabla^{\partial M} & \text{if } n \text{ is odd}, \end{cases}$$  \hspace{1cm} (7)

$$\gamma^\mathbb{S} \cong \begin{cases} \gamma^{\partial M} & \text{if } n \text{ is even} \\ \gamma^{\partial M} \oplus -\gamma^{\partial M} & \text{if } n \text{ is odd}. \end{cases}$$  \hspace{1cm} (8)

On the space of smooth sections $\psi \in \Gamma(\mathbb{S})$, we have a Dirac operator $D$ associated with the connection $\nabla^\mathbb{S}$ and the Clifford multiplication $\gamma^\mathbb{S}$, locally given by

$$D\psi = \sum_{j=1}^{n} \gamma^\mathbb{S}(u_j)\nabla^\mathbb{S}_{u_j} \psi = \frac{n}{2} H\psi - \gamma(N)\sum_{j=1}^{n} \gamma(u_j)\nabla_{u_j} \psi,$$
where \(\{u_1, \ldots, u_n\}\) is a local orthonormal frame tangent to the boundary \(\partial M\) and 
\(H = (1/n)\text{trace } A\) is its mean curvature function. In the particular case where the field \(\psi \in \Gamma(S)\) is the restriction of a spinor field \(\psi \in \Gamma(SM)\) on \(M\), one gets

\[D\psi = \frac{n}{2}H\psi - \gamma(N)D\psi - \nabla_N \psi.\]  

With the above identifications, this Dirac operator is related to the intrinsic Dirac operator \(D^{\partial M}\) of the boundary by

\[D \cong \begin{cases} 
D^{\partial M} & \text{if } n \text{ is even} \\
D^{\partial M} \oplus -D^{\partial M} & \text{if } n \text{ is odd}
\end{cases}\]

Note that we have always the anticommutativity property

\[D\gamma(N) = -\gamma(N)D\]  

and so, when \(\partial M\) is compact, the spectrum of \(D\) is symmetric with respect to zero and coincides with the spectrum of \(D^{\partial M}\), if \(n\) is even, and with \(\text{Spec}(D^{\partial M}) \cup -\text{Spec}(D^{\partial M})\), if \(n\) is odd.

3. Parallel spinors and extrinsic Killing spinors

Recall that a spinor field \(\psi \in \Gamma(SQ)\), defined on a \(k\)-dimensional Riemannian spin manifold \(Q\), is called a Killing spinor field when the following over-determined first order equation is satisfied

\[\nabla_u \psi = -\frac{\lambda}{k}\psi(u)\psi, \quad \text{for each } u \in TQ,\]  

(K)

where \(\lambda \in \mathbb{C}\) is a constant, called the associated Killing number. It is well-known [Fr1, Fr2, BFGK] that the existence of a non-trivial solution to (K) implies that the manifold \(Q\) has constant scalar curvature \(R^Q = 4(k-1)\lambda^2/k\). In particular, Killing numbers associated with non-trivial Killing spinors are either real or imaginary. The spinor field \(\psi\) is called a real Killing spinor if \(\lambda \in \mathbb{R} - \{0\}\) (hence \(R^Q\) is a positive constant), or an imaginary Killing spinor if \(\lambda \in i\mathbb{R} - \{0\}\) (hence \(R^Q\) is a negative constant). When the manifold \(Q\) is compact, only the first case can occur. Of course, the remaining case \(\lambda = 0\) corresponds to parallel spinor fields on \(Q\), which can be viewed as real Killing spinors with vanishing Killing number. Note that, if we take \(\lambda\) to be a real function in (K), then necessarily \(R^Q = 4(k-1)\lambda^2/k\) is constant, i.e. such a spinor is a real Killing spinor (see [H]).

It is a clear that a Killing spinor \(\psi\) with associated Killing number \(\lambda\) is an eigenspinor for the Dirac operator corresponding to the eigenvalue \(\lambda\). Moreover, if \(Q\) is compact and so \(R^Q \geq 0\), the Friedrich inequality [Fr1] shows that \(\lambda\) is an eigenvalue of the Dirac operator with the least absolute value.

Assume that \(M\) is a connected \((n+1)\)-dimensional manifold, with boundary \(\partial M\), on which there exists a non-trivial parallel spinor field \(\psi\). Note that the existence of a parallel spinor imposes strong restrictions on the holonomy group (see, for example, [Wa, Ba1]). In particular, the manifold is Ricci-flat, hence its scalar
curvature vanishes. Then, using (5), it follows that the restriction of $\psi$ (denoted by the same symbol) to the boundary hypersurface $\partial M$ satisfies

$$\nabla_u^S \psi = -\frac{1}{2} \gamma^S (Au) \psi, \quad \text{for all } u \in T\partial M.$$ 

This restriction is a non-trivial spinor field on $\partial M$ since the length of a parallel spinor field is trivially constant on the connected manifold $M$. If the boundary $\partial M$ is a totally umbilical hypersurface, then

$$\nabla_u^S \psi = -\frac{H}{2} \gamma^S (u) \psi, \quad \text{for all } u \in T\partial M,$$

where $H$ is the mean curvature of $\partial M$ which will be a fortiori constant. That is, the restriction of $\psi$ to $\partial M$ satisfies a first order equation which is formally the same as (K) with respect to the extrinsic connection $\nabla^S$ and the extrinsic Clifford multiplication $\gamma^S$. We will refer to this type of spinor fields on $\partial M$ as extrinsic Killing spinors. They are closely related, on any hypersurface, to intrinsic Killing spinors. In fact, using (7) and (8), one easily proves the following:

**Lemma 1.** Let $\Sigma$ be a hypersurface of an $(n+1)$-dimensional Riemannian spin manifold $M$ and denote by $\nabla^S$ and $\gamma^S$ respectively the induced connection and the induced Clifford multiplication on the restriction $\nabla^S$ to $\Sigma$ of the spinor bundle $\nabla^S M$. If $\mathcal{E}K_\lambda (\Sigma)$ denotes the linear space of extrinsic Killing spinors on $\Sigma$ with associated Killing number $\lambda$, that is, the space of solutions of

$$\nabla_u^S \psi = -\frac{\lambda}{n} \gamma^S (u) \psi, \quad \text{for each } u \in T\Sigma,$$  

we have a natural isomorphism

$$\mathcal{E}K_\lambda (\Sigma) \cong \begin{cases} \mathcal{K}_\lambda (\Sigma) \cong \mathcal{K}_{-\lambda} (\Sigma) & \text{if } n \text{ is even} \\ \mathcal{K}_\lambda (\Sigma) \oplus \mathcal{K}_{-\lambda} (\Sigma) & \text{if } n \text{ is odd,} \end{cases}$$

where, on the Riemannian spin manifold $\Sigma$, the space $\mathcal{K}_\mu (\Sigma)$ stands for the space of intrinsic Killing spinors with Killing number $\mu$.

Thus, if the boundary of a manifold $M$ is umbilical, the extrinsic Killing spinor $\psi$ can be identified, for $n$ even, with an intrinsic Killing spinor field on $\partial M$ corresponding to the Killing number $nH/2$ (and, according to (6), the extrinsic spinor field $\gamma(N)\psi$ with an intrinsic Killing spinor field with Killing number $-nH/2$), and for $n$ odd, with a pair of intrinsic Killing spinor fields corresponding to the Killing numbers $nH/2$ and $-nH/2$. In both cases, we have necessarily

$$\frac{n^2}{4} H^2 = \frac{n}{4(n-1)} R^{\partial M},$$

where $R^{\partial M}$ is the scalar curvature of $\partial M$. We summarize these observations in the following result, where we will choose, in order to induce the extrinsic spin structure on the boundary, the unit field $N$ normal to $\partial M$ so that the (constant) mean curvature $H$ is non-negative.
Lemma 2. Let $M$ be an $(n + 1)$-dimensional Riemannian spin manifold whose boundary $\partial M$ is totally umbilical. Then the restriction to the boundary of a non-trivial parallel spinor is a non-trivial extrinsic real Killing spinor field with non-negative Killing number (precisely $nH/2$). In other words, there exists a linear injective map

$$\mathcal{P}(M) \hookrightarrow \mathcal{E}^\pm(\partial M) \cong \begin{cases} \mathcal{K}^+(\partial M) & \text{if } n \text{ is even} \\ \mathcal{K}^+(\partial M) \oplus \mathcal{K}^-(\partial M) & \text{if } n \text{ is odd}, \end{cases}$$

where, on a Riemannian spin manifold $Q$, we denote by $\mathcal{P}(Q)$ and $\mathcal{K}^+(Q)$ (resp. $\mathcal{K}^-(Q)$) the space of parallel spinors and the space of Killing spinors with non-negative (resp. non-positive) Killing number.

Remark 1. In fact, Lemma 2 is also valid for any umbilical hypersurface immersed in a Riemannian spin manifold. For example, when the ambient manifold $M$ is the cone $(\mathbb{R} \times P, dr^2 + r^2\langle , \rangle_P)$ constructed on an $n$-dimensional Riemannian spin manifold $P$, the manifold $P$ can be viewed as an umbilical hypersurface on each level $\{r\} \times P$. Hence, parallel spinors on $M$ give rise to Killing spinors on $P$. In this case, every Killing spinor on $P$ can be obtained in such a way. This fact was proved by Bär in [Bä1] and was used to characterize Riemannian spin manifolds carrying non-trivial Killing spinors.

4. Extension of extrinsic Killing spinor fields

In this Section, we shall prove that the linear injections of Lemma 2 are in fact isomorphisms, provided that some geometrical assumptions on $M$ and on its boundary $\partial M$ are satisfied.

The basic tool to relate the spin geometries of the compact manifold $M$ and of its compact boundary $\partial M$, is the integral version of the Schrödinger-Lichnerowicz formula

$$D^2 = \nabla^* \nabla + \frac{1}{4} R,$$

where $R$ is the scalar curvature of $M$. One possible expression of this formula, applied to any spinor field $\psi \in \Gamma(SM)$, is the following

$$\int_{\partial M} \left( \langle D\psi, \psi \rangle - \frac{n}{2} H |\psi|^2 \right) = \int_M \left( |\nabla \psi|^2 - |D\psi|^2 + \frac{1}{4} R |\psi|^2 \right).$$

By the spinorial Cauchy-Schwarz inequality, one gets the following integral inequality, called Reilly inequality [Re, HMZ1, HMZ2],

$$\int_{\partial M} \left( \langle D\psi, \psi \rangle - \frac{n}{2} H |\psi|^2 \right) \geq \int_M \left( \frac{1}{4} R |\psi|^2 - \frac{n}{n+1} |D\psi|^2 \right). \quad (11)$$

Equality in (11) is true only for the so-called twistor spinors, that is, those satisfying the following over-determined first order equation

$$\nabla_{X^*} \psi = - \frac{1}{n+1} \gamma(X)D\psi, \quad \forall X \in TM.$$

Inequality (11) will be the key fact to prove the main result.
Theorem 3. Let $M$ be an $(n+1)$-dimensional compact Riemannian spin manifold with boundary $\partial M$. Assume that the scalar curvature of $M$ and the mean curvature of $\partial M$ (w.r.t. the inner normal) are non-negative and that the Einstein tensor $\text{Ric} - \frac{\kappa}{2} \langle \cdot, \cdot \rangle$ of $M$ is non-negative along the normal direction $N$ of $\partial M$. Then each extrinsic real Killing spinor field on $\partial M$ associated with a non-negative Killing number, is the restriction to the boundary of a parallel spinor field on $M$.

Moreover, if $\mathcal{E}K^+(\partial M)$ is non-trivial, the boundary is umbilical and the linear injection

$$ \mathcal{P}(M) \hookrightarrow \mathcal{E}K^+(\partial M) \cong \begin{cases} K^+(\partial M) \cong K^- (\partial M) & \text{if } n \text{ is even} \\ K^+(\partial M) \oplus K^- (\partial M) & \text{if } n \text{ is odd} \end{cases} $$

given by the restriction to $\partial M$, is an isomorphism.

Proof. Take a non-trivial extrinsic real Killing spinor field $\phi \in \Gamma_1(S)$ corresponding to a Killing number $\lambda \geq 0$. Then,

$$ \lambda^2 = \frac{n}{4(n-1)} R^{\partial M} \quad \text{and} \quad D\phi = \lambda \phi, \quad (12) $$

where $R^{\partial M}$ is the (necessarily constant) scalar curvature of $\partial M$. Consider the following boundary problem

$$ \begin{cases} D\psi = 0 & \text{on } M \\ P_+\psi|_{\partial M} = P_+\phi & \text{along } \partial M, \end{cases} \quad (BP) $$

where $P_\pm$ denote the pointwise orthogonal projections, of a spinor field on $\partial M$, on the $\pm 1$-eigenspaces of the endomorphism

$$ i\gamma(N) : \Gamma(S) \longrightarrow \Gamma(S). $$

It is proved in [HMZ2] (see also [HMR]) that, for any $\phi \in \Gamma(S)$, the system (BP) has a smooth unique solution $\psi \in \Gamma(\tilde{S}M)$. The Reilly inequality (11) applied to the unique solution $\psi$, together with the assumption $R \geq 0$, imply

$$ \int_{\partial M} \left( (D\psi, \psi) - \frac{n}{2} H|\psi|^2 \right) \geq 0. \quad (13) $$

But, using (10), we see that $D$ interchanges the pointwise orthogonal $\pm 1$-subspaces of the endomorphism $i\gamma(N)$. Then

$$ \Re(D\psi, \psi) = \Re(DP_+\psi, P_-\psi) + \Re(DP_-\psi, P_+\psi) $$

and, taking into account that $D$ is a $L^2$-self-adjoint operator,

$$ \int_{\partial M} (D\psi, \psi) = 2 \int_{\partial M} \Re(DP_+\psi, P_-\psi). $$

But $P_+\psi = P_+\phi$ and the second equality in (12) imply

$$ DP_+\psi = DP_+\phi = \lambda P_-\phi.$$
Hence, from this equation, the Cauchy-Schwarz inequality and the fact that $\lambda \geq 0$, one has
\[
\int_{\partial M} (D\psi, \psi) = 2\lambda \int_{\partial M} \Re(P_-\varphi, P_-\psi) \leq \lambda \int_{\partial M} \left( |P_-\varphi|^2 + |P_-\psi|^2 \right).
\]
Now by (12), it follows
\[
DP_+\varphi = \lambda P_-\varphi, \quad DP_-\varphi = \lambda P_+\varphi,
\]
hence
\[
\lambda \int_{\partial M} |P_-\varphi|^2 = \lambda \int_{\partial M} |P_+\varphi|^2.
\]
Then, since $P_+\varphi = P_-\varphi$, we finally get
\[
\int_{\partial M} (D\psi, \psi) \leq \lambda \int_{\partial M} \left( |P_+\varphi|^2 + |P_-\psi|^2 \right) = \lambda \int_{\partial M} |\psi|^2,
\]
and equality holds only if $P_+\varphi = P_-\psi$, that is, the Killing spinor field $\psi$ is the restriction to $\partial M$ of the spinor field $\psi$ defined on $M$. Using this information in Inequality (13), it follows
\[
\int_{\partial M} \left( \lambda - \frac{n}{2}H \right) |\psi|^2 \geq 0 \tag{14}
\]
with equality if and only if $\psi$ is the restriction to the boundary of the ambient (harmonic twistor) spinor field $\psi$, that is, a parallel spinor. Hence the proof is complete if one proves that equality in (14) occurs. But, from the first relation in (12), the Gauss equation relating the curvature tensors of $M$ and the hypersurface $\partial M$, the assumption $\text{Ric}(N, N) - \frac{1}{2}R \geq 0$ and the Cauchy-Schwarz inequality, one has
\[
\frac{4(n-1)}{n} \lambda^2 = R^\partial M = R - 2\text{Ric}(N, N) + n^2H^2 - |A|^2 \leq n(n-1)H^2
\]
with equality only if $\partial M$ is a totally umbilical hypersurface. Hence, with the assumption $H \geq 0$, one has
\[
\lambda \leq \frac{n}{2}H
\]
on $\partial M$. Hence, equality in (14) is achieved, the boundary $\partial M$ is umbilical and the extrinsic Killing spinor field is the restriction of a parallel spinor. \hfill \Box

**Corollary 4.** Under the assumptions of Theorem 3, if a connected component of $\partial M$ carries a non-trivial Killing spinor (w.r.t. the induced spin structure), then

1. the boundary $\partial M$ is connected and totally umbilical, and
2. there is a non-trivial parallel spinor on $M$ (in particular, $M$ is Ricci-flat).

**Proof.** By Lemma 1, the existence of a non-trivial Killing spinor on a connected component, say $\partial M_0$ of $\partial M$, implies that there exists a non-trivial extrinsic Killing spinor $\varphi$ on $\partial M$ with non-negative Killing number, which vanishes along the remaining components. Then Theorem 3 could be applied to conclude that there is a non-trivial parallel spinor on $M$ whose restriction to $\partial M$ is $\varphi$. In particular, as $M$ is connected, $\varphi$ has constant non-zero length and so $\partial M$ is connected. Moreover, the umbilicity of $\partial M = \partial M_0$ follows from the proof of Theorem 3. \hfill \Box
Remark 2. Assume that $M$ is a Ricci-flat compact spin manifold whose boundary $\partial M$ is mean-convex. Corollary 4 can be applied to deduce that supersymmetries along the boundary give rise to supertranslations on the interior of the manifold. This fact could be useful in the context of the Hartle–Hawking no-boundary proposal [Ha,HH], since given a supersymmetric metric on the boundary $\partial M$, the class of Euclidean (i.e., Riemannian) geometries on $M$ satisfying the Einstein vacuum equations, is considerably reduced.

Corollary 5. Let $M$ be an $(n+1)$-dimensional complete Riemannian spin manifold with $n \geq 2$, non-negative Ricci curvature, mean-convex boundary $\partial M$ and non-negative Einstein tensor along the normal direction of $\partial M$. Assume that $\partial M$ is isometric to a round sphere. Then $M$ is isometric to a flat disc bounded by $\partial M$.

Proof. Let $\tilde{M}$ be the universal covering of $M$. We lift the Riemannian and spin structures of $M$ to $\tilde{M}$. We know that every geometric assumption on $M$ remains true for $\tilde{M}$ except eventually for the connectedness of $\partial \tilde{M}$. Note that the Gauss equation and the non-negativity of the Einstein tensor, imply

$$R_{\partial \tilde{M}} \leq n(n-1)H^2_{\partial \tilde{M}}.$$  

But each possible component of $\partial \tilde{M}$ is isometric to a sphere and so $R_{\partial \tilde{M}} = n(n-1)/r^2$, where $r > 0$, is the corresponding radius. Then, as $H_{\partial \tilde{M}} \geq 0$, one has $H_{\partial \tilde{M}} \geq 1/r$. Now one might use [Ka, Theorem A] to deduce the compacity of $\tilde{M}$ so that Theorem 3 and Corollary 4 could be applied to $\tilde{M}$. Then $\partial \tilde{M}$ is connected and so $\tilde{M}$ is a 1-fold covering, that is, $M$ is 1-connected.

Since, for $n \geq 2$, the spin structure on a round sphere is unique, one has on $\partial M$ a maximal number of independent Killing spinor fields, namely $2\left[\frac{n+1}{2}\right]$ (see [BFGK,Fr2], although this fact is also a trivial consequence of Theorem 3 above). Then, using Theorem 3 and Corollary 4, it follows that the space of parallel spinor fields on $M$ has also maximal dimension $2\left[\frac{n+1}{2}\right]$. Hence (see [Bâ2,Wa]) $M$ is a flat disc with spherical boundary. 

Corollary 6. Suppose that $M$ is a complete Ricci-flat spin manifold of dimension at least 3, with mean-convex boundary isometric to a round sphere. Then $M$ is a flat disc.

Remark 3. This last result is also true for non-spin Riemannian manifolds. A proof could follow the lines of [Ro, Theorem 2]. In fact, if one assumes that $\partial M$ is a round sphere, a suitable use of a Reilly inequality (see [Re]) allows to prove that each first eigenfunction of the Laplacian on $\partial M$ extends to a function on $M$ whose Hessian vanishes. Taking such functions as coordinate functions, a map $M \to \mathbb{R}^{n+1}$ could be constructed and show that it is an isometry onto a flat disc in $\mathbb{R}^{n+1}$.

Remark 4. Corollary 6 can be viewed as a uniqueness result for Einstein’s equations as a boundary-value problem. This type of uniqueness has been sometimes required in the context of the Hartle-Hawking no-boundary proposal in quantum gravity. Indeed, Gibbons and Hartle [GH] asked for information about solutions to
Einstein’s equations on compact four-dimensional manifolds with prescribed three-metric on its boundary. Precisely, some renormalization requirements on Ricci-flat (Riemannian) manifolds imply, as in the set-up of this paper, that they must be flat (see, for example, [G, GI]).

On the other hand, from a strictly geometrical point of view, Schlenker proved in [Sch] that each metric on a sphere, which is close enough to the standard round metric, can be realized as the restriction to the boundary of a Ricci-flat metric on the disc, bounded by that sphere. He also asked for the corresponding uniqueness. Corollary 6 gives an affirmative answer when the metric on the initial sphere is the standard one.

**Corollary 7.** Let $M$ be a complete Ricci-flat spin manifold without boundary. If a round sphere could be embedded in $M$ as a hypersurface, then $M$ is flat.

**Proof.** Assume that $\Sigma$ is a hypersurface embedded in $M$ and that $\Sigma$ is isometric to a round sphere, say of radius $r > 0$. As $\Sigma$ is one-connected, we may lift the embedding to the universal cover $\tilde{M}$ of $M$, which is also a complete boundaryless Ricci-flat manifold. Then $\Sigma$ determines in $\tilde{M}$ two domains with common boundary $\Sigma$. The same approach as in the proof of Corollary 5, implies that the mean curvature $H$ of $\Sigma$ satisfies $H \geq 1/r$, with respect to a certain choice of unit normal. Now, consider the inner domain $\Omega$, for this normal field along $\Sigma$. Then Corollary 6 can be applied to $\Omega$. Hence, $\Omega$ is flat. But every Einstein manifold is analytic (see [Be]), and so $\tilde{M}$ and $M$ are both flat. \(\square\)

**References**


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Hayward, S.A.: Signature change in general relativity. Class. Quantum Grav. 9, 1851–1862 (1992)


