ON THE EXISTENCE OF A PROPER MINIMAL SURFACE IN $\mathbb{R}^3$ WITH A CONFORMAL TYPE OF DISK

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1 Introduction

Proper minimal surfaces in $\mathbb{R}^3$ have peculiar properties that are not shared by general minimal surfaces; especially in the embedded case.

It has been proved that, under additional conditions, this family of surfaces has strong restrictions on their conformal structures. For instance, Huber and Osserman proved that if $M$ is a complete minimal surface with finite total curvature, then $M$ has the conformal type of a compact Riemann surface minus a finite number of points. In particular it is parabolic, that is, $M$ is not compact and $M$ does not carry a negative non-constant subharmonic function.

In the same context, Collin, Kusner, Meeks, and Rosenberg [CKMR] have proved that if $M$ is a properly immersed minimal surface in $\mathbb{R}^3$, then $M(+) = \{(x_1, x_2, x_3) \in M : x_3 \geq 0\}$ is parabolic. Furthermore, López [L] have shown that a properly immersed minimal surface with compact boundary and finite topology omitting some special subset of $\mathbb{R}^3$ is parabolic.

This kind of results have motivated the following conjecture:

Conjecture (Meeks, Sullivan [Me]). If $f : M \to \mathbb{R}^3$ is a complete proper minimal immersion where $M$ is a Riemannian surface without boundary and with finite genus, then $M$ is parabolic.

The main goal of this paper is to show a counterexample to the conjecture.

Theorem. There exists $\chi : \mathbb{D} \to \mathbb{R}^3$, a conformal proper minimal immersion defined on the open unit disk.

The immersion $\chi$ is obtained as the limit of a sequence of minimal immersions with boundary. These boundaries go uniformly to infinity in such a way that we get properness at the limit.

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The aforementioned sequence is constructed in a recursive way. In this process, the lemma on page 1283 is crucial. In this lemma, we modify a given minimal surface $X$ near its boundary to obtain a new minimal surface $Y$ such that the norm of $Y$ along the boundary increases with respect to the norm of $X$ and, at same time, the norm of $Y$ is controlled by a large enough lower bound in a small neighborhood of the boundary.

The tools we have utilized in the proof of this lemma are those that Nadirashvili [N] used to obtain a complete bounded minimal surface in $\mathbb{R}^3$: Runge’s theorem and the Lópex–Ros transformation.

It is important to remark that the geometry of the surface described in the theorem is very complicated; the convex hull of the image under $\chi$ of any closed set of $D$ containing an open arc of $\partial D$ is $\mathbb{R}^3$. Moreover, we can deduce more information about the geometry of the surface from the paper [L].

I would like to point out that the same technique as used in this paper can be used to construct a complete minimal surface properly immersed in a ball of $\mathbb{R}^3$, [MM].

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2 Background and Notation

The purpose of this section is to fix the notation used in the paper and to summarize some results about minimal surfaces.

We set $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$. By a polygon $P$ we mean a closed simple curve in $\mathbb{R}^2$ formed by a finite number of straight segments verifying $0 \in \text{Int } P$, where $\text{Int } P$ denotes the interior domain bounded by the curve $P$.

Let $X : D \to \mathbb{R}^3$ be a conformal minimal immersion defined on a simply connected domain $D$, and let $S = \{e_1, e_2, e_3\}$ be a set of orthogonal coordinates in $\mathbb{R}^3$. We label $(X(z))_{j,S} = \langle X(z), e_j \rangle$, $j = 1, 2, 3$. We write $(X(z))_j$ instead of $(X(z))_{j,S}$ when it is clear which orthogonal frame we are using.

We define the Weierstrass representation of the minimal immersion $X$ in $S$, $\phi_{(X,S)} = \{\phi_{1,S}, \phi_{2,S}, \phi_{3,S}\}$, as

$$\phi_{j,S}(z) = \frac{\partial(X(z))_{j,S}}{\partial u} - i \frac{\partial(X(z))_{j,S}}{\partial v}, \quad j = 1, 2, 3, \quad z = u + iv,$$
The functions $\phi_1, \phi_2, \phi_3$ are holomorphic on $D$, verifying $\sum_{j=1}^{3} |\phi_j|^2 \equiv 0$ and $\sum_{j=1}^{3} |\phi_j|^2 \neq 0$. As usual, we define

$$f(X,S) = \phi_1 - i\phi_2, \quad g(X,S) = \phi_3 / (\phi_1 - i\phi_2) .$$

For $\phi$ the Weierstrass representation of $X$, we also denote $f(\phi, S) = f(X, S)$ and $g(\phi, S) = g(X, S)$.

Conversely, if we consider $S = \{e_1, e_2, e_3\}$ a set of Cartesian orthogonal coordinates in $R^3$, and $f$ and $g$ are (respectively) holomorphic and meromorphic functions on $D$ such that

$$\phi_1 = \frac{1}{2} f(1 - g^2), \quad \phi_2 = \frac{1}{2} f(1 + g^2), \quad \phi_3 = fg$$

are holomorphic functions on $D$, then

$$X : D \rightarrow R^3,$$

$$X(z) = \sum_{j=1}^{3} \left( \text{Re} \int_{z_0}^{z} \phi_j(w) dw \right) e_j + c, \quad z_0 \in D, \quad c \in R^3,$$

is a conformal minimal immersion.

We can write the conformal metric associated to the immersion $X$, $\lambda_X^2(z) \langle \cdot, \cdot \rangle$, in terms of the Weierstrass representation as follows:

$$\lambda_X(z) = \frac{1}{2} |f(X,S)(z)|^2 \langle 1 + |g(X,S)(z)|^2 \rangle = \frac{1}{\sqrt{2}} \|\phi(X,S)(z)\| .$$

Observe that the above formula does not depend on the orthogonal frame $S$.

### 3 Proof of the Theorem

In order to prove the theorem, we need the following lemma.

**Lemma.** Let $X : O \rightarrow R^3$ be a conformal minimal immersion defined on a simply connected domain $O$ with $X(0) = 0$. Consider $r > 0$, $0 < s < r/100$ and a polygon $P$ with $P \subset O$, satisfying

$$r < \|X(z)\| < r + s/2, \quad \forall z \in O \setminus \text{Int} P .$$

Then, for any $b_1, b_2 > 0$, there exist a polygon $Q$ and a conformal minimal immersion $Y : U \rightarrow R^3$ defined on an open neighborhood of $\text{Int} Q$ with $Y(0) = 0$, such that

(a) $P \subset \text{Int} Q \subset \overline{\text{Int} Q} \subset U \subset O$;
(b) $\|Y(z) - X(z)\| < b_1, \quad \forall z \in \overline{\text{Int} P}$;
(c) $|||Y(z)|| - (r + s)| < b_2, \quad \forall z \in Q$;
(d) $\|Y(z)\| > r - 3\sqrt{sr} > r/2, \quad \forall z \in \text{Int} Q \setminus \text{Int} P$. 


Roughly speaking, in this lemma, we modify a minimal surface $X$ near
the boundary (property (b)), to obtain a new minimal surface $Y$ such that
the norm of $Y$ along a polygon increases with respect to the norm of $X$
(property (c)), and, at same time, the norm of $Y$ is controlled by a lower
bound (property (d)). Properties (c) and (d) are crucial to obtain proper-
ness.

This lemma will be proved in section 4.

We use the lemma to construct two sequences $\{X_n\}_n$ and $\{P_n\}_n$, where
$P_n$ is a polygon, and $X_n$ is a conformal minimal immersion defined on a
neighborhood of $\text{Int } P_n$ with $X_n(0) = 0$, satisfying the following properties
for all $n \in \mathbb{N}$:

(T1)$_n$ $P_{n-1} \subset \text{Int } P_n$ and $P_n \subset D(0,3)$;
(T2)$_n$ $\|X_n(z)\| - r_n < \frac{1}{2(n+1)^2}$, $\forall z \in P_n$;
(T3)$_n$ $\|X_n(z)\| \geq \frac{r_{n-1}}{2} - \frac{1}{2n}$, $\forall z \in \text{Int } P_n \setminus \text{Int } P_{n-1}$;
(T4)$_n$ $\|X_n(z) - X_{n-1}(z)\| < \frac{1}{n^2}$, $\forall z \in \text{Int } P_{n-1}$;
(T5)$_n$ $\lambda_{X_n}(z) \geq \alpha_n \lambda_{X_{n-1}}(z)$, $\forall z \in \text{Int } P_{n-1}$
where $\{\alpha_k\}_k$ is a sequence such
that $0 < \alpha_k < 1$ and $\prod_{k=1}^2 \alpha_k$ converges to $1/2$.

where $r_k = r_{k-1} + 2/k$ for $k > 1$ and $r_1 > 301$.

Sequences $\{X_n\}$ and $\{P_n\}$ are constructed in a recursive way. We
can take $X_1(u + iv) = r_1(u,v,0)$ and $P_1 \subset D(0,3)$ a suitable polygon
satisfying (T2)$_1$. Suppose that we have got $X_1, \ldots, X_{n-1}$ and $P_1, \ldots, P_{n-1}$.

Now we construct the $n$th term. We choose $\{\hat{\epsilon}_k\} \searrow 0$, with $\hat{\epsilon}_k < 1/n^2$
for all $k$. For each $k$ we consider $Y_k : U_k \to \mathbb{R}^3$ and $Q_k$, given by the lemma,
for the following data:

$$X = X_{n-1}, \quad P = P_{n-1}, \quad r = r_{n-1} - \frac{1}{n}, \quad s = \frac{2}{n},$$

$$b_1 = \hat{\epsilon}_k, \quad b_2 = \frac{1}{2(n+1)^2},$$

and $O$ a simply connected domain with $\text{Int } P_{n-1} \subset O \subset D(0,3)$ and verifying (1).

From (b) in the lemma, we deduce that the sequence $\{Y_k\}$
uniformly converges to $X_{n-1}$ on $\text{Int } P_{n-1}$. This implies that $\{\lambda_{Y_k}\}$
uniformly converges to $\lambda_{X_{n-1}}$ on $\text{Int } P_{n-1}$, and hence there is a $k_0 \in \mathbb{N}$ such that

$$\lambda_{Y_{k_0}}(z) \geq \alpha_n \lambda_{X_{n-1}}(z), \quad \forall z \in \text{Int } P_{n-1}.$$

We define $X_n = Y_{k_0}$, $P_n = Q_{k_0}$. It is easy to check (using the lemma) that
$X_n$ and $P_n$ verify (T1)$_n$, ..., (T5)$_n$. This concludes the construction of the sequences $\{X_n\}$ and $\{P_n\}$.

Now, we define $\Delta = \bigcup_{n \in \mathbb{N}} \text{Int } P_n$. $\Delta$ is a proper simply connected
domain of $\mathbb{C}$ (see (T1)$_n$); thus $\Delta$ is biholomorphic to a disk.
From (T4)\(_n\), we have that \(\{X_n\}\) is a Cauchy sequence on compact sets in \(\Delta\). Then there exists \(\chi: \Delta \to \mathbb{R}^3\) a harmonic map such that \(\{X_n\} \to \chi\), uniformly on compact sets in \(\Delta\). \(\chi\) has the following properties:

- \(\chi\) is minimal and conformal.
- \(\chi\) is an immersion. Indeed, for any \(z \in \Delta\) there exists \(n_0 \in \mathbb{N}\) such that \(z \in \text{Int} P_{n_0}\). From property (T5) one has that, \(\forall k > n_0\),
  \[
  \lambda_{X_k}(z) \geq \alpha_k \lambda_{X_{k-1}}(z) \geq \cdots \geq \alpha_k \cdots \alpha_{n_0+1} \lambda_{X_{n_0}}(z) \geq \alpha_k \cdots \alpha_1 \lambda_{X_{n_0}}(z).
  \]
  Taking the limit as \(k \to \infty\), we deduce that
  \[
  \lambda_{\chi}(z) \geq \frac{1}{2} \lambda_{X_{n_0}}(z) > 0,
  \]
  and so \(\chi\) is an immersion.
- \(\chi\) is proper. Let \(B\) be a compact subset of \(\mathbb{R}^3\). Consider \(n_0\) such that for all \(k > n_0\), one has
  \[
  \|p\| < \frac{r_{k-1}}{2} - \frac{1}{2k} - 2, \quad \forall p \in B
  \]
  and (using (T3)\(_n\) and (T4)\(_n\)), for all \(z \in \text{Int} P_k \setminus \text{Int} P_{k-1}\),
  \[
  \|\chi(z)\| \geq \frac{r_{k-1}}{2} - \frac{1}{2k} - \|X_k(z) - \chi(z)\| > \frac{r_{k-1}}{2} - \frac{1}{2k} - 2.
  \]
  From (2) and (3), we have \(\chi^{-1}(B) \cap (\text{Int} P_k \setminus \text{Int} P_{k-1}) = \emptyset\), \(\forall k > n_0\).
  Then \(\chi^{-1}(B) \subset \text{Int} P_{n_0}\). This implies that \(\chi^{-1}(B)\) is compact.

This completes the proof of the theorem.

## 4 Proof of the Lemma

To prove the lemma, we modify the immersion \(X\) such that it changes almost anything on \(\text{Int} P\). The aim of this modification is increasing the norm of \(X\) to get (c) and (d) in the lemma at the same time.

The proof of the lemma consists of two inductive process. In the first one, we obtain a new minimal immersion \(X_n\) defined on a simply connected domain \(\Omega\) containing \(\text{Int} P\). The new immersion \(X_n\) is close to \(X\) on \(\text{Int} P\) and its norm is greater than \(r + s\) at \(a_1, \ldots, a_n\), a finite collection of points around the boundary of \(\Omega\).

In the second process, from \(X_n\) we obtain a new minimal immersion \(Y\) defined on \(\Omega\) that proves the lemma. To construct \(Y\), we increase the norm of \(X_n\) along \(n\) curves of \(\partial \Omega\) that join the points \(a_i\). As a consequence, \(Y\) verify (c) for \(Q\) a polygon near \(\partial \Omega\). Also, \(Y\) is close to \(X_n\) on \(\text{Int} P\), and then \(Y\) and \(X\) are close on \(\text{Int} P\).

In both processes, we need to control the immersions in such a way that (d) holds.
First, we prepare the first process. We are going to fix some constants and a finite collection of points \( p_1, \ldots, p_n \) around the polygon \( P \) that we will use along this proof. Our purpose is to add poles at the points \( p_i \) to the Weierstrass representation \( \phi^0 \) of \( X \), in order to increase the norm of the immersion at the aforementioned points \( a_1, \ldots, a_n \).

We define the following constants:

1.a) \( \lambda > 1 \) such that \( \lambda^3 < 2 \) and \( \lambda(r + s) - 2\lambda^3\sqrt{\lambda^3(r + s)^2 - r^2} > r - 2\sqrt{(r + s)^2 - r^2} \);
1.b) \( s' = +\sqrt{\lambda^4(r + s)^2 - r^2} \);
1.c) \( 1 > \epsilon_0 > 0, \) chosen small enough so that certain inequalities that we are going to use in this paper are true. \( \epsilon_0 \) only depends on \( r, s, b_1, \lambda \) and \( s' \).

Let \( P' \) be a new polygon and \( W \) a simply connected open set such that \( P \subset \text{Int} P' \subset \overline{\text{Int} P'} \subset W \subset \overline{W} \subset O \). For a small enough neighborhood \( E \) of \( P' \) (with \( E \subset W \setminus \text{Int} P \)), we can define a continuous map \( S \), where \( S(z) = \{e_1(z), e_2(z), e_3(z)\} \) with \( e_3(z) = X(z)/\|X(z)\| \).

We choose a finite collection of points \( \{p_1, \ldots, p_n\} \subset E \) (we label \( p_{n+1} = p_1 \)), such that the segments \( p_1p_2, \ldots, p_{n-1}p_n, p_np_{n+1} \) form a new polygon \( \hat{P} \subset E \) and verifying the following properties:

2.a) for all \( i = 1, \ldots, n \), there exists a disk \( B^i \subset E \) with \( p_i, p_{i+1} \in B^i \), small enough so that \( X \) is close to a constant map on \( B^i \), i.e.
\[
\|X(z) - X(w)\| < \epsilon_0, \quad \forall z, w \in B^i; \tag{4}
\]
2.b) for all \( i = 1, \ldots, n \), the sets of orthogonal coordinates \( S(p_i) \) and \( S(p_{i+1}) \) are close, i.e.
\[
\|e^i_j - e^{i+1}_j\| < \epsilon_0, \quad \forall j \in \{1, 2, 3\}; \tag{5}
\]
where we label \( e^i_j = e_j(p_i) \).
2.c) for all \( i = 1, \ldots, n \),
\[
f_{(X,S(p_i))}(p_i) \neq 0. \tag{6}
\]

We can choose \( p_1, \ldots, p_n \) verifying (6) because \( f_{(X,S(p))}(p) = 0 \) can not hold on any open subset. If we have \( f_{(X,S(p))}(p) = 0 \) on an open subset then, \( X(p) \) is parallel to the normal vector of \( X \) at \( p \), \( X(p) = \|X(p)\|N(p) \), but this is impossible on an open set for \( X \) a minimal immersion.

Note that \( n \) diverges as \( \epsilon_0 \to 0 \).
We define \( n \) complex numbers \( \theta_1, \ldots, \theta_n \), with \( |\theta_i| = 1 \) and \( \text{Im} \theta_i \neq 0 \) such that
\[
\left| \frac{\theta_i f(X,S(p_i))(p_i)}{f(X,S(p_i))(p_i)} - \theta_{i+1} \frac{f(X,S(p_i+1))(p_i+1)}{|f(X,S(p_i+1))(p_i+1)|} \right| < \epsilon_0, \quad \forall i = 1, \ldots, n. \quad (7)
\]
Remark that 2.a), 2.b) and (7) hold for \( i = n \) where \( p_{n+1} = p_1 \).

![Figure 1: Distribution of the points \( p_i \).](image)

For every point \( p_i, i = 1, \ldots, n \), we consider a disk \( D(p_i, \delta) \) where \( 0 < \delta < \epsilon_0^2 \) chosen small enough so that (see Figure 1):

3.a) \( \text{Int} \hat{\mathcal{P}} \setminus \bigcup_{k=1}^n D(p_k, \delta) \) is a simply connected set;

3.b) \( D(p_i, \delta) \cup D(p_{i+1}, \delta) \subset B^1, \forall i = 1, \ldots, n; \)

3.c) \( D(p_i, \delta) \cap D(p_k, \delta) = \emptyset, \forall i = 1, \ldots, n, \) and \( k \neq i; \)

3.d) \( \sqrt{\delta} \max_{D(p_i, \delta)} \|f(X,S(p_i))\| < 1, \forall i = 1, \ldots, n; \)

3.e) \( \sqrt{\delta} \max_{D(p_i, \delta)} \|f(X,S(p_i))g_2(X,S(p_i))\| < 1, \forall i = 1, \ldots, n; \)

3.f) \( \sqrt{\delta} \max_{D(p_i, \delta)} \|\phi^0\| < 1, \forall i = 1, \ldots, n. \)

We finish these previous steps defining \( l \) as
\[
l = \sup_{z \in \text{Int} \hat{\mathcal{P}} \setminus \bigcup_{k=1}^n D(p_k, \delta)} \inf \{ \text{Euclidean length of } \alpha : \alpha \text{ is a curve in } \text{Int} \hat{\mathcal{P}} \setminus \bigcup_{k=1}^n D(p_k, \delta) \} + 2\pi \delta + \delta + 1. \quad (8)
\]
Observe that \( \delta \) depend on \( \epsilon_0 \) and \( \theta_1, \ldots, \theta_n. \)
4.1 The first inductive process. In this process, we will modify the Weierstrass data of $X$, $\phi^0$, to produce a sequence of $n$ new Weierstrass data $\phi^1, \ldots, \phi^n$. In the $i$th step, we modify $\phi^{i-1}$ on the disk $D(p_i, \delta)$ to obtain a new Weierstrass representation $\phi^i$ with a pole at $p_i$ that is close to $\phi^{i-1}$ outside some neighborhoods of $p_1, \ldots, p_i$ (see next property (A6)). To do this we will use the López–Ros transformation in the orthogonal frame $S(p_i)$. At the same time, we have to have a control on $\phi^i$ along the segment $p_i(p_i + \delta)$ (see property (A7)).

When we have $\phi^1, \ldots, \phi^n$, we are going to define a simply connected domain $\Omega$ that contains $\mathrm{Int} \mathcal{P}$ and does not contain any poles of $\phi^1, \ldots, \phi^n$. Then $X_i(z) = \int_0^z \phi^i, z \in \Omega$ is well defined for all $i = 1, \ldots, n$. From (A6), we have that $X_i$ and $X_{i-1}$ are close outside $D(p_i, \delta)$ (see property (P1)). In particular $X_n$ are close to $X_1$ outside $\Omega \setminus \cup_{k=1}^n D(p_k, \delta)$. As $\phi^i$ has a pole at $p_i$, we obtain a point $a_i$ that is close to $p_i$ and verifies (A7), that is, $\|X_i(a_i)\|$ is greater than $r + s_i$ (see property (P4)). The control that we have on $\phi^i$ along the segment $p_i(p_i + \delta)$ implies that $X_n(a_i)$ is close to $X_n(a_{i-1})$ (see property (P3)). This fact will be crucial along the second inductive process, section 4.2.

We are going to construct in a recursive way a sequence $\Psi_i = \{\phi^i, k_i, a_i, C_i, G_i, D_i\}, i = 1, \ldots, n$, where

4.a) $\phi^i : \mathbb{W} \to \mathbb{C}^3$ is a Weierstrass representation with poles at $p_1, \ldots, p_i$;
4.b) $k_i$ is a suitable positive constant;
4.c) $a_i$ is a point lying on the segment $\overline{p_i q_i}$, where $q_i = p_i + \delta$;
4.d) $C_i$ is an open arc of a circumference centered at $p_i$ with $a_i \in C_i$;
4.e) $G_i$ is the closed annular sector bounded by $C_i$, a piece of $\partial D(p_i, \delta)$ and two radii as Figure 2 indicates;
4.f) $D_i$ is an open simply connected subset of $\mathbb{C}$ verifying $\overline{D_i} \cap G_i = \emptyset$ and $\{p_i, w_i = p_i - k_i \theta_i\} \subset D_i \subset \overline{D_i} \subset D(p_i, \delta)$.

Remark 1. In what follows, we will use the convention that $\Psi_{n+1} \overset{\text{def}}{=} \Psi_1$.

The above sequence is constructed in order to satisfy the following properties:

(A1) $\sqrt{\lambda} \max_{D(p_i, \delta)} |f(\phi^i, S(p_k))| < 1, k = i + 1, \ldots, n$;
(A2) $\sqrt{\lambda} \max_{D(p_i, \delta)} \{f(\phi^i, S(p_k)) g^2(\phi^i, S(p_k)) \} < 1, k = i + 1, \ldots, n$;
(A3) $2 \lambda^2 s_i |f(\phi^i, S(p_k))| |(p_k - f(\phi^i, S(p_k)))/(p_k)| < \epsilon_0, k = i + 1, \ldots, n$;
(A4) $\| \Re \int_{\alpha_z} \phi^i \| < \epsilon_0, \forall z \in C_i$ where $\alpha_z$ is a piece of $C_i$ joining $a_i$ and $z$;
(A5) In the orthogonal frame \( S(p_i) \), one has \( \phi_i^1 = \phi_i^{i-1} \);
(A6) \[ \| \phi^i(z) - \phi^{i-1}(z) \| < \epsilon_0/n! \quad \forall z \in \bar{W} \setminus (D(p_i, \delta) \cup (\cup_{k=1}^{i-1} D_k)) ; \]
(A7) Using the frame \( S(p_i) \), we have \[ \| (\text{Re} \int_{q_i} a_i \phi^i) e_1^i + (\text{Re} \int_{q_i} a_i \phi^i) e_2^i \| > \lambda s^i ; \]
(A8) \[ \| (\text{Re} \int_{q_i} \phi^i) e_1^i + (\text{Re} \int_{q_i} \phi^i) e_2^i \| < \lambda^3 s', \forall z \in G_i ; \]
(A9) \[ \| \text{Re} \int_{q_i} \phi^i - \text{Re} \int_{q_i-1} \phi^{i-1} \| < \epsilon_0 (4 \lambda^2 s' + 12), i = 2, \ldots, n + 1 . \]

All the above properties are true for \( i = 1, \ldots, n \), except for properties (A1), (A2), and (A3) that hold only for \( i = 1, \ldots, n - 1 \). In the same way, property (A9) is valid for \( i = 2, \ldots, n + 1 \) (see Remark 1).

As we have mentioned at the beginning of this section, we proceed by recursion.

Let \( \phi^0 \) be the Weierstrass representation of \( X = X_0 \). Assume we have constructed \( \Psi_1, \ldots, \Psi_{i-1} \). We are going to construct \( \Psi_i \).

Define the meromorphic data \( \phi^i \), in the orthogonal frame \( S(p_i) \), as follows:
\[
\begin{align*}
 f_{(\phi^i, S(p_i))} &= f_{(\phi^{i-1}, S(p_i))} h_i, \\
g_{(\phi^i, S(p_i))} &= g_{(\phi^{i-1}, S(p_i))}/h_i,
\end{align*}
\]
where
\[
h_i(z) = \frac{k_i \theta_i}{z - p_i} + 1 .
\]

Furthermore, we assume that \( k_i > 0 \) is small enough to verify
\[
 k_i \max_{w \in D(p_i, \delta)} \left\{ \left| \frac{f_{(\phi^{i-1}, S(p_i))}(w) - f_{(\phi^{i-1}, S(p_i))}(p_i)}{w - p_i} \right| \right\} < 1 , \tag{9}
\]
and (A1)\(_i\), (A2)\(_i\), (A3)\(_i\) and (A6). It is possible because \( \phi^i \to \phi^{i-1} \), as \( k_i \to 0 \), uniformly on \( \bar{W} \setminus (D(p_i, \delta) \cup (\cup_{k=1}^{i-1} D_k)) \) and (A1)\(_i-1\), (A2)\(_i-1\) and
Remark 2. The meromorphic function $h_i$ is close to 1 outside a neighborhood of $p_i$. The constant $\theta_i$ has the effect of a rotation over $h_i$, when $z$ is close to $p_i$. Outside a neighborhood of $p_i$ this "rotation effect" almost disappears.

From the definition of $\phi^i$, (A5)$_i$ trivially holds.

We define $a_i$ as the first point in $\overline{q_ip_i}$ when we move along this segment from $q_i$ to $p_i$, and such that

$$\frac{1}{2} |f_{(\phi^0, S(p_i))}(p_i)| \int_{q_ip_i} \frac{k_i dw}{w-p_i} = \lambda^2 s_i.$$

(10)

Observe that $f_{(\phi^0, S(p_i))}(p_i) \neq 0$ and $\int_{q_ip_i} \frac{k_i dw}{w-p_i} \in \mathbb{R}^+$, $\forall z \in \overline{q_ip_i}$.

Our next step consist of seeing the following:

$$\| \left( \text{Re} \int_{q_ip_i} \phi_1^i(w) dw \right) e_1^i + \left( \text{Re} \int_{q_ip_i} \phi_2^i(w) dw \right) e_2^i - \frac{1}{2} \left( \int_{q_ip_i} \frac{k_i dw}{w-p_i} \right) \left( \text{Re} f_{(\phi^0, S(p_i))}(p_i) \theta_i e_1^i + \text{Im} f_{(\phi^0, S(p_i))}(p_i) \theta_i e_2^i \right) \| < 2\epsilon_0,$$

(11)

where $z \in \overline{q_ip_i}$, and $\phi_1^i$ and $\phi_2^i$ are expressed in the orthogonal frame $S(p_i)$.

In other words, the first two coordinates in $S(p_i)$ of the curve $z \mapsto \text{Re} \int_{q_ip_i} \phi^i(w) dw$, $z \in \overline{q_ip_i}$, approximate the segment starting at 0 in the direction of $\text{Re} f_{(\phi^0, S(p_i))}(p_i) \theta_i e_1^i + \text{Im} f_{(\phi^0, S(p_i))}(p_i) \theta_i e_2^i$.

To get (11), for $z \in \overline{q_ip_i}$, we write

$$\left( \text{Re} \int_{q_ip_i} \phi_1^i(w) dw \right) + i \left( \text{Re} \int_{q_ip_i} \phi_2^i(w) dw \right)$$

$$= \frac{1}{2} \left( \int_{q_ip_i} f_{(\phi^0, S(p_i))}(w) h_1(w) dw \right.$$

$$- \int_{q_ip_i} f_{(\phi^0, S(p_i))}(w) g_{(\phi^0, S(p_i))}(w) \frac{dw}{h_1(w)} \left. \int_{q_ip_i} f_{(\phi^0, S(p_i))}(p_i) \frac{k_i \theta_i}{w-p_i} dw \right.$$

$$+ \frac{1}{2} \int_{q_ip_i} (f_{(\phi^0, S(p_i))}(p_i) - f_{(\phi^0, S(p_i))}(p_i)) \frac{k_i \theta_i}{w-p_i} dw \right)$$

(12)

$$+ \frac{1}{2} \int_{q_ip_i} (f_{(\phi^0, S(p_i))}(w) - f_{(\phi^0, S(p_i))}(p_i)) \frac{k_i \theta_i}{w-p_i} dw.$$

(13)
\[ + \frac{1}{2} \int_{\Omega^2} f_{(\varphi^{-1},S(p_i))}(w)dw \]

\[ - \frac{1}{2} \int_{\Omega^2} f_{(\varphi^{-1},S(p_i))}(w)g_{(\varphi^{-1},S(p_i))}^2(w)\frac{dw}{h_i(w)}. \]

To obtain an upper bound in (12) we use the definition of \( a_i \) and property (A3) in the following sense:

\[ \left| \int_{\Omega^2} \left( f_{(\varphi^{-1},S(p_i))}(p_i) - f_{(\varphi^{-1},S(p_i))}(p_i) \right) \frac{k_i\theta_i}{w - p_i} dw \right| \]

\[ \leq 2\lambda^2 s^p \left| \frac{f_{(\varphi^{-1},S(p_i))}(p_i)}{|f_{(\varphi^{-1},S(p_i))}(p_i)|} \right| < \epsilon_0. \]  

(16)

Now, we deal with (13). Taking (9) into account one has

\[ \left| \int_{\Omega^2} \left( f_{(\varphi^{-1},S(p_i))}(w) - f_{(\varphi^{-1},S(p_i))}(p_i) \right) \frac{k_i\theta_i}{w - p_i} dw \right| \]

\[ \leq \delta k_i \max_{w \in \Omega(p_i,\delta)} \left\{ \left| \frac{f_{(\varphi^{-1},S(p_i))}(w)}{w - p_i} \right| \right\} \leq \delta < \epsilon_0. \]  

(17)

To get the same upper bound for (14), we use (A1)_{i-1},

\[ \left| \int_{\Omega^2} f_{(\varphi^{-1},S(p_i))}(w)dw \right| \leq \delta \max_{D(p_i,\delta)} \{ |f_{(\varphi^{-1},S(p_i))}| \} < \sqrt{\delta} < \epsilon_0. \]  

(18)

In order to bound (15), observe that \(|h_i(w)| = \sqrt{(\frac{k_i}{w-p_i})^2 + 2 \Re \theta_i(\frac{k_i}{w-p_i}) + 1} > \sqrt{1 - \Re^2 \theta_i} = |\Im \theta_i|, \forall w \in \overline{\Omega(p_i)}\). Therefore, (A2)_{i-1} leads us to

\[ \left| \int_{\Omega^2} f_{(\varphi^{-1},S(p_i))}(w)g_{(\varphi^{-1},S(p_i))}^2 (w)\frac{dw}{h_i(w)} \right| \]

\[ \leq \delta \max_{D(p_i,\delta)} \{ |f_{(\varphi^{-1},S(p_i))}g_{(\varphi^{-1},S(p_i))}^2| \} \leq \sqrt{\delta} < \epsilon_0. \]  

(19)

Inequalities (16), (17), (18) and (19) give (11).

From (11) and the definition of \( a_i \) we deduce (A7)_i, and

\[ \left\| \left( \Re \int_{\Omega^2} \phi_1 \right) e_1 + \left( \Re \int_{\Omega^2} \phi_2 \right) e_2 \right\| < \lambda^2 s^p + 2\epsilon_0 < \lambda^3 s^p, \quad \forall z \in \Omega(p_i). \]  

(20)

At this point, we are able to define \( D_i, C_i \) and \( G_i \).

Let \( D_i \) be a simply connected domain with \( \{p_i, p_i - k_i \theta_i\} \subset D_i \subset \overline{D_i} \subset D(p_i, \delta) \) and \( D_i \cap \overline{\Omega(p_i)} = \emptyset \) (this is possible because \( \Im \theta_i \neq 0 \), thus we have \( p_i - k_i \theta_i \notin \overline{\Omega(p_i)} \)). Since (20), we can take \( C_i \) an open arc of a circumference centered at \( p_i \) with \( a_i \in C_i \) and small enough to verify (A4)_i and (A8)_i.
on $G_i$, where $G_i$ is the closed annular sector bounded by $C_i$, a piece of $\partial D(p_i, \delta)$ and two radii, as indicated in Figure 2.

To check (A9), we write:

$$\left\| \text{Re} \int_{q_{\alpha_1}} \phi^i - \text{Re} \int_{q_{\alpha_{1-1}}} \phi^{i-1} \right\|$$

$$= \left\| \sum_{k=1}^{3} \left( \text{Re} \int_{q_{\alpha_1}} \phi_{k,S(p_i)}^i \right) e_k^i - \sum_{k=1}^{3} \left( \text{Re} \int_{q_{\alpha_{1-1}}} \phi_{k,S(p_{i-1})}^{i-1} \right) e_k^{i-1} \right\|$$

$$\leq \sum_{k=1}^{3} \left\| \left( \text{Re} \int_{q_{\alpha_1}} \phi_{k,S(p_i)}^i \right) e_k^i - \left( \text{Re} \int_{q_{\alpha_{1-1}}} \phi_{k,S(p_{i-1})}^{i-1} \right) e_k^{i-1} \right\|. \quad (21)$$

Next step consists of getting an upper bound for the three addends in (21):

From (11), we have

$$\left\| \left( \text{Re} \int_{q_{\alpha_1}} \phi_{1,S(p_i)}^i (w) dw \right) e_1^i - \left( \text{Re} \int_{q_{\alpha_{1-1}}} \phi_{1,S(p_{i-1})}^{i-1} (w) dw \right) e_1^{i-1} \right\|$$

$$\leq \left\| \left( \frac{1}{2} \text{Re} \int_{(\phi, S(p_i))}(p_i) \theta_i \int_{q_{\alpha_1}} k_i dw \right) e_1^i \right\|$$

$$- \left( \frac{1}{2} \text{Re} \int_{(\phi, S(p_{i-1}))}(p_{i-1}) \theta_{i-1} \int_{q_{\alpha_{1-1}}} k_{i-1} dw \right) e_1^{i-1} \right\| + 4\epsilon_0$$

using (10),

$$= \lambda^2 s' \frac{\text{Re} \int_{(\phi, S(p_i))}(p_i) \theta_i}{\text{Re} \int_{(\phi, S(p_{i-1}))}(p_{i-1}) \theta_{i-1}} \left\| e_1^i \right\| - \lambda^2 s' \frac{\text{Re} \int_{(\phi, S(p_{i-1}))}(p_{i-1}) \theta_{i-1}}{\text{Re} \int_{(\phi, S(p_{i-1}))}(p_{i-1}) \theta_{i-1}} \left\| e_1^{i-1} \right\| + 4\epsilon_0$$

$$\leq \lambda^2 s' \left\| \left( \frac{\text{Re} \int_{(\phi, S(p_i))}(p_i) \theta_i}{\text{Re} \int_{(\phi, S(p_{i-1}))}(p_{i-1}) \theta_{i-1}} \right) e_1^i - \left( \frac{\text{Re} \int_{(\phi, S(p_{i-1}))}(p_{i-1}) \theta_{i-1}}{\text{Re} \int_{(\phi, S(p_{i-1}))}(p_{i-1}) \theta_{i-1}} \right) e_1^{i-1} \right\| + 4\epsilon_0$$

$$\leq \lambda^2 s' \left\| \left( \frac{\text{Re} \int_{(\phi, S(p_{i-1}))}(p_{i-1}) \theta_{i-1}}{\text{Re} \int_{(\phi, S(p_{i-1}))}(p_{i-1}) \theta_{i-1}} \right) e_1^i \right\| - \left( \frac{\text{Re} \int_{(\phi, S(p_{i-1}))}(p_{i-1}) \theta_{i-1}}{\text{Re} \int_{(\phi, S(p_{i-1}))}(p_{i-1}) \theta_{i-1}} \right) e_1^{i-1} \right\| + 4\epsilon_0$$

$$+ \lambda^2 s' \left\| e_1^i - e_1^{i-1} \right\| \frac{\text{Re} \int_{(\phi, S(p_{i-1}))}(p_{i-1})}{\text{Re} \int_{(\phi, S(p_{i-1}))}(p_{i-1})} + 4\epsilon_0$$

(7) and (5) apply and conclude

$$\leq 2\lambda^2 s' \epsilon_0 + 4\epsilon_0 = \epsilon_0(2\lambda^2 s' + 4).$$

In the same way, we obtain a bound for the second addends in (21),
Ω is a simply connected domain; let

\[ \phi_i \text{ be a disk centered at } \alpha, \text{ and let } \Phi_i \text{ be a simply connected domain.} \]

We define Ω as follows (see Figure 3): Let \( D^i \) be a disk centered at \( p_i \), containing \( D(p_i, \delta) \) for all \( i = 1, \ldots, n \), and let \( \alpha_i \subset D^i \setminus D(p_i, \delta) \) be a simple curve connecting \( \partial D^i \cap \partial (\text{Int } \hat{P} \setminus \bigcup_{k=1}^n D^k) \) to the point \( q_i \). Let \( N_i \) be a small neighborhood of \( \alpha_i \cup q_i \) in \( G_i \cup (D^i \setminus D(p_i, \delta)) \). We define

\[ \Omega = \left( \text{Int } \hat{P} \setminus \bigcup_{k=1}^n D^k \right) \cup \left( \bigcup_{k=1}^n N_k \right). \]

For suitable \( D^i, \alpha_i \) and \( N_i \), the following properties hold:

5.a) \( \Omega \) is a simply connected domain;

\[ \left( \text{Re} \int_{q_1}^\alpha \phi_2 S(p_i) dw \right) e_2^i - \left( \text{Re} \int_{q_1}^{\phi_2^{-1} S(p_i)} \phi_2^{-1} dw \right) e_2^{-1} \]

\[ \leq \left( \frac{1}{2} \text{Im} \int_{\phi_i S(p_i)} (p_i) \frac{\bar{k}_i}{w-p_i} e_2^i \right) - \left( \frac{1}{2} \text{Im} \int_{\phi_i^{-1} S(p_i)} (p_i) \frac{\bar{k}_i}{w-p_i} e_2^{-1} \right) + 4\epsilon_0 \leq \epsilon_0 (2\lambda^2 s' + 4). \]

To estimate the third addend in (21), we use the properties (A6), (A7), and (A8), to obtain

\[ \left( \text{Re} \int_{q_1}^\alpha \phi_3 S(p_i) dw \right) e_3^i - \left( \text{Re} \int_{q_1}^{\phi_3^{-1} S(p_i)} \phi_3^{-1} dw \right) e_3^{-1} \]

\[ \leq \left( \text{Re} \int_{q_1}^\alpha \phi_3 S(p_i) \right) + \left( \text{Re} \int_{q_1}^{\phi_3^{-1} S(p_i)} \phi_3^{-1} \right) \]

\[ \leq \delta \left( \max_{D(p_i, \delta)} \{ ||\phi^{-1}|| \} + \max_{\tilde{D}(p_i, \delta)} \{ ||\phi^{-2}|| \} \right) \]

\[ \leq \delta \left( \max_{D(p_i, \delta)} \{ ||\phi^0|| \} + \max_{\tilde{D}(p_i, \delta)} \{ ||\phi^0|| + 2\epsilon_0 \} \right) \leq 2\sqrt{\delta} + 2\epsilon_0 < 4\epsilon_0. \]

The above bounds imply (A9).
5.b) \(\overline{\Omega} \backslash a_i \subset \overline{\Omega}\) and \(\text{Int} P \subset \Omega\);
5.c) \(\overline{\Omega}\) does not contain any \(p_i\) point and any zero of \(h_i\) for all \(i = 1, \ldots, n\);
5.d) \(\sup_{z \in \Omega} \inf \{\text{Euclidean length of } \alpha : \alpha \text{ is a curve in } \overline{\Omega} \text{ with origin } 0 \text{ and ending at } z\} < l\) where \(l\) has been defined on (8);
5.e) \(\overline{\Omega} \cap D(p_i, \delta) \subset G_i\).

Taking 5.a) and 5.c) into account, we can define \(n\) minimal immersions \(X_1, \ldots, X_n\),
\[
X_i : \Omega' \to \mathbb{R}^3
\]
\[
X_i(z) = \text{Re} \int_0^z \phi^i(w)dw, \quad i = 1, \ldots, n,
\]
where \(\Omega'\) is a suitable open neighborhood of \(\overline{\Omega}\).

Consider an orthogonal frame \(T_i = \{w_i^1, w_i^2, w_i^3\}\) for every \(i = 1, \ldots, n\) in such a way that
\[
w_i^3 = \frac{X_n(a_i)}{\|X_n(a_i)\|}. \tag{22}
\]
These orthogonal frames will use in the second process.

We write \(Q_i, i = 1, \ldots, n\) as simple curves such that
6.a) \(Q_i\) is connected, \(\overline{Q_i} = Q_i\) and \(Q_i \cap Q_j = \emptyset\) for \(j \neq i\);
6.b) \(Q_i\) connects \(C_i\) with \(C_{i+1}\);
6.c) \(\partial \Omega\) is formed by the curves \(Q_i\) and pieces of \(C_i\);
6.d) \(Q_i \subset B'\);
6.e) \( Q_i \cap D(p_k, \delta) = \emptyset \) for \( k \notin \{i, i+1\} \);

6.f) We can suppose that \( f_{(X_n, T_k)}(z) \neq 0 \), \( \forall z \in Q_i \), \( \forall i = 1, \ldots, n \). To obtain this, we only need to make slight local modifications on the choice of \( \Omega \). Note that the above modifications can be made in such a way that the properties of \( \Omega \) and \( Q_i \) remain.

Now, we see some properties of the immersions \( X_1, \ldots, X_n \) for \( i = 1, \ldots, n \),

(P1) \[ \|X_i(z) - X_{i-1}(z)\| < \epsilon_0/n, \forall z \in \overline{\Omega} \setminus D(p_i, \delta); \]

(P2) \( (X_i)_3 = (X_{i-1})_3 \), in the orthogonal frame \( S(p_i) \);

(P3) \[ \|X_n(a_i) - X_n(a_{i+1})\| < \epsilon_0(4\lambda^2 s' + 17). \]

Observe that \( a_{n+1} = a_1 \) when \( i = n \);

(P4) \[ \|X_n(a_i)\| > \lambda(r + s); \]

(P5) \[ \|X_n(q_i) - X_n(z)\| < \lambda^3 s' + 5\epsilon_0, \forall z \in G_i \cap \Omega. \]

Property (P1) is a easy consequence of (A6) and 5.d). On the other hand, it is straightforward to check (P2) from (A5).

Taking into account (P1), (A9)\( \iota \) and (4), we get (P3), as follows:

\[
\|X_n(a_i) - X_n(a_{i+1})\| \leq \|X_i(a_i) - X_{i+1}(a_{i+1})\| + 2\epsilon_0
\]
\[
\leq \left| \text{Re} \int_{\gamma_{x_i}} \phi_i' - \text{Re} \int_{\gamma_{x_{i+1}}} \phi_i'^{i+1} \right| + \|X_i(q_i) - X_{i+1}(q_{i+1})\| + 2\epsilon_0
\]
\[
\leq \epsilon_0(4\lambda^2 s' + 12) + \|X_0(q_i) - X_0(q_{i+1})\| + 4\epsilon_0
\]
\[
\leq \epsilon_0(4\lambda^2 s' + 12) + 5\epsilon_0 = \epsilon_0(4\lambda^2 s' + 17).\]

In order to prove (P4), recall that

\[
\|X_n(a_i)\| > \|X_i(a_i)\| - \epsilon_0.
\]

We give bounds for the coordinates of \( X_i(a_i) \) in \( S(p_i) \) separately. Firstly, we bound the first and second coordinates.

\[
\|(X_i(a_i))_1, S(p_i)e_1^1 + (X_i(a_i))_2, S(p_i)e_2^2\|
\]
\[
> \left( \left| \text{Re} \int_{\gamma_{x_i}} \phi_i^i e_1^1 + \text{Re} \int_{\gamma_{x_i}} \phi_i^i e_2^2 \right| - \|X_i(q_i)_1 e_1^1 + (X_i(q_i))_2 e_2^2\| \right.
\]

using (A7)\( \iota \), (4) and the definition of \( S(p_i) \) in page 1286, we have

\[
> \lambda s' - \|X_0(q_i)_1 e_1^1 + (X_0(q_i))_2 e_2^2\| - \epsilon_0
\]
\[
\geq \lambda s' - \|X_0(q_i)_1 e_1^1 + (X_0(q_i))_2 e_2^2\| - 2\epsilon_0 = \lambda s' - 2\epsilon_0 > s'.
\]

Now, we get a bound for the third coordinate. We use (P2)\( \iota \) and hypothesis (1) in the lemma,

\[
\|(X_i(a_i))_3\| = \|(X_{i-1}(a_i))_3\| \geq \|(X_0(a_i))_3\| - \epsilon_0
\]
then, one has
\[
\|X_n(a_i)\| \geq \sqrt{(s')^2 + (r - 2\epsilon_0)^2} - \epsilon_0
= \sqrt{\lambda^2(r + s)^2 - 4\epsilon_0(r - \epsilon_0)} - \epsilon_0 > \lambda(r + s),
\]
that proves (P4).

To obtain (P5), we consider the orthogonal frame \(S(p_i)\). Let \(z \in \overline{G_i} \cap \Omega\). Using (A8)\(_i\), we have
\[
\|X_n(q_i) - X_n(z)\| \leq \|X_i(z)\| + 2\epsilon_0
\leq \left(\operatorname{Re} \int_{q_i^z} \phi_{i1}^z e_1^z + \operatorname{Re} \int_{q_i^z} \phi_{i2}^z e_2^z\right) + |(X_i(q_i) - X_i(z))_{S(p_i)}| + 2\epsilon_0
< \lambda^3 s' + |(X_0(q_i) - X_0(z))_{S(p_i)}| + 4\epsilon_0 \leq \lambda^3 s' + 5\epsilon_0.
\]

**4.2 The second inductive process.** In the first process, we have obtained an immersion \(X_n\); and we have only guaranteed that its norm is greater than \(r + s\) at the points \(a_1, \ldots, a_n\), (property (P4)). In this second process, our objective is increasing the norm of \(X_n\) around the whole boundary of \(\Omega\) in order to satisfy its norm is greater than \(r + s\). So, we will modify the immersion \(X_n\) to produce a sequence of minimal immersions \(Y_1, \ldots, Y_n\), defined on \(\overline{\Omega}\). In each step, we take certain small neighborhoods, \(Q_i^\ell\) of the curve \(Q_i\), and modify the Weierstrass representation of \(Y_{i-1}, \psi^{i-1}\), on \(Q_i^\ell\), to get a new minimal immersion \(Y_i\). \(Y_i\) and \(Y_{i-1}\) are close on \(\overline{\Omega} \setminus Q_i^\ell\), and the intrinsic metric of \(Y_i\) increase on \(Q_i^\ell\) with respect to \(Y_{i-1}\). From this, we obtain that the norm of \(Y_i\) is greater than \(r + s\) around \(Q_i\). To do this, we use the López–Ros transformation in the set of Cartesian coordinates \(T_i\) and Runge’s theorem. So, note that the third coordinates of \(X_n(a_i)\) and \(X_n(a_{i+1})\), in the frame \(T_i\), are close and greater than \(r + s\). This fact will be decisive in section 4.3.

First, we define a collection of neighborhoods of the arcs \(C_i\) and a collection of neighborhoods of the curves \(Q_i\):

For all \(i = 1, \ldots, n\), let \(\widehat{C}_i\) be an open neighborhood of \(C_i\), small enough to satisfy
\[
\|X_n(z) - X_n(a_i)\| < 3\epsilon_0, \quad \forall z \in \widehat{C}_i \cap \overline{\Omega}.
\]
We can take \(\widehat{C}_i\) verifying the above because, using (A4)\(_i\), for all \(z \in C_i\),
\[
\|X_n(z) - X_n(a_i)\| < \|X_i(z) - X_i(a_i)\| + 2\epsilon_0 < 3\epsilon_0.
\]
We define \(0 < \epsilon_1\) such that
\[
2\epsilon_1 < \frac{1}{2} \min_{Q_i} \{|f_{(X_n, T_{i})}|\}, \quad \forall i = 1, \ldots, n.
\]
Observe that from 6.f), \(\min_{Q_i} \{|f_{(X_n, T_{i})}|\} \neq 0\).
For all $i = 1, \ldots, n$, we define $Q_i^\xi = \{ z \in \mathbb{C} : \text{dist}(z, Q_i) \leq \xi \}$, where \text{dist}(z, A) means the Euclidean distance between point $z$ and set $A$. We assume that $0 < \xi$ is small enough to satisfy

7.a) $Q_i^\xi \subset \Omega'$;
7.b) $Q_i^\xi \cap Q_j^\xi = \emptyset, i \neq j$;
7.c) $Q_i^\xi \cap \overline{D(p_k, \delta)} = \emptyset, \text{ for } k \not\in \{i, i + 1\}$;
7.d) $Q_i^\xi \subset B_i$;
7.e) $Q_i^{\xi/2}$ and $\Omega \setminus Q_i^\xi$ are simply connected sets;
7.f) For all $z \in Q_i$, we have $|f_{(X_n, T_i)}(z) - f_{(X_n, T_i)}(x)| < \epsilon_1, \forall x \in B(z, \xi/2)$;
7.g) $\sup_{z \in \Omega \setminus Q_i^\xi} \inf_{\alpha} \text{Euclidean length of } \alpha : \alpha$ is a curve in $\Omega \setminus Q_i^\xi$ with origin 0 and ending at $z < l$.

Label $Y_0 \overset{\text{def}}{=} X_n$. We are going to construct in a recursive way $Y_1, \ldots, Y_n$, where $Y_i : \Omega' \to \mathbb{R}^3$ is a conformal minimal immersion with $Y_i(0) = 0$ verifying the following properties:

(B1)$_i$ $(Y_i)_{3,T_i} = (Y_{i-1})_{3,T_i}$;
(B2)$_i$ $\|Y_i(z) - Y_{i-1}(z)\| < \frac{\epsilon_i}{n}, \forall z \in \Omega \setminus Q_i^\xi$;

Figure 4: Properties of $X_n$. 

Figure 4: Properties of $X_n$. 

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(B3) \[ |f(Y_i; T_k)(z) - f(Y_{i-1}; T_k)(z)| < \frac{4|z|}{n}, \forall z \in \Omega \setminus Q_i^\xi, \text{ for } k = i + 1, \ldots, n; \]
(B4) \[ \frac{\xi}{2} \left( \frac{1}{n} + \frac{\nu_i}{\tau_i(\tau_i - \nu_i)} \right) \max Q_i^\xi \{ |f(Y_{i-1}; T_i)g^2(Y_{i-1}; T_i)| + \nu_i \max Q_i^\xi \{ |f(Y_{i-1}; T_i)| \} \} < 1; \]
(B5) \[ \frac{1}{2} \left( \frac{\xi^2}{2} \min Q_i \{ |f(Y_0; T_i)| \} - 1 \right) > 2(r + s) + 1; \]
where \( \tau_i \) and \( \nu_i \) are certain positive constants.

Assume we have \( Y_0, Y_1, \ldots, Y_{i-1} \). We define \( Y_i \) as
\[ f(Y_i; T_i) = f(Y_{i-1}; T_i)l_i, \quad g(Y_i; T_i) = g(Y_{i-1}; T_i)/l_i, \]
where \( l_i : \mathbb{C} \to \mathbb{C} \) is a holomorphic non-zero function verifying
- \( |l_i(z) - \tau_i| < \nu_i, \forall z \in Q_i^{\xi/2} \);
- \( |l_i(z) - 1| < \nu_i, \forall z \in \Omega \setminus Q_i^\xi \).

The existence of this function \([N]\) is a consequence of Runge’s theorem. We define \( Y_i \) as the minimal immersion that becomes from the above meromorphic data.

We have assumed that \( \tau_i > 0 \) is large enough and \( \nu_i > 0 \) is small enough to satisfy (B2), (B3), (B4), and (B5). It is possible because the Weierstrass representation of \( Y_{i-1} \) uniformly converges to the Weierstrass representation of \( Y_i \), as \( \nu_i \to 0 \) and \( \tau_i \to \infty \) on \( \Omega \setminus Q_i^\xi \). We take into account property 7, to obtain (B2).

Observe that (B1) trivially holds.

### 4.3 Definition of \( Y \)

At last, we define the open set \( U = \Omega \) and \( Y : U \to \mathbb{R}^3 \) as \( Y = Y_n \). In this section, we are going to check that the immersion \( Y \) verifies all the claims of the lemma.

Bearing in mind \( \text{Int} \mathcal{P} \subset \Omega \setminus (\bigcup_{k=1}^n D(p_k, \delta) \cup \bigcup_{k=1}^n Q_k^\xi) \), property (b) is a easy consequence of (P1) and (B2).

The proof of the rest of the claims of the lemma are nontrivial.

Properties (a) and (c). To prove the existence of a polygon \( Q \) verifying (a) and (c), we only need to get the following:

for all curve \( \beta \) with origin 0 and ending at a point of \( \partial \Omega \), there exists \( z' \in \beta \) such that \( \|Y(z')\| > r + s \). \hspace{1cm} (25)

We are going to see the above property. Let \( \beta \subset \mathcal{M} \) be a curve with \( \beta(0) = 0 \) and \( \beta(1) = z_0 \in \partial \Omega \).

Note, we want to show that \( \|Y_n(z)\| > r + s, \forall z \in \hat{C}_i \), for \( i = 1, \ldots, n \).

Let \( z \in \hat{C}_i \), we study three possibilities:

- Suppose that \( z \in \hat{C}_i \cap Q_i^\xi, i \in \{1, \ldots, n\} \). Then, considering \( T_i \), and using (B2), (B1), (23), (22) and (P4), we have
  \[ \|Y_n(z)\| \geq \|Y_i(z)\| - \epsilon_0 \geq |(Y_i(z))_3| - \epsilon_0 \]
\[ = \left| (Y_{i-1}(z))_{3} \right| - \varepsilon_0 \geq \left| (Y_0(z))_{3} \right| - 2\varepsilon_0 \]
\[ > \left| (Y_0(a_i))_{3} \right| - 5\varepsilon_0 = \|Y_0(a_i)\| - 5\varepsilon_0 > \lambda(r+s) - 5\varepsilon_0 > r + s. \]

- Suppose that \( z \in \hat{C}_i \cap Q^\varepsilon_{i-1}. \) Now, considering \( T_{i-1} \), one has
  \[ \|Y_0(z)\| \geq \left| (Y_0(z))_{3,T_{i-1}} \right| - 2\varepsilon_0 \]
  and from (23), (P3)_{i-1} and (P4)_{i-1}, we deduce
  \[ \geq \|Y_0(a_i)\|_{3,T_{i-1}} - 5\varepsilon_0 \geq \|Y_0(a_{i-1})\|_{3,T_{i-1}} - \varepsilon_0(4\lambda^2 s' + 17) - 5\varepsilon_0 \]
  \[ \geq \lambda(r + s) - \varepsilon_0(4\lambda^2 s' + 17) - 5\varepsilon_0 \geq r + s. \]

Note that case \( \hat{C}_i \cap Q^\varepsilon_0 \) can be proved in the same way.

- Suppose that \( z \in \hat{C}_i \setminus \bigcup_{k=1}^n Q^\varepsilon_k \). Therefore,
  \[ \|Y_0(z)\| \geq \|Y_0(z)\| - \varepsilon_0 > \|Y_0(a_i)\| - 4\varepsilon_0 > \lambda(r + s) - 4\varepsilon_0 > r + s. \]

This proves that \( \|Y_0(z)\| > r + s, \forall z \in \hat{C}_i \). Hence, we can suppose that \( \beta \cap \hat{C}_i = \emptyset, \forall i = 1, \ldots, n. \)

From 6.c), \( z_0 \in Q_i, i = 1, \ldots, n. \) Let \( z_1 \in \beta \cap \partial B(z_0, \xi/2). \) Observe that \( \overline{z_1 \infty L} \subset Q_i^{\xi/2}. \) Next, we give a lower bound for \( \|Y_0(z_0) - Y_0(z_1)\|. \) We consider \( T_i, \) and write \( f^{i-1} = f_{(Y_1 \cup \cdots \cup Y_{i-1})}, g^{i-1} = g_{(Y_1 \cup \cdots \cup Y_{i-1})}. \) Then,

\[ \|Y_0(z_0) - Y_0(z_1)\| \geq \|Y_0(z_0) - Y_0(z_1)\| - 2\varepsilon_0 \]
\[ \geq \left| \left( \text{Re} \frac{1}{2} \int_{z_1 \infty L} f^{i-1} l_i - \frac{f^{i-1}(g^{i-1})^2}{l_i} \right) + i \left( \text{Re} \frac{1}{2} \int_{z_1 \infty L} f^{i-1} l_i + \frac{f^{i-1}(g^{i-1})^2}{l_i} \right) \right| - 2\varepsilon_0 \]
\[ = \frac{1}{2} \left| \int_{z_1 \infty L} f^{i-1} l_i - \frac{f^{i-1}(g^{i-1})^2}{l_i} \right| - 2\varepsilon_0 \]
\[ \geq \frac{\tau_i}{2} \left| \int_{z_1 \infty L} f^{i-1} l_i \right| - \frac{1}{2\tau_i} \left| \int_{z_1 \infty L} f^{i-1}(g^{i-1})^2 \right| \]
\[ - \frac{1}{2} \left| \int_{z_1 \infty L} f^{i-1}(l_i - \tau_i) \right| - \frac{1}{2} \left| \int_{z_1 \infty L} f^{i-1}(g^{i-1})^2 \left( \frac{1}{l_i} - \frac{1}{\tau_i} \right) \right| - 2\varepsilon_0 \]
\[ \geq \frac{\tau_i}{2} \left| \int_{z_1 \infty L} f^{i-1} l_i \right| - \frac{\xi}{4} \left( \frac{1}{\tau_i} \max_{Q_i^\varepsilon} \left\{ |f^{i-1}(g^{i-1})^2| \right\} \right) \]
\[ + \nu_i \max_{Q_i^\varepsilon} \left\{ |f^{i-1}| \right\} + \frac{\nu_i}{\tau_i(\tau_i - \nu_i)} \max_{Q_i^\varepsilon} \left\{ |f^{i-1}(g^{i-1})^2| \right\} \right| - 2\varepsilon_0 \]
using (B4)i, we continue

\[ \geq \frac{1}{2} \left( \tau_i \left| \int_{z_1 \infty L} f^{i-1} - 1 \right| \right) - 2\varepsilon_0. \]
On the other hand, from (B3)$_1,\ldots,(B3)_{i-1},$ 7.f), and (24), we deduce
\[
\left|\int_{\overline{z_{120}}} f^{n-1}(w)dw\right|
\geq \left|f(Y_0,T_1)(z_0)\int_{\overline{z_{120}}} dw\right| - \left|\int_{\overline{z_{120}}} (f(Y_0,T_1)(z_0) - f(Y_0,T_1)(w))dw\right|
- \left|\int_{\overline{z_{120}}} (f(Y_0,T_1)(w) - f^{n-1}(w))dw\right|
\geq \frac{\xi}{\text{min}_Q} (|f(Y_0,T_1)(z_0)| - \epsilon_1 - \epsilon_1) \geq \frac{\xi}{\text{min}_Q} (\min\{|f(Y_0,T_1)|\} - 2\epsilon_1)
\geq \frac{\xi}{\text{min}_Q} \min\{|f(Y_0,T_1)|\}.
\]

Therefore,
\[
\|Y_n(z_0) - Y_n(z_1)\| \geq \frac{1}{2}\left(\tau + \min\{|f(Y_0,T_1)|\} - 1\right) - 2\epsilon_0
\]
and using (B5)$_i$, we obtain
\[
\|Y_n(z_0) - Y_n(z_1)\| \geq 2(r + s) + 1 - 2\epsilon_0 > 2(r + s).
\tag{26}
\]
Indeed, one has \(\|Y_n(z_0)\| > r + s\) or \(\|Y_n(z_1)\| > r + s\).

This concludes the proof of (25). Thus, there is a polygon \(Q\) verify (a) and (c).

**Property (d).** Let \(z \in \text{Int} Q \setminus \text{Int} P\). We distinguish the five possible cases to prove this property:

- **Suppose that** \(z \notin (\bigcup_{k=1}^p D(p_i, \delta)) \cup (\bigcup_{k=1}^n Q_k^r)\). (P1), (B2) and (1) imply \(\|Y_n(z)\| \geq \|X(z)\| - 2\epsilon_0 > r - 2\epsilon_0 > r - 3\sqrt{sr}\).

- **Suppose that** \(z \in D(p_i, \delta) \setminus \bigcup_{k=1}^n Q_k^r\). We consider \(S(p_i)\) and use (P1), (B2), (P2)$_i$, then
\[
\|Y_n(z)\| \geq \|X_i(z)\| - 2\epsilon_0 \geq \|(X_i(z))_3| - 2\epsilon_0 = \|(X_{i-1}(z))_3| - 2\epsilon_0 \geq \|(X(z))_3| - 3\epsilon_0
\]
and, from (4) and the definition of \(S(p_i)\), one has
\[
\geq \|(X(p_i))_3| - 4\epsilon_0 = \|X(p_i)\| - 4\epsilon_0 > r - 4\epsilon_0 > r - 3\sqrt{sr}.
\]

- **Suppose that** \(z \in D(p_i, \delta) \cap Q_k^r\). Now, we consider the orthogonal frame \(T_1\), and obtain
\[
\|Y_n(z)\| \geq \|Y(z)\| - \epsilon_0 \geq \|(Y(z))_3| - \epsilon_0 = \|(Y_{i-1}(z))_3| - \epsilon_0 \geq \|(Y_0(z))_3| - 2\epsilon_0
\geq \|(Y_0(a_i))_3| - \|(Y_0(a_i))_3 - (Y_0(q_i))_3| - \|(Y_0(q_i))_3 - (Y_0(z))_3| - 2\epsilon_0
\]
Note that $|(Y_{0}(a_{i}))_{3,T_{i}}| = |Y_{0}(a_{i})|$. Since $z \in G_{i}$, we can use (P4) and (P5) to get the following:

$$> \lambda(r + s) - 2\lambda^{3}s' - 12\epsilon_{0} = \lambda(r + s) - 2\lambda^{3}\sqrt{\lambda^{4}(r + s)^{2} - r^{2} - 12\epsilon_{0}}$$

next we recall the definition of $\lambda$ on 1.b), and $s < r/100$, then

$$> r - 2\sqrt{(r + s)^{2} - r^{2} - 12\epsilon_{0}} > r - 3\sqrt{sr}.$$  

- Suppose that $z \in D(p_{i+1}, \delta) \cap Q_{i}$. We consider $T_{i}$. This case is similar to the former case:

$$\|Y_{n}(z)\| \geq |(Y_{0}(a_{i}))_{3,T_{i}}| - |(Y_{0}(a_{i+1}))_{3,T_{i}} - (Y_{0}(a_{i+1}))_{3,T_{i}}|| - |(Y_{0}(q_{i+1}))_{3,T_{i}} - (Y_{0}(z))_{3,T_{i}}| - 2\epsilon_{0}$$

Here, (P3) gives

$$> \lambda(r + s) - \epsilon_{0}(4\lambda^{2} + 17) - 2\lambda^{3}s' - 12\epsilon_{0} \geq r - 3\sqrt{sr}.$$  

Note that case $z \in D(p_{1}, \delta) \cap Q_{0}$ can be proved in the same way, considering $T_{n}$.

- Suppose that $z \in Q_{i} \setminus \cup_{k=1}^{n} D(p_{k}, \delta)$. We consider $T_{i}$. A similar computation gives

$$\|Y_{n}(z)\| \geq |(Y_{0}(a_{i}))_{3}| - |(Y_{0}(a_{i}))_{3} - (Y_{0}(q_{i}))_{3}| - |(Y_{0}(q_{i}))_{3} - (Y_{0}(z))_{3}| - 2\epsilon_{0}$$

$$\geq \lambda(r + s) - \lambda^{3}s' - 7\epsilon_{0} - |(Y_{0}(q_{i}))_{3} - (Y_{0}(z))_{3}|$$

and using properties (P1) and (4), we obtain

$$\geq \lambda(r + s) - \lambda^{3}s' - 10\epsilon_{0} \geq r - 3\sqrt{sr}.$$  

This completes the proof of the lemma.

**References**


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