STUDY OF THE OPTIMAL HARVESTING CONTROL AND THE OPTIMALITY SYSTEM FOR AN ELLIPTIC PROBLEM

M. DELGADO†, J. A. MONTERO‡, AND A. SUÁREZ†

Abstract. An optimal harvesting problem with a concave nonquadratic cost functional and a diffusive degenerate elliptic logistic state equation type is investigated. Under certain assumptions, we prove the existence and uniqueness of an optimal control. A characterization of the optimal control via the optimality system is also derived, which leads to approximating the optimal control.

Key words. degenerate logistic equation, singular eigenvalue problems, optimal control, optimality system

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1. Introduction. In this work we consider the optimal harvesting control of a species whose state is governed by the degenerate elliptic logistic equation; i.e.,

\begin{align}
-\Delta u &= (a - f)u^\alpha - bu^\beta \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega,
\end{align}

(1.1)

where \( \Omega \) is a bounded and regular domain of \( \mathbb{R}^N \), \( N \geq 1 \). Here \( a, f, \) and \( b \) are bounded functions. In particular, \( a \) is strictly positive, \( b \) is nonnegative and nontrivial, \( a - f \) can change sign, and \( \alpha \) and \( \beta \) satisfy

\( 0 < \alpha < 1, \quad \alpha < \beta. \)

The solutions of (1.1) can be regarded as the steady states solutions of the corresponding time-dependent model. In such a case, \( u(x) \) stands for the population density and \( \Omega \) for the inhabiting area. Since the population is subject to homogeneous Dirichlet boundary conditions, we are assuming that the environment surrounding \( \Omega \) is lethal. In such a model, the positive function \( b(x) \) describes the interspecific pressure of the species and \( a(x) \) represents the growth rate of the species. The function \( f(x) \) will be considered nonnegative and denotes the distribution of control harvesting of the species by reducing the growth rate. Equation (1.1), under the change of variables \( w^m = u \), is a particular case of

\begin{align}
-\Delta w^m &= (a - f)w - bw^2 \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial \Omega.
\end{align}

(1.3)

This model was introduced in population dynamics by Gurtin and MacCamy in [11] for describing the dynamics of biological populations whose mobility depends upon their

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†Dpto. Ecuaciones Diferenciales y Análisis Numérico, Fac. Matemáticas, C/ Tarfia s/n C. P. 41012, Univ. Sevilla, Spain (madelgado@us.es, suarez@us.es). The work of these authors was supported by the Spanish Ministry of Science and Technology under grant BFM2000-0797.
‡Dpto. Análisis Matemático, C. P. 18071, Univ. Granada, Spain (jmontero@goliat.ugr.es). The work of this author was partially supported by “Junta de Andalucía” (FQM116) and DGESIC (PB98-1343).
density. In this context, $m > 1$ (nonlinear slow diffusion) means that the diffusion is slower than in the linear case $m = 1$, giving rise to more realistic biological results; see [11].

One of the main differences between the degenerate case ($m > 1$) and the nondegenerate one ($m = 1$) is that in the first case the strong maximum principle does not hold in general. So, unlike the nondegenerate case, three kinds of solutions appear: the trivial solution, the strictly positive solutions (the species can survive in the whole domain), and the nonnegative and nontrivial solutions, which are zero in a region of $\Omega$. This region is called dead core.

Equation (1.1) has been studied previously for $b = 0$ in [1] and [2] and for $b$ strictly positive in [8] and [19] and references therein. However, very little is known in the case that $b$ can vanish in some region. In our knowledge, this problem has been analyzed only in [9] in the particular case $a - f$ equals a constant. We generalize these results and prove that there exists a maximal nonnegative solution of (1.1), which will be denoted by $u_f$. Moreover, when $f$ is such that the function $a - f$ is positive, we show that (1.1) possesses a unique positive solution which is linearly asymptotically stable.

After studying in detail the state equation, our main goal is to analyze the optimal control criteria, that is, maximize the payoff functional

$$J(f) := \int_{\Omega} (\lambda u_f h(f) - k(f)),$$

where $h$ and $k$ are regular functions, and $\lambda > 0$ will be considered as a parameter. Here, $J$ represents the difference between economic revenue measured by $\int_{\Omega} \lambda u_f h(f)$ and the control cost measured by $\int_{\Omega} k(f)$. The parameter $\lambda$ describes the quotient between the price of the species and the cost of the control. This functional includes the special case (quadratic functional)

$$h(t) = t \quad \text{and} \quad k(t) = t^2,$$

which seems to have been introduced in population dynamics in [17] (see also [6], [15], and references therein).

We say that $f \in L^\infty_+(\Omega)$ is an optimal control if

$$J(f) = \sup_{g \in L^\infty_+(\Omega)} J(g).$$

This control problem is a generalization of the one studied in detail in [6], [17], and [18], where $\alpha = 1$, $\beta = 2$, $h(t) = t$, and $k(t) = t^2$.

In [7], the authors analyzed the case $0 < \alpha < 1 \leq \beta$, $b$ strictly positive, and the cost functional (1.4) under more restrictive monotony assumptions on functions $h$, $k$. There, the controls are restricted to the set

$$\mathcal{D} := \{f \in L^\infty_+(\Omega) : f \leq a \quad \text{a.e. in } \Omega\}.$$ 

If $f \in \mathcal{D}$, then the maximal solution of (1.1) is strictly positive. In such a case, the existence and uniqueness of optimal control in $\mathcal{D}$ for $\lambda$ sufficiently small are proved.

In this work, we assume only (1.2), $b$ nonnegative and nontrivial, and our control space is $L^\infty_+(\Omega)$. So, $u_f$ can have dead cores depending on the control $f \in L^\infty_+(\Omega)$ chosen. In this framework, we show that there exists an optimal control in $L^\infty_+(\Omega)$ for any $\lambda > 0$. When $\lambda$ is smaller than a determined bound, we can express the optimal
control in terms of $u_f$ and, if $\lambda$ is small enough, then the optimal control is unique. In such a case, our assumptions imply that if $f$ is an optimal control, then the dead core for $u_f$ is empty. See [20], where a related problem is studied and where the dead core is allowed to exist.

In order to obtain the uniqueness result, we will use two different ways. First, we follow an argument described in [6] proving that the map $f \mapsto J(f)$ is Fréchet differentiable and strictly concave. The Fréchet derivability of the map $f \mapsto J(f)$ is rather more difficult than in the case $m = 1$, because it involves both linear elliptic and eigenvalue problems with potentials which blow up in a neighborhood of $\partial \Omega$. These difficulties have been solved by using results of singular eigenvalue problems from [4]; see also [12]. Second, we express the unique optimal control in terms of the solution of the optimality system, and we give an alternative proof of the uniqueness for the optimal control via the optimality system. This is an interesting point in the optimal control problems, because it allows us to approximate the optimal control by a constructive scheme which provides us with a sequence of functions converging to some special solutions of the optimality system. The uniqueness of solution of the optimality system was not considered in [17], but it was studied in [6] in the particular case $m = 1$ and the quadratic functional. Here, we present an alternative and shorter proof of the uniqueness, which can be applied to the case studied in [6]. Again, the second alternative presents another technical difficulty that must be overcome: the optimality system is a reaction-diffusion system with a singular reaction term. We present the subsupersolution method for these kinds of systems which provides us with an iterative method to approach the solution of the nonlinear system; see [5], [12] for the case of one equation.

An outline of this work is as follows: in Section 2 we introduce some notations and collect some results concerning the existence and uniqueness of the principal eigenvalue and the corresponding solution for linear elliptic problems with unbounded potentials. In Section 3 we study (1.1). We show the existence of a maximal nonnegative solution and, under stronger restrictions on the coefficients, the existence and uniqueness of a positive solution of (1.1). In Section 4 we prove the existence of optimal control for functional $J$ and show that for $\lambda$ sufficiently small the functional $J$ is Fréchet differentiable and strictly concave. Then, we deduce easily the uniqueness of optimal control. In the last section we characterize the optimal control. This characterization provides us with the optimality system. Finally, we prove the uniqueness of the positive solution of the optimality system and an iterative scheme based on alternating monotone sequences to approach its solution. As is remarked in recent related works (see [16], [17, Remark 4.1]), it is interesting to give conditions to guarantee the convergence of the method to the solution of the optimal control problem.

2. Preliminaries and notations. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial \Omega$. For any $f \in L^\infty(\Omega)$ we denote

$$
 f_M := \text{ess sup } f, \quad f_L := \text{ess inf } f,
$$

and define the sets

$$
 L^\infty_+(\Omega) := \{ f \in L^\infty(\Omega) : f_L \geq 0 \} \quad L^\infty_-(\Omega) := \{ f \in L^\infty(\Omega) : f_M \leq 0 \}.
$$

Moreover, we denote $C^1_0(\overline{\Omega}) = \{ u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial \Omega \}$ and by $P_+$ its nonnegative cone, whose interior is

$$
 \text{int}(P_+) := \{ u \in C^1_0(\overline{\Omega}) : u > 0 \text{ in } \Omega, \partial u/\partial n < 0 \text{ on } \partial \Omega \},
$$
where \( n \) is the outward unit normal at \( \partial \Omega \).

In this section we primarily consider the singular eigenvalue problem

\[
\begin{aligned}
\begin{cases}
-\Delta u + M(x)u = \sigma u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]

where

\[ (HM) \quad M \in L^\infty_{\text{loc}}(\Omega) \text{ verifying } M(x)d_\Omega(x) \in L^\infty(\Omega), \]

and \( d_\Omega(x) \) := \text{dist}(x,\partial \Omega).

The following result, whose proof can be found in [13], shows that (2.1) is well defined in \( H^1_0(\Omega) \).

**Lemma 2.1.** Let \( \varphi \in W^{1,q}_0(\Omega) \) for some \( 1 < q < \infty \). Then there exists a constant \( C > 0 \) such that

\[
\left\| \frac{\varphi}{d_\Omega} \right\|_q \leq C \| \varphi \|_{W^{1,q}(\Omega)}.
\]

Although (2.1) is not included in the singular eigenvalue problem studied in [4], we can do some minor changes to the proofs of Theorem 3.4 and Lemma 3.5 in [4] to conclude the existence and uniqueness of the principal eigenvalue of (2.1) and its associated eigenfunction. In the following result, we collect these results and some properties of the principal eigenvalue; see [7].

**Theorem 2.2.** Assume that \( M \) satisfies (HM). Then there exists a unique principal eigenvalue \((i.e., a real eigenvalue with an associated positive eigenfunction \( \varphi_1(-\Delta + M) \)) \. We denote it by \( \sigma_1(-\Delta + M) \). Moreover, \( \varphi_1(-\Delta + M) \in W^{2,p}(\Omega) \) for all \( p > 1 \), and so \( \varphi_1(-\Delta + M) \in \text{int}(P_+) \). Furthermore, we have the following:

1. Assume that \( M_i, i = 1, 2, \) satisfy (HM) and \( M_1 \leq M_2 \). Then

\[ \sigma_1(-\Delta + M_1) \leq \sigma_1(-\Delta + M_2). \]

2. Assume that \( M_n, M, n \in \mathbb{N}, \) satisfy (HM) with

\[
(2.2) \quad \int_\Omega M_n \varphi^2 \to \int_\Omega M \varphi^2 \quad \text{as } n \to \infty \quad \text{and for all } \varphi \in H^1_0(\Omega).
\]

Then,

\[ \sigma_1(-\Delta + M_n) \to \sigma_1(-\Delta + M) \quad \text{as } n \to \infty. \]

In the particular case \( M \equiv 0 \), we denote \( \sigma_1 := \sigma_1(-\Delta) \) and \( \varphi_1 = \varphi_1(-\Delta) \) normalized such that \( \| \varphi_1 \|_\infty = 1 \).

When \( M \) verifies (HM), the following strong maximum principle is satisfied.

**Lemma 2.3.** Let \( u \in W^{2,p}(\Omega) \cap C^1(\overline{\Omega}), p > 1, \) be such that \( u \geq 0 \) in \( \Omega, u \neq 0, \) and

\[ (-\Delta + M)u \geq 0 \quad \text{a.e. in } \Omega, \quad u \geq 0 \quad \text{on } \partial \Omega. \]

Then \( u(x) > 0 \) for all \( x \in \Omega \) and \( (\partial u/\partial n)(x_0) < 0 \) for all \( x_0 \in \partial \Omega, \) where \( u(x_0) = 0 \).

**Proof.** Assume there exists \( x_0 \in \Omega \) such that \( u(x_0) = 0 \). By hypothesis, we can take \( x_1 \in \Omega, \) where \( u(x_1) > 0 \) and a subdomain regular \( \Omega_1 \subset \Omega \) such that \( x_0, x_1 \in \Omega_1 \). But \( M \in L^\infty(\Omega_1), \) and so the strong maximum principle leads us to a contradiction.
On the other hand, applying Lemma 3.6 in [4] with \( \rho(s) = s^{-1} \), we get that \( (\partial u/\partial n)(x_0) < 0 \) for all \( x_0 \in \partial \Omega \) such that \( u(x_0) = 0 \). \( \square \)

The following technical result will help us to prove the positivity of the principal eigenvalue.

**Proposition 2.4.** Assume that \( M \) satisfies (HM) and that there exists \( \varphi \in W^{2,p}_0(\Omega) \cap C^0(\overline{\Omega}) \), \( p > N \) such that \( \varphi > 0 \) in \( \Omega \) and for all subdomain \( \Omega' \subset \overline{\Omega} \subset \Omega \) it holds \( (-\Delta + M)\varphi := F \) with \( F \in C^0(\Omega) \). Then, \( \sigma_1(-\Delta + M) > 0 \).

**Proof.** From the Krein–Rutman theorem, it is well known that if \(-\Delta + M \) satisfies the strong maximum principle in \( \Omega \), then \( \sigma_1(-\Delta + M) > 0 \). Let \( v \in W^{2,p}(\Omega) \cap C^1(\overline{\Omega}) \) be such that \( v \neq 0 \), and

\[
(-\Delta + M)v \geq 0 \quad \text{in a.e. } \Omega, \quad v \geq 0 \quad \text{on } \partial \Omega.
\]

We have to prove that \( v > 0 \) in \( \Omega \) and \( \partial v/\partial n(x) < 0 \) for all \( x \in \partial \Omega \) such that \( v(x) = 0 \). For each \( \epsilon > 0 \) and \( K > 0 \), we define

\[
w := v + \epsilon + \epsilon K \varphi \in C^0(\overline{\Omega})
\]

And so, for any \( \epsilon > 0 \), there exists \( \gamma(\epsilon) > 0 \) such that \( w > 0 \) in \( \Omega_\epsilon := \{ x \in \Omega : d_\Omega(x) < \gamma(\epsilon) \} \). Moreover,

\[
(-\Delta + M)w \geq \epsilon(M + KF) > 0 \quad \text{a.e. in } \Omega \setminus \overline{\Omega}_\epsilon
\]

for \( K \) sufficiently large. Moreover, since \( \varphi \) is a strict supersolution in \( \Omega \setminus \overline{\Omega}_\epsilon \), we can apply Corollary 2.4 in [3] and obtain that \( w > 0 \) in \( \Omega \setminus \overline{\Omega}_\epsilon \). Thus, we get that \( w > 0 \) in \( \Omega \setminus \overline{\Omega}_\epsilon \). Hence, \( w > 0 \) in \( \Omega \) for all \( \epsilon > 0 \), and we obtain that \( v \geq 0 \) in \( \Omega \). Now, it suffices to apply Lemma 2.3. \( \square \)

Given \( M \) verifying (HM) and \( f \in L^\infty(\Omega) \) we consider the problem

\[
\begin{cases}
-\Delta u + M(x)u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Observe that by Lemma 2.1, (2.4) is well defined in \( H^1_0(\Omega) \). The following result (whose proof can be found in [7]) shows that (2.4) possesses a unique solution in \( C^0(\overline{\Omega}) \); it provides a useful estimate and properties of the solution.

**Theorem 2.5.** Assume that \( M \) satisfies (HM) and \( \sigma_1(-\Delta + M) > 0 \). Then, there exists a unique solution \( u \in C^{1,\kappa}(\overline{\Omega}) \) for some \( \kappa \in (0,1) \) of (2.4). Moreover, there exists a constant \( K > 0 \) (independent of \( f \)) such that

\[
\|u\|_{C^{1,\kappa}(\overline{\Omega})} \leq K\|f\|_\infty.
\]

Furthermore, the following properties hold:

1. Consider \( f_i \in L^\infty(\Omega), i = 1,2, \) with \( f_1 \leq f_2 \), and let \( u_i, i = 1,2, \) be the respective solutions of (2.4). Then, \( u_1 \leq u_2 \).

2. Assume that \( M_i, i = 1,2, \) satisfy (HM), \( \sigma_1(-\Delta + M_1) > 0 \), and \( M_1 \leq M_2 \). Let \( u_i, i = 1,2, \) be the respective solutions of (2.4) with \( f \in L^\infty_+(\Omega) \). Then, \( u_1 \leq u_2 \).

Note: Similar results to the previous ones have been obtained in [12] when \( M \in C^1(\Omega) \), \( Md'_\Omega \in L^\infty(\Omega) \) for \( \gamma \in (0,2) \), and the operator is not necessarily self-adjoint.
3. The state equation. Consider the equation
\[
\begin{cases}
-\Delta u = (a - f)u^\alpha - bu^\beta & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases}
\] (3.1)
and assume that

(H1) $0 < \alpha < 1$, $\alpha < \beta$, $a, b \in L^\infty_+(\Omega)$, $f \in L^\infty(\Omega)$, $a_L > 0$, $(a - f)_M > 0$.

Observe that if $(a - f)_M \leq 0$, then, by the maximum principle, (3.1) does not possess a nonnegative and nontrivial solution. This justifies the hypothesis $(a - f)_M > 0$.

In order to study (3.1), we consider the porous medium equation
\[
\begin{cases}
-\Delta w = \mu w^\alpha & \text{in } \Omega, \\
\quad w = 0 & \text{on } \partial \Omega,
\end{cases}
\] (3.2)
where $\mu \in \mathbb{R}$. The following lemma holds.

Lemma 3.1. Assume $0 < \alpha < 1$. The porous medium equation (3.2) has a nontrivial and nonnegative solution if and only if $\mu > 0$. If $\mu > 0$, there exists a unique solution, denoted $w_\mu$, which is strictly positive and $w_\mu \in C^{2,\alpha}(\Omega)$. Moreover, it verifies
\[
\epsilon_0 \varphi_1 \leq w_\mu \leq K_0 e \quad \text{in } \Omega,
\] (3.3)
where $e$ is the unique positive solution of
\[
-\Delta e = 1 \quad \text{in } \Omega, \quad e = 0 \quad \text{on } \partial \Omega,
\]
and $\epsilon_0^{1-\alpha} = \mu/\sigma_1$, $K_0^{1-\alpha} = \mu \|e\|_\infty^\alpha$.

The results of the existence and uniqueness of a positive solution of (3.2) are well known; see [1], for instance. Estimate (3.3) can be obtained easily by the subsuper-solution method.

The following result shows that (3.1) has a maximal nonnegative solution.

Theorem 3.2. Assume (H1). There exists a unique maximal nonnegative solution $u_f$ of (3.1). Moreover, by elliptic regularity $u_f \in W^{2,p}(\Omega)$, for all $p > 1$, and so $u_f \in C^{1,\kappa}(\Omega)$, with $0 < \kappa \leq 1 - N/p$. Furthermore, we have the following a priori bound:
\[
\|u_f\|_\infty \leq ((a - f)_M \|e\|_\infty)^{1/(1-\alpha)}.
\] (3.4)
Finally, the map $f \mapsto u_f$ is nonincreasing.

Proof. Let $u$ be a weak solution of (3.1); then by (H1) and elliptic regularity it follows that $u \in C^1_0(\Omega)$. So, there exists $K > 0$ sufficiently large such that
\[
u \leq Ke \quad \text{in } \Omega,
\]
and the pair $(u, Ke)$ is a subsupersolution of (3.2) with $\mu = (a - f)_M$. By the uniqueness of positive solution of (3.2) it follows that
\[
u \leq w(a - f)_M.
\]
The existence of positive a priori bounds and that $u \equiv 0$ is a solution of (3.1) imply the existence of a nonnegative maximal solution of (3.1). By (3.3) we get the bound (3.4).
Let \( f_1, f_2 \in L^\infty(\Omega) \) be such that \( f_1 \leq f_2 \). It is clear that the pair \((u_{f_2}, Ke)\) is a subsupersolution of (3.1) for \( f = f_1 \) for \( K > 0 \) sufficiently large. So, there exists a solution \( u \) such that \( u_{f_2} \leq u \leq Ke \). The maximality of \( u_{f_1} \) completes the proof.

\( \square \)

Note: From (3.4) we get, for each maximal nonnegative solution of (3.1), a uniform upper bound, i.e.,

\[
(3.5) \quad u_f \leq ((a - f)_M \|e\|_\infty)^{1/(1-\alpha)} \leq (a_M \|e\|_\infty)^{1/(1-\alpha)} := K,
\]

for any \( f \in L^\infty(\Omega) \).

Observe that \( u_f \) would be eventually the trivial solution. The following result shows that this cannot occur in a subset of \( L^\infty(\Omega) \). We define

\[
C := \{ f \in L^\infty(\Omega) : (a - f)_L > 0 \}.
\]

In the following result we prove the existence and uniqueness of positive solution of (3.1) when \( f \in C \).

**Proposition 3.3.** Assume (H1), and let \( f \in C \). Then, there exists a unique nontrivial and nonnegative solution, \( u_f \), of (3.1). Moreover, \( u_f \) is strictly positive; in fact,

\[
(3.6) \quad \epsilon_f \varphi_1 \leq u_f \quad \text{in } \Omega,
\]

where \( \epsilon_f \) satisfies

\[
(3.7) \quad \epsilon_f^{1-\alpha} \sigma_1 + \epsilon_f^{\beta-\alpha} b_M = (a - f)_L.
\]

Moreover, \( u_f \) is linearly asymptotically stable, i.e.,

\[
(3.8) \quad \sigma_1 (-\Delta + M_f) > 0,
\]

where

\[
(3.9) \quad M_f := -\alpha(a - f)u_f^{\alpha-1} + \beta b u_f^{\beta-1}.
\]

Furthermore, the map \( f \in C \mapsto u_f \) is continuous.

Note: Observe that by (H1), (3.7) possesses a unique positive solution.

**Proof.** For the existence of solution, it is not hard to show that \((\epsilon_f \varphi_1, w_{(a - f)_M})\) is a subsupersolution of (3.1) for \( \epsilon_f > 0 \) defined in (3.7).

Observe that by the strong maximum principle for \( f \in C \), any nontrivial and nonnegative solution \( u \) of (3.1) is strictly positive; this means that \( u \in \text{int}(P_+) \).

The uniqueness of positive solution follows as in Theorem 1 of [9] and the continuity of the map \( f \mapsto u_f \) as in Theorem 3.3 of [7].

It remains to prove (3.8). First observe that \( M_f \) satisfies (HM). Indeed, by (3.6), there exists a positive constant \( C \) (independent of \( f \)) such that

\[
(3.10) \quad C \epsilon_f d_\Omega \leq u_f \quad \text{in } \Omega.
\]

Thus, since \( \alpha < 1 \), we have that

\[
|M_f|d_\Omega = u_f^{\alpha-1}d_\Omega - \alpha(a - f) + \beta b u_f^{\beta-\alpha} \leq C^{\alpha-1}u_f^{\alpha-1}d_\Omega - \alpha(a - f) + \beta b u_f^{\beta-\alpha} \leq K
\]

for some \( K > 0 \). Therefore, \( M_f \) satisfies (HM), and \( \sigma_1 (-\Delta + M_f) \) is well defined. Observe that \( u_f^p \in W^{2,p}_\text{loc}(\Omega) \cap C_0^0(\Omega) \) for all \( p > 1 \), and it satisfies

\[
(-\Delta + M_f)(u_f^p) = \alpha(1 - \alpha)u_f^{\alpha-2}\|\nabla u_f\|^2 + (\beta - \alpha)Bu_f^{\alpha+\beta-1} > 0 \quad \text{in } \Omega,
\]

and thus we can apply Proposition 2.4 and conclude that \( \sigma_1 (-\Delta + M_f) > 0 \). \( \square \)
4. Existence and uniqueness of optimal control. For \( \lambda > 0 \) we consider the functional \( J : L_+^\infty(\Omega) \to \mathbb{R} \),

\[
J(g) := \int_{\Omega} (\lambda h(g)u_g - k(g)),
\]

where \( h \in C^1(\mathbb{R}^+; \mathbb{R}^+) \), \( k \in C^2(\mathbb{R}^+; \mathbb{R}^+) \); \( h(s) = 0 \) if and only if \( s = 0 \), and \( k(s) = 0 \) if and only if \( s = 0 \). Function \( h \) is concave, and \( k \) is a strictly convex function satisfying \( k''(s) \geq k_0 > 0 \) for some \( k_0 \). Note that \( h', k' \) are Lipschitz continuous functions on a bounded set. We assume

\[
(H2) \quad \lim_{t \to 0} \frac{k(t)}{h(t)} = 0, \quad \lim_{t \to +\infty} \frac{k(t)}{h(t)} = +\infty.
\]

Observe that the particular case \( h(t) = t \) and \( k(t) = t^2 \), studied in \([6], [15], [17], \) and \([18]\), is in the setting of our functional. Also, we remove some hypotheses of nonmonotone type involving functions \( k \) and \( h \) considered in \([7]\). The idea will be to show that the integrand of functional \( J(f) \) must be positive if \( f \) is an optimal control.

In the first part of this section we want to prove the existence of the optimal control under hypothesis \((H2)\). First, we prove that the optimal controls are bounded.

**Lemma 4.1.** Assume \((H2)\). If \( f \in L_+^\infty(\Omega) \) is an optimal control, then

\[
(4.1) \quad \lambda u_f(x)h(f(x)) \geq k(f(x)) \text{ a.e. in } \Omega.
\]

Moreover, if \( f \in L_+^\infty(\Omega) \) is an optimal control, then

\[
0 \leq f \leq T_\lambda,
\]

where

\[
T_\lambda := \sup \left\{ t \in \mathbb{R}^+ : \frac{k(t)}{h(t)} = \lambda \mathcal{K} \right\},
\]

and \( \mathcal{K} \) is the uniform bound defined in \((3.5)\).

**Note:** By the hypotheses imposed on \( h \) and \( k \) and \((H2)\), it follows that \( T_\lambda > 0 \) and that \( T_\lambda \to 0 \) as \( \lambda \downarrow 0 \).

**Proof.** Suppose that \( f \in L_+^\infty(\Omega) \) is an optimal control and \((4.1)\) is not true. Then, there exists \( \Omega_1 \subset \Omega \) with \( |\Omega_1| > 0 \) (positive measure) such that

\[
(4.2) \quad \lambda u_f(x)h(f(x)) < k(f(x)) \quad \forall x \in \Omega_1.
\]

Now, by defining a new control \( \mathcal{f} \) as

\[
\mathcal{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega \setminus \Omega_1, \\ 0 & \text{if } x \in \Omega_1, \end{cases}
\]

and taking into account that \( u_{\mathcal{f}} \geq u_f \) in \( \Omega \), we obtain

\[
J(f) = \int_{\Omega_1} \lambda u_f(x)h(f(x)) - k(f(x)) + \int_{\Omega_1} \lambda u_f(x)h(f(x)) - k(f(x))
< \int_{\Omega_1} \lambda u_f(x)h(f(x)) - k(f(x)) \leq \int_{\Omega_1} \lambda u_{\mathcal{f}}(x)h(f(x)) - k(f(x))
= \int_{\Omega_1} \lambda u_{\mathcal{f}}(x)h(\mathcal{f}(x)) - k(\mathcal{f}(x)) = J(\mathcal{f}).
\]
But $f$ is an optimal control. So, previous inequality shows that (4.2) is absurd. It also shows that $f \leq T_\lambda$ follows from the definition of $T_\lambda$, Theorem 3.2, and (4.1).

**Theorem 4.2.** Assume (H2). There exists an optimal control; i.e., $f \in L^\infty_+(\Omega)$ such that

$$J(f) = \sup_{g \in L^\infty_+(\Omega)} J(g).$$

Moreover, the benefit is positive, i.e., $\sup_{g \in L^\infty_+(\Omega)} J(g) > 0$.

**Proof.** By (3.5) and Lemma 4.1, it follows that

$$s := \sup_{g \in L^\infty_+(\Omega)} J(g) < +\infty,$$

and so there exists a maximizing sequence $f_n \in L^\infty_+(\Omega)$. By similar reasoning to that used in the previous lemma, we can suppose that $0 \leq f_n \leq T_\lambda$. Then, there exists a subsequence, relabelled by $f_n$, such that

$$f_n \to f \in [0, T_\lambda] \text{ in } L^2(\Omega).$$

By (3.5), we can prove that

(4.3) \quad $u_{f_n} \to u_* \text{ in } H^1_0(\Omega),$

where $u_*$ is a positive solution of (3.1) (possibly not the maximal positive solution). In any case, we have $u_f \geq u_*$. Now, taking into account the concavity of the functions $h$ and $-k$, it follows that

$$J(f) \geq \lim \sup \int_\Omega \lambda h(f_n) u_{f_n} - k(f_n) = s,$$

and so we have the existence of an optimal control.

The optimal benefit is positive by following an argument like that used in [7]. In fact, it is clear, from the asymptotic properties of the functions $h$ and $k$, that $J(\epsilon) > 0$ by taking $\epsilon \in \mathbb{R}^+$ small enough.

Now, we are going to prove that, for $\lambda$ sufficiently small, there exists a unique optimal control. For that we will use the argument described in section 6 in [6]. In summary, by Lemma 4.1 we know that the optimal controls belong to a convex, $[0, T_\lambda]$. Moreover, we will show that $J$ is Fréchet continuously differentiable and strictly concave in $[0, T_\lambda]$. Hence, the uniqueness of optimal control is a direct consequence. The first step is the following result, which provides us with the Gâteaux derivative of the map $f \in \mathcal{C} \mapsto u_f \in \text{int}(P_+)$. Its proof is similar to Lemma 3.5 in [7], and so we omit it.

**Lemma 4.3.** Let $f \in \mathcal{C}$, $g \in L^\infty(\Omega)$, and $\epsilon \simeq 0$ be such that $f + \epsilon g \in \mathcal{C}$. Then,

$$\frac{u_{f+\epsilon g} - u_f}{\epsilon} \to \xi_{f,g} \text{ in } H^1_0(\Omega) \text{ as } \epsilon \to 0,$$

where $\xi_{f,g}$ is the unique solution of

(4.4) \quad $\begin{cases} -\Delta \xi + M_f(x) \xi = -gu_f^\alpha \text{ in } \Omega, \\ \xi = 0 \text{ on } \partial\Omega. \end{cases}$
Observe that (4.4) has a unique solution because \( \sigma_1(-\Delta + M_f) > 0 \) (see (3.8)) and Theorem 2.5.

Now, we can prove the following proposition (see Proposition 4.4 in [7]).

**Proposition 4.4.** Let \( J : \mathcal{C} \subset L^\infty(\Omega) \to \mathbb{R} \) be. Then \( J \) is Fréchet continuously differentiable and

\[
J'(f)(g) = \int_{\Omega} (\lambda h'(f) u_f - \lambda u_f^\alpha P_f - k'(f)) \, g \quad \forall f \in \mathcal{C} \forall g \in L^\infty(\Omega),
\]

where for any \( f \in \mathcal{C}, P_f \in C^1_0(\Omega) \) is the unique solution of

\[
\begin{aligned}
-\Delta P_f + M_f(x) P_f &= h(f) \quad \text{in } \Omega, \\
P_f &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

and \( M_f \) is defined in (3.9).

Note: Since \( M_f \) satisfies (HM) and by (3.8), it follows from Theorem 2.5 that the existence and uniqueness of \( P_f \in C^1_0(\Omega) \).

Observe that by the note following Lemma 4.1, there exists \( \lambda_0 > 0 \) such that

\[
a_L > T_\lambda \quad \text{for } \lambda < \lambda_0.
\]

Following the argument of Theorem 3.1 in [17] (using now (4.7) and Proposition 4.4) we obtain the following corollary.

**Corollary 4.5.** Assume (H2). Let \( f \in L^\infty_+(\Omega) \) be an optimal control. Then for \( \lambda < \lambda_0 \),

\[
k'(f) = \lambda(h'(f) u_f - u_f^\alpha P_f)^+.
\]

In order to prove that \( J \) is strictly concave in \([0, T_\lambda]\), we will show that maps involved in \( J' \) are Lipschitz continuous. This result was proven in [7] when \( \beta \geq 1 \). Since the Lipschitz character of the maps involved is crucial in this work (see, for example, the proof of Lemma 5.4), we present a complete proof of this result for the reader’s convenience.

**Theorem 4.6.** Assume (H2). There exists \( \Lambda > 0 \) such that for \( 0 < \lambda < \Lambda \) the maps

\[
f \in [0, T_\lambda] \mapsto u_f, \ P_f, \ u_f^\alpha P_f \in L^\infty(\Omega),
\]

are Lipschitz continuous, with the Lipschitz constants independent of \( \lambda \).

**Proof.** Let \( f, g \in [0, T_\lambda] \) be; by the monotony of the map \( f \mapsto u_f \), it follows that

\[
0 < u_{T_\lambda} \leq u_f, u_g \leq u_0.
\]

Moreover, for \( \lambda < \lambda_0 \) (defined in (4.7)), \( u_{T_\lambda} > 0 \), and so

\[
0 < u_{T_\lambda} \leq u_f, u_g \leq u_0
\]

for \( \lambda < \lambda_0 \). Hereafter, we take \( \lambda < \lambda_0 \). By the mean value theorem,

\[
u_f^\alpha - u_g^\alpha = \alpha \theta^\alpha - 1(f, g)(u_f - u_g), \quad u_f^\beta - u_g^\beta = \beta \eta^\beta - 1(f, g)(u_f - u_g)
\]

with

\[
0 < u_{T_\lambda} \leq \min\{u_f, u_g\} \leq \theta(f, g), \eta(f, g) \leq \max\{u_f, u_g\} \leq u_0.
\]

Let \( w := u_f - u_g \) be. Then, \( w \) satisfies

\[
\begin{aligned}
(-\Delta + N(f, g))w &= (g - f)u_f^\alpha \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where

\[ N(f, g) := -\alpha(a - f)\theta^{\alpha - 1}(f, g) + \beta b\eta^{\beta - 1}(f, g). \]

Using \( f \geq 0 \) and (4.8), it follows that

\[ N(f, g) \geq -\alpha a\theta^{\alpha - 1}(f, g) + \beta b\eta^{\beta - 1}(f, g) \geq m_\lambda, \]

where

\[ (4.9) \quad m_\lambda := \begin{cases} -\alpha au_\lambda^{\alpha - 1} + b\beta u_0^{\beta - 1} & \text{if } \beta \geq 1, \\ -\alpha au_\lambda^{\alpha - 1} + b\beta u_0^{\beta - 1} & \text{if } \beta < 1. \end{cases} \]

It is not hard to show that \( m_\lambda \) satisfies (HM). Moreover, we claim that as \( \lambda \downarrow 0 \),

\[ (4.10) \quad \int_\Omega m_\lambda \varphi^2 \to \int_\Omega (-\alpha au_\lambda^{\alpha - 1} + b\beta u_0^{\beta - 1}) \varphi^2 \quad \forall \varphi \in H^1_0(\Omega). \]

Indeed, for \( \varphi \in H^1_0(\Omega) \) and using (3.10) we have

\[ \int_\Omega (u_\lambda^{\alpha - 1} - u_0^{\alpha - 1}) \varphi^2 = \int_\Omega (u_\lambda^{\alpha} - u_\lambda u_0^{\alpha - 1}) \frac{\varphi}{u_\lambda} \leq C\epsilon \|u_\lambda^{\alpha} - u_\lambda u_0^{\alpha - 1}\|_\infty \int_\Omega \frac{\varphi}{u_\lambda} \varphi, \]

where \( \epsilon \) is defined in (3.7). By the continuity of the map \( f \mapsto u_f \), Lemma 2.1, and the fact that \( \epsilon \) does not tend to 0 as \( \lambda \downarrow 0 \), we obtain that

\[ \int_\Omega (u_\lambda^{\alpha - 1} - u_0^{\alpha - 1}) \varphi^2 \to 0 \quad \text{as } \lambda \downarrow 0. \]

Reasoning similarly with the other terms, (4.10) is proved. So, by Theorem 2.2 we obtain that

\[ (4.11) \quad \sigma_1(-\Delta + N(f, g)) \geq \sigma_1(-\Delta + m_\lambda) \to \sigma_1(-\Delta - \alpha au_0^{\alpha - 1} + b\beta u_0^{\beta - 1}) > 0 \]

as \( \lambda \downarrow 0 \). This last inequality follows by (3.8) because \( f \equiv 0 \in C \). Hence, using the monotony of the map \( \lambda \mapsto T_\lambda \), there exists \( \lambda_1 > 0 \) such that

\[ (4.12) \quad N(f, g) \geq m_\lambda \geq m_{\lambda_1}, \]

and

\[ (4.13) \quad \sigma_1(-\Delta + N(f, g)) \geq \sigma_1(-\Delta + m_{\lambda_1}) > 0 \quad \text{for } \lambda < \lambda_1. \]

So, by (4.12) we get

\[ (-\Delta + m_{\lambda_1})w \leq (g - f)u_g^\alpha, \]

and hence, using (4.13), Theorem 2.5, and (3.5), it follows that

\[ (4.14) \quad \|u_f - u_g\|_\infty = \|w\|_\infty \leq \|w\|_{C^1(\overline{\Omega})} \leq C\|f - g\|_\infty. \]

This shows that the map \( f \mapsto u_f \) is Lipschitz.

Now, take \( f \in [0, T_\lambda] \). Using the monotony of the map \( f \mapsto u_f \), we have that

\[ (4.15) \quad M_f \geq m_{\lambda_1}. \]
Thus, by Theorem 2.5 we obtain that

\[(4.16)\quad P_f \leq \mathcal{P} \quad \text{in } \Omega,\]

where \(\mathcal{P} \in C^1_0(\overline{\Omega})\) is the unique solution of

\[
\begin{cases}
-\Delta u + m_\lambda u = T \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

and \(T := \max_{r \in [0,T_\lambda]} h(r)\).

We will prove now that the map \(f \mapsto P_f\) is Lipschitz. Let \(f, g \in [0,T_\lambda]\) and \(z := P_f - P_g\) be. Then \(z\) satisfies

\[-\Delta z + M_f z = T(f,g) \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \partial \Omega,
\]

where

\[T(f,g) = h(f) - h(g) + P_g[\alpha(a-f)(u_f^{\alpha-1} - u_g^{\alpha-1}) - \beta b(u_f^{\beta-1} - u_g^{\beta-1})] + \alpha(g-f)P_g u_g^{\alpha-1}.\]

Applying again the mean value theorem, we get

\[(4.17)\quad u_f^{\alpha-1} - u_g^{\alpha-1} = (\alpha - 1)\xi^{\alpha-2}(f,g)(u_f - u_g),
\]

\[u_f^{\beta-1} - u_g^{\beta-1} = (\beta - 1)\eta^{\beta-2}(f,g)(u_f - u_g),
\]

\[0 < u_{T_\lambda} \leq \min\{u_f, u_g\} \leq \xi(f,g), \eta(f,g) \leq \max\{u_f, u_g\} \leq u_0.
\]

Hence,

\[T(f,g) = h(f) - h(g) + P_g[\alpha(a-1)(a-f)\xi^{\alpha-2} - \beta(\beta-1)b\eta^{\beta-2}](u_f - u_g) + \alpha(g-f)P_g u_g^{\alpha-1}.\]

By a similar argument to the one used in the proof of (4.14), we obtain

\[(4.18)\quad ||P_f - P_g||_\infty = ||z||_\infty \leq C||T(f,g)||_\infty.
\]

Since \(\mathcal{P} \in C^1_0(\overline{\Omega})\), it follows that

\[(4.19)\quad |\mathcal{P}(x)| \leq d_\Omega(x)||\mathcal{P}||_{C^1(\overline{\Omega})}.
\]

So, using (3.10), (4.16), and (4.19), we obtain

\[\|(f-g)P_g u_g^{\alpha-1}\|_\infty \leq C\|f - g\|_\infty\|P_g u_g^{\alpha-1}\|_\infty \leq C\|f - g\|_\infty||\mathcal{P}||_{C^1(\overline{\Omega})} \leq C\|f - g\|_\infty,
\]

with \(C\) independent of \(f\) and \(g\).

On the other hand, since \(u_f - u_g \in C^1_0(\overline{\Omega})\) and using (4.14), (4.16), (4.17), and (4.19),

\[\|(a-f)P_g \xi^{\alpha-2}(u_f - u_g)\|_\infty \leq C\||\mathcal{P}||_{C^1(\overline{\Omega})}||\xi^{\alpha-2}\|_\infty\|u_f - u_g\|_{C^1(\overline{\Omega})} \leq C\|f - g\|_\infty,
\]

with \(C\) independent of \(f\) and \(g\).

Analogously it can be treated the term \(P_g \eta^{\beta-2}(u_f - u_g)\). Then, since \(h\) is Lipschitz in \([0,T_\lambda]\) and by (4.18), it follows that the map \(f \mapsto P_f\) is Lipschitz.

Let \(f, g \in [0,T_\lambda]\) be. By (4.8), we have

\[\|(u_f - u_g)P_f\|_\infty = \|\alpha \xi^{\alpha-1} P_f(u_f - u_g)\|_\infty \leq C\||\mathcal{P}||_{C^1(\overline{\Omega})} f - g\|_\infty \leq C\|f - g\|_\infty,
\]
and so
\[ \| u_f^a P_f - u_g^a P_g \|_\infty \leq \| (u_f^a - u_g^a) P_f \|_\infty + \| u_g^a (P_f - P_g) \|_\infty \leq C \| f - g \|_\infty. \]

This completes the proof. \( \square \)

We can conclude the main result about uniqueness of optimal control of this section with the following theorem.

**Theorem 4.7.** Assume (H2). Then, there exists \( \Lambda_0 > 0 \) such that if \( \lambda < \Lambda_0 \), there exists a unique optimal control.

5. **The optimality system and the approximation to the optimal control.** In this section, we deduce the optimality system in the special cases \( h(t) = t \) and \( k(t) = t^2 \), which satisfy clearly (H2). The optimality system will be used to demonstrate the uniqueness of the optimal control in a different way and provides an iterative method to approach it. In this case, we know that
\[ T_\lambda = \lambda K \quad \text{and} \quad \lambda_0 = \frac{a_F}{K}, \]
where \( K \) is defined in (3.5). Moreover, by Corollary 4.5, for \( \lambda < \lambda_0 \), if \( f \) is an optimal control, then
\[ f = \frac{\lambda}{2} u_f^1 - u_f^{\alpha - 1} P_f. \] (5.1)

Let \( \psi \) be the unique positive solution of
\[ \begin{cases} -\Delta \psi + m_{\lambda_1} \psi = K \\ \psi = 0 \end{cases} \quad \text{in} \Omega, \quad \text{on} \partial \Omega, \] (5.2)
where \( m_{\lambda_1} \) is defined in (4.9) and satisfies (4.12) and (4.13). So, if \( f \) is an optimal control it follows by Lemma 4.1 that \( f \in [0, \lambda K] \). On the other hand, by (4.15) and Theorem 2.5, we get that
\[ P_f \leq \lambda \psi \quad \text{for} \lambda < \lambda_1. \] (5.3)

As a consequence of (5.3) we obtain the following proposition (see Proposition 5.2 and Corollary 5.3 in [7]).

**Proposition 5.1.** Assume (H1). There exists a constant \( \Lambda_1 > 0 \) such that if \( \lambda \leq \Lambda_1 \), then
\[ P_f \leq u_f^{1-\alpha}. \] (5.4)

So, if \( f \) is an optimal control, we have that
\[ f = \frac{\lambda}{2} u_f^1 - u_f^{\alpha - 1} P_f. \] (5.5)

As a consequence, any optimal control \( f \) may be expressed as in (5.5), where the pair \((u_f, P_f) := (u, P)\) satisfies
\[ \begin{cases} -\Delta u = u^a (a - \frac{1}{2} u + \frac{1}{2} u^\alpha P - bu^{\beta - \alpha}) \quad \text{in} \Omega, \\ -\Delta P + (-a uu^{\alpha - 1} + \beta bu^{\beta - 1}) P = \frac{1}{2} (u - u^0 P (1 + \alpha) + \alpha uu^{\alpha - 1} P^2) \quad \text{in} \Omega, \\ u = P = 0 \quad \text{on} \partial \Omega, \end{cases} \] (5.6)
and \( u > 0 \).

The former result says that, when \( \lambda \) is small enough, if \( f \) is an optimal control, then \((u_f, P_f)\) is a solution of (5.6). We are going to prove now that, for a range of \( \lambda \), there exists a unique positive solution of (5.6) verifying \( u^{1-\alpha} \geq P \), and so the unique optimal control will be

\[
f = \frac{\lambda}{2}(u - u^\alpha P).
\]

**Theorem 5.2 (uniqueness of optimal control).** Assume (H1). There exists \( \Lambda_2 > 0 \) such that for \( \lambda \leq \Lambda_2 \), (5.6) possesses a unique positive solution \((u, P)\) satisfying \( u^{1-\alpha} \geq P \).

**Proof.** We define the following map:

\[
T : I := [0, \lambda\mathcal{K}] \subset L^\infty_+(\Omega) \mapsto L^\infty_+(\Omega), \quad f \mapsto T(f) = \frac{\lambda}{2}(u_f - u^\alpha_f P_f).
\]

By Theorem 4.6, for \( \lambda < \Lambda \), \( T \) is a Lipschitz continuous function with Lipschitz constant of type \( \lambda\mathcal{L}/2 \), where \( \mathcal{L} \) is the corresponding one for the function \( f \mapsto u_f - u^\alpha_f P_f \). So, we can choose \( \Lambda_2 := \min\{\Lambda, \frac{\lambda}{2}\} \) such that for \( \lambda \leq \Lambda_2 \), \( T \) is a contractive function.

Assume that there exist two positive solutions \((u_i, P_i), i = 1, 2\), of (5.6) with \( u^{1-\alpha} \geq P \). We define

\[
f_i = \frac{\lambda}{2}(u_i - u^\alpha_i P_i) \in I, \quad i = 1, 2.
\]

Hence, by (5.6) and Proposition 3.3 we have that

\[
u_i = u_{f_i}, \quad P_i = P_{f_i}, \quad \Rightarrow T(f_i) = f_i, \quad i = 1, 2.
\]

Since \( T \) is contractive, it follows that \( f_1 = f_2 \), and again by Proposition 3.3 we have that \( u_{f_1} = u_{f_2} \); hence \( u_1 = u_2 \), and so \( P_1 = P_2 \). This completes the proof. \( \square \)

Now, we use the optimal control characterization obtained by formula (5.5) to give an iterative procedure to approach it. The idea is to be near the solution of the optimality system by sub- and supersolutions (see other papers related with similar problems [6], [14], [15], [17]). The interest here, besides the degeneration of the second equation of the optimality system, is that we prove the convergence of the method by a different argument than that used in the mentioned references. We start this part with some notation. We define, for simplicity, the following functions:

\[
B(x, u, p) = \left[ a(x) - \frac{\lambda}{2}(u - u^\alpha p) \right] u^\alpha - b(x)u^\beta \text{ for } x \in \Omega,
\]

and, taking into account the monotony properties of the second equation of optimality system (5.6), we define

\[
D(x, u, p) = \begin{cases}
\frac{\lambda}{3} u - \beta pb(x) u^{\beta-1} & \text{if } \beta < 1, \ 2\alpha - 1 < 0, \\
\frac{\lambda}{3} u - \beta pb(x) u^{\beta-1} + \frac{\lambda}{2}\alpha u^{2\alpha-1} p^2 & \text{if } \beta < 1, \ 2\alpha - 1 \geq 0, \\
\frac{\lambda}{2} u & \text{if } \beta \geq 1, \ 2\alpha - 1 < 0, \\
\frac{\lambda}{2} u + \frac{\lambda}{2}\alpha u^{2\alpha-1} p^2 & \text{if } \beta \geq 1, \ 2\alpha - 1 \geq 0
\end{cases}
\]
and

\[
C(x, u, p) = \begin{cases} 
  p(\alpha(x)u^{\alpha-1} - \frac{1}{2}u^\alpha(\alpha + 1)) + \frac{1}{2}\alpha u^{2\alpha-1}p^2 & \text{if } \beta < 1, \ 2\alpha - 1 < 0, \\
  p(\alpha(x)u^{\alpha-1} - \frac{1}{2}u^\alpha(\alpha + 1)) & \text{if } \beta < 1, \ 2\alpha - 1 \geq 0, \\
  p(\alpha(x)u^{\alpha-1} - \beta b(x)u^{\beta-1}) & \text{if } \beta \geq 1, \ 2\alpha - 1 < 0, \\
  -\frac{1}{2}u^\alpha(\alpha + 1)) + \frac{1}{2}\alpha u^{2\alpha-1}p^2 & \text{if } \beta \geq 1, \ 2\alpha - 1 \geq 0.
\end{cases}
\]

We are interested in the solutions, \((u, p)\), of optimality system (5.6) that satisfy \(u_{\lambda,C} \leq u \leq u_0\) and \(0 \leq p \leq \lambda \psi\) (recall (5.3)). Consequently, we can reduce the study for solutions that satisfy \((u, p) \in [u_{\lambda,C}, u_0] \times [0, \lambda \psi]\). Therefore, there exist a constant \(K > 0\) and a function \(M_1(x), x \in \Omega\), satisfying hypothesis (HM) such that

\[
\begin{align*}
B(x, u, p) + Ku^\alpha & \quad (\nearrow u, \not\nearrow p), \\
C(x, u, p) + M_1(x)p & \quad (\nearrow u, \not\nearrow p), \\
D(x, u, p) + M_1(x)p & \quad (\nearrow u, \not\nearrow p);
\end{align*}
\]

i.e., \(B(x, u, p) + Ku^\alpha\) is increasing in \(u\) for fixed \(x \in \Omega\) and \(0 \leq p \leq \lambda \psi\) and increasing in \(p\) for fixed \(x \in \Omega\) and \(u_{\lambda,C} \leq u \leq u_0\). The other terms are interpreted analogously.

Definition 5.3 (subsupersolutions). The functions \(u_1, p_1, u^1, p^1 \in L^\infty(\Omega) \cap H^1(\Omega)\) are said to be a system of subsupersolutions for optimality system (5.6) if they verify

\[
\begin{cases}
  u_1(x) \leq u^1(x), & p_1(x) \leq p^1(x) \quad \text{a.e. in } \Omega, \\
  p_1 \leq 0 & \leq p^1 \quad \text{on } \partial \Omega,
\end{cases}
\]

and there exists a positive constant \(k\) such that

\[
0 < kd_\Omega(x) \leq u_1(x) \leq u^1(x) \quad \text{a.e. in } \Omega
\]

and, for any \(\phi \in H^1_0(\Omega), \phi \geq 0,\)

\[
\begin{align*}
\int_{\Omega} \nabla u^1 \cdot \nabla \phi & \geq \int_{\Omega} B(x, u^1, p^1)\phi, \\
\int_{\Omega} \nabla u_1 \cdot \nabla \phi & \leq \int_{\Omega} B(x, u_1, p_1)\phi, \\
\int_{\Omega} \nabla p^1 \cdot \nabla \phi & \geq \int_{\Omega} C(x, u^1, p^1)\phi + \int_{\Omega} D(x, u^1, p^1)\phi, \\
\int_{\Omega} \nabla p_1 \cdot \nabla \phi & \leq \int_{\Omega} C(x, u_1, p_1)\phi + \int_{\Omega} D(x, u_1, p_1)\phi.
\end{align*}
\]

Recall that a function \(v \in H^1(\Omega)\) is said to be less than or equal to \(w \in H^1(\Omega)\) on \(\partial \Omega\) when \((v - w) \uparrow = \max\{0, v - w\} \in H^1_0(\Omega)\).

It is not difficult to prove that, under the hypothesis of Theorem 5.2, there exists a \(\Lambda_3 > 0\) such that if \(\lambda \leq \Lambda_3\), then the functions

\[
u_1 = u_{\lambda,C}, \quad p_1 \equiv 0, \quad u^1 = u_0, \quad p^1 = \lambda \psi,
\]

are a system of subsupersolutions for the optimality system (5.6) in the sense of Definition 5.3. We show only the case \(p^1 = \lambda \psi\) when \(\beta \geq 1, \ 2\alpha - 1 \geq 0\). The other cases are similar. It is not hard to show that \(p^1 = \lambda \psi\) is a supersolution if

\[
\lambda(K - m_{\lambda, \psi}) \geq \lambda \psi \left[ \alpha u_{\lambda,C}^{\alpha-1} - \beta b_{\lambda,C}^{\beta-1} - \frac{\lambda}{2} u_{\lambda,C}^\alpha(\alpha + 1) \right] + \frac{\lambda}{2} u_0 + \frac{\lambda}{2} \alpha u_0^{2\alpha-1}(\lambda \psi)^2.
\]
or, equivalently,
\[ K \geq \psi \left[ m_{\lambda_1} + \alpha uu_{x,k}^{\alpha-1} - \beta uu_{x,k}^{\beta-1} \right] - \frac{\lambda}{2} u_{x,k}^{2} (\alpha + 1) \psi + \frac{\alpha}{2} \lambda^{2} u_{0}^{2\alpha-1} \psi^{2}. \]

Recalling the definition of \( m_{\lambda_1} \), for \( \lambda \leq \lambda_1 \) we have that \( m_{\lambda_1} + \alpha uu_{x,k}^{\alpha-1} - \beta uu_{x,k}^{\beta-1} \leq 0 \), and by (3.5) \( u_0 \leq K \); it is enough to take \( \lambda \) small to obtain that \( p^1 \) is a supersolution.

Now, we define by induction, for \( n \geq 2 \), the sequences \( \{u_n\}, \{u^n\}, \{p_n\}, \{p^n\} \in H_0^1(\Omega) \) as
\[
\begin{align*}
(5.10) & \quad -\Delta u_n + K(u_n)^\alpha = B(x, u_{n-1}, p_{n-1}) + K(u_{n-1})^\alpha \quad \text{in } \Omega, \\
(5.11) & \quad -\Delta u^n + K(u^n)^\alpha = B(x, u^{n-1}, p^{n-1}) + K(u^{n-1})^\alpha \quad \text{in } \Omega, \\
(5.12) & \quad -\Delta p_n + M_1(x)p_n = C(x, u^n, p_{n-1}) + D(x, u_n, p_{n-1}) + M_1(x)p_{n-1} \quad \text{in } \Omega, \\
(5.13) & \quad -\Delta p^n + M_1(x)p^n = C(x, u_n, p^{n-1}) + D(x, u^n, p^{n-1}) + M_1(x)p^{n-1} \quad \text{in } \Omega.
\end{align*}
\]

Observe that sequences \( \{u_n\}, \{u^n\} \), are well defined because the map \( u \mapsto Ku^\alpha \) is continuous and strictly increasing and such that \( B(x, u, p) + Ku^\alpha \) is also increasing in \( u \) when the other variables are fixed. (See more details in [5], [10].)

On the other hand, fixed \( u_1, u^1 \) and, thanks to (5.8), the problems (5.12) and (5.13) are in the setting of (2.4), and so by Theorem 2.5 it follows the existence and uniqueness of \( p_2 \) and \( p^2 \) and such that \( p_2 \leq p^2 \) and so on. We note that for (5.10)–(5.11) and (5.12)–(5.13), the sub supersolutions method works (cf. [12]). The standard method gives us the following order relation:
\[
\begin{align*}
& \quad u_1 \leq u_2 \leq \cdots \leq u_n \leq u^n \leq u^{n-1} \leq \cdots \leq u^1, \\
& \quad p_1 \leq p_2 \leq \cdots \leq p_n \leq p^n \leq p^{n-1} \leq \cdots \leq p^1
\end{align*}
\]
and
\[
u_n \not\rightarrow u, \ u^n \searrow v, \ p_n \not\rightarrow p, \ p^n \searrow q \quad \text{(pointwise)},
\]
where \( u, v, p, q \) belong to \( C^{1,\delta}(\Omega) \), for any \( \delta \in (0, 1) \), and satisfy the system
\[
\begin{align*}
\begin{cases}
-\Delta u = B(x, u, p) & \text{in } \Omega, \\
-\Delta v = B(x, v, q) & \text{in } \Omega, \\
-\Delta q = C(x, u, q) + D(x, u, p) & \text{in } \Omega, \\
u = v = p = q = 0 & \text{on } \partial\Omega
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
& \quad u_1 = u_{x,k} \leq u, \ v \leq u^1 = u_0 \quad \text{in } \Omega \\
& \quad p_1 = 0 \leq p, \ q \leq p^1 = \lambda \psi \leq u_{x,k}^{-1} \quad \text{in } \Omega.
\end{align*}
\]
Clearly, if \( (u, p) \) is the solution of optimality system (5.6), then \( (u, u, p, p) \) is a solution of (5.14). So, to complete the iterative approximation and the convergence of the sequences \( \{u_n\}, \{u^n\}, \{p_n\}, \{p^n\} \) to the unique solution, \( (u, p) \), of the optimality system, it is sufficient to prove the uniqueness of the solution for system (5.14), under conditions (5.15). To do it, we need the following technical lemma.

**Lemma 5.4.** Assume (H1). Then
\[
\forall f, g \in [0, \lambda k] \subset L^\infty_+(\Omega),
\]
it is possible to define the function $P_{f,g}$ as the unique positive solution of the problem

\begin{equation}
\begin{cases}
-\Delta P = C(x, u_f, P) + D(x, u_g, P) & \text{in } \Omega, \\
P = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

satisfying

\begin{equation}
0 \leq P_{f,g} \leq \lambda \psi,
\end{equation}

provided that the parameter $\lambda$ is small enough and the function $\psi$ is defined in (5.2). Moreover, the map defined before, $(f, g) \in [0, \lambda K] \times [0, \lambda K] \mapsto P_{f,g} \in L^\infty(\Omega)$, is Lipschitz continuous.

An analogous result is obtained interchanging $u_f$ and $u_g$ in (5.16).

**Proof.** We consider the case $\beta \geq 1$, $2\alpha - 1 \geq 0$. The other cases have similar proofs. Observe that, in this case, (5.16) is

\begin{equation}
\begin{cases}
-\Delta P + \left[ -\alpha a u_f^{\alpha - 1} + \beta b u_f^{\beta - 1} + \frac{\lambda}{2} u_f^2 (1 + \alpha) \right] P = \frac{\lambda}{2} u_g + \frac{\lambda}{2} \alpha u_f^{2 \alpha - 1} P^2 & \text{in } \Omega, \\
P = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}

Taking into account Theorem 2.5, condition (5.17), and the definition of the function $\psi$, we can use the subsupersolution method with $p^* \equiv 0$ as subsolution and $p^* \equiv \lambda \psi$ as supersolution, provided $\lambda > 0$ small. Thus, the existence of positive solution of (5.18) is proved. The uniqueness of a contradiction argument follows. Suppose that $P, Q$ are two solutions under above requirements; then $P - Q$ satisfies

\begin{equation}
(-\Delta + M_1(x))(P - Q) = 0,
\end{equation}

where

\begin{equation}
M_1 = -\alpha a u_f^{\alpha - 1} + \beta b u_f^{\beta - 1} + \frac{\lambda}{2} u_f^2 (1 + \alpha) - \frac{\lambda}{2} \alpha u_f^{2 \alpha - 1} (P + Q).
\end{equation}

Observe that $M_1$ satisfies (HM). Now, using that $P, Q \leq \lambda \psi$, we obtain that

\begin{equation}
M_1 \geq -\alpha a u_f^{\alpha - 1} + \beta b u_f^{\beta - 1} + \frac{\lambda}{2} u_f^2 (1 + \alpha) - \lambda^2 \alpha \psi^2 u_f^{2 \alpha - 1}.
\end{equation}

and so we can prove the existence of a function $N$ satisfying (HM) and $\lambda_0 > 0$ such that for $\lambda \leq \lambda_0$

\begin{equation}
M_1 \geq N \quad \text{in } \Omega \quad \text{and} \quad \sigma_1(-\Delta + N) > 0.
\end{equation}

Hence, the previous equation has the unique solution $(P - Q) \equiv 0$.

To show the Lipschitzian character of the map $(f, g) \mapsto P_{f,g}$, let $P_{f,g}$ be the solution of problem (5.18) satisfying (5.17). Denote $\overline{q} = P_{f,\overline{g}}$ and $q = P_{f,g}$. Then, some calculus gives

\begin{equation}
(-\Delta + M(x))(q - \overline{q}) = R_{f,\overline{g},\overline{g}} := \alpha a q u_f^{\alpha - 1} - u_f^{2 \alpha - 1} - \beta b q u_f^{\beta - 1} - \beta b \overline{q}^{\beta - 1} - \frac{\lambda}{2} (\alpha + 1) q (u_f^{\alpha - 1} - u_f^{\alpha - 1}) + \frac{\lambda}{2} (u_g - u_f) + \frac{\lambda}{2} \alpha q^2 (u_f^{2 \alpha - 1} - u_f^{2 \alpha - 1}).
\end{equation}
where

\[ M = -\alpha a v^{-1} + \beta b u^{-1} + \frac{\lambda}{2} (\alpha + 1) w^{-1} - \frac{\lambda}{2} \alpha (q + \eta) u^{-1}. \]

As in the proof of (5.19), we can prove the existence of a function \( N \) satisfying (HM) such that for \( M \geq N \) in \( \Omega \) and \( \sigma_1(-\Delta + N) > 0 \) for small \( \lambda \). Thus, by Theorem 2.5 we get that

\[ \|g - \eta\| = \|P_{f,g} - P_{\overline{f},\overline{g}}\| \leq \|P_{f,g} - P_{\overline{f},\overline{g}}\|_{C^1(\overline{\Omega})} \leq C\|R_{f,\overline{f},g,\overline{g}}\|. \]

Now, using a similar argument to the one used in the proof of Theorem 4.6 to obtain a bound of \( \|T(f,g)\|_\infty \), we have

\[ \|\alpha a (u_{f}^{\alpha - 1} - u_{\overline{f}}^{\alpha - 1}) + \beta b (u_{f}^{\beta - 1} - u_{\overline{f}}^{\beta - 1}) + \frac{\lambda}{2} (\alpha + 1) q (u_{f}^{\alpha} - u_{\overline{f}}^{\alpha})\| \leq C\|f - \overline{f}\|_\infty, \]

\[ \|\alpha a q (u_{f}^{\alpha - 1} - u_{\overline{f}}^{\alpha - 1}) + \beta b q (u_{f}^{\beta - 1} - u_{\overline{f}}^{\beta - 1}) + \frac{\lambda}{2} (\alpha + 1) q (u_{f}^{\alpha} - u_{\overline{f}}^{\alpha})\| \leq C\|g - \overline{g}\|_\infty, \]

and so \( \|R_{f,\overline{f},g,\overline{g}}\| \leq C \{\|f - \overline{f}\|_\infty + \|g - \overline{g}\|_\infty\} \). Finally, we have

\[ (5.20) \quad \|P_{f,g} - P_{\overline{f},\overline{g}}\| \leq C \{\|f - \overline{f}\|_\infty + \|g - \overline{g}\|_\infty\} \]

for a convenient positive constant \( C \). \( \square \)

**Theorem 5.5.** Assume (H1). There exists a positive constant \( \Lambda_4 \) such that if \( \lambda \leq \Lambda_4 \), then the system (5.14)-(5.15) has a unique solution.

**Proof.** The main idea is simple. We will use the Lipschitzian character of the solutions of system (5.14)-(5.15) with respect to the controls and an argument similar to the one used in Theorem 5.2.

Suppose \((u_i, v_i, p_i, q_i)\), for \( i = 1, 2 \), are two solutions of system (5.14)-(5.15). We define, for \( i = 1, 2 \),

\[ f_i = \frac{\lambda}{2} [u_i - u_{i,\alpha} p_i], \quad g_i = \frac{\lambda}{2} [v_i - v_{i,\alpha} q_i]. \]

Now, taking into account the previous notation, we have for \( i = 1, 2 \),

\[ u_i = u_{f_i}, \quad v_i = u_{g_i}, \quad p_i = P_{g_i, f_i}, \quad q_i = P_{f_i, g_i}, \]

and

\[ f_i = \frac{\lambda}{2} [u_i - u_{g_i} P_{f_i, g_i}], \quad g_i = \frac{\lambda}{2} [u_i - u_{f_i} P_{g_i, f_i}]. \]

We know (recall Theorem 4.6 and Lemma 5.4) that the operator \( T : [0, \Lambda K] \times [0, \Lambda K] \to L^\infty(\Omega) \times L^\infty(\Omega) \), defined as

\[ T(f,g) = \left( \frac{\lambda}{2} [u_f - u_{\overline{g}} P_{f, g}], \frac{\lambda}{2} [u_g - u_{\overline{f}} P_{g, f}] \right), \]

is Lipschitz continuous, with constant \( \Lambda K/2 \), where \( C > 0 \) is the Lipschitz constant of \( (f, g) \mapsto (u_f - u_{\overline{g}} P_{f, g}, u_g - u_{\overline{f}} P_{g, f}) \) and verifies \( T(f_i, g_i) = (f_i, g_i) \). Therefore, by taking \( \lambda < \min\{\Lambda_1, \frac{C}{2}\} \), \( T \) is a contraction and consequently has an unique fixed point. So, \((f_1, g_1) = (f_2, g_2)\). Then, we have \( u_1 = u_2, v_1 = v_2, \) and finally \( p_1 = p_2 \) and \( q_1 = q_2 \). \( \square \)
REFERENCES