ON THE DEFINITION OF COHERENCE MEASURE FOR FUZZY SETS

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Received 17 March 2003
Revised 5 November 2003

Studying comparison methods for fuzzy sets is an essential task for the good understanding of the underlying theory in this field. Most of these tools deal with fuzzy sets from the view of similarity, order relationships and so forth. In this paper however, based on a former comparison measures introduced by the authors, the so called Coherence Measures, the extension and analysis of these tools to a measurable Lebesgue set X is carried out. Furthermore we present how coherence measures could be linked to the Fishburn-Yager’s ambiguity measures. Besides, two methods for constructing coherence measures, one from ambiguity measures and another from metrics on \( P(X) \), the set of fuzzy sets on X, are shown and exemplified by a variety of measures and metrics. Finally some illustrative examples testing the coherence measures introduced are provided.

Keywords: Coherence measures; ambiguity measures; fuzzy sets.

1. Introduction

Comparison methods for either finite or infinite fuzzy sets present a number of missfunctions that depends on a variety of different causes. One of these causes is because of the fact that generally fuzziness measures and ambiguity measures [4,5,9,10] are studied independently of the similarity measures. However these kind of measures can not be isolated among them, rather it would be necessary to provide tools allowing for measuring similarity, in the sense of likeness, between two fuzzy sets taking into
account from the beginning the fuzziness degree of the fuzzy sets as a very important point. In this way it would be needed to have a kind of measure that would assess this similarity, and simultaneously would take into account the degree of fuzziness between the concerned fuzzy sets. The case in which the fuzzy sets involved in this problem are finite ones was studied in [6,8] by introducing the so called coherence measures. These measures provide information on both the similarity and the ambiguity at once. Basically, these measures explore the significance of the following idea: “coherence between A and the opposite to B is the opposite to coherence between A and B”. The aim of this paper, based upon [7], is to extend the concept of coherence measures from finite sets to measurable Lebesgue fuzzy sets, and then to study the properties and characteristics of the corresponding measures. From this point of view, Section 2 introduces Coherence Measures and shows some properties being valid for both finite and non finite fuzzy sets. In Section 3 the relationship between coherence measures and ambiguity ones is analysed. Coherence measures construction methods for the non finite case are studied, and conditions for the extension of an ambiguity measure to a coherence one are provided by the so called Theorem of Extension. As an illustration of these results, last section shows four applications of this Theorem of Extension by presenting some theoretic and practical examples. Among them, the analysis carried out on the ambiguity measures of Yager [8,9,10] and of De Luca and Termini [2] is to be remarked. Finally, last example, shows the meaning of the coherence measures by means of a practical example.

2. Coherence Measures

**Definition 1.** Let $X$ be a measurable set and $\mathcal{P}(X)$ the set of fuzzy sets on $X$. One says that $\cohe: \mathcal{P}(X) \times \mathcal{P}(X) \to [0,1]$ is a **coherence measure** on $\mathcal{P}(X)$ if and only if the three following axioms hold:

C1) $\cohe(A,B) = \cohe(B,A)$

C2) $\cohe(A,B^c) = 1 - \cohe(A,B)$

C3) $\cohe(\emptyset,X) = 0$

It is clear that C1 shows the map $\cohe$ as a symmetric measure. C2 presents the basic idea on the concept of coherence measure: if the term coherence is meant as the possibility of coexistence of two evaluations, then it is clear that (in the $[0,1]$ interval) such possibility between $A$ and $B^c$ is the opposite to such another between $A$ and $B$. Finally C3 shows that the minimum coherence is to be attained when the sets $\emptyset$ and $X$ are compared. In the following the main results that can be obtained from these axioms are shown (an asterisk will be used to distinguish those results valid for both the finite and non finite case). The proofs are trivial. 

**Remark.** Let $X$ be a measurable Lebesgue set with a finite positive measure $m$. If $\chi(X)$ denotes the characteristic function of $X$,

$$m = \lambda (X) = \int_X \chi(x)dx$$
Then, as it is well known, there is an interval $I \subseteq X$ such that

$$0 < \lambda (I) = \int_I dx < \lambda (X) = \int_X dx$$

**Lemma 2 (\text{\textdagger}).** Let $\text{cohe}:\mathcal{P}^c(X) \times \mathcal{P}^c(X) \to [0,1]$ be a coherence measure then:

a) $\text{cohe}(A^c, B^c) = \text{cohe}(A, B)$  
b) $\text{cohe}(\emptyset, \emptyset) = \text{cohe}(X, X) = 1$  
c) If $A^c(x) = 0.5 \ \forall x$, then $\forall A \in \mathcal{P}^c(X)$ $\text{cohe}(A, A^c) = 0.5$

The following lemma is a negative result on the monotony of these measures.

**Lemma 3 (\text{\textdagger}).** Let $\text{cohe}:\mathcal{P}^c(X) \times \mathcal{P}^c(X) \to [0,1]$ be a coherence measure. Then $\forall A, B, C, D \in \mathcal{P}^c(X)$, neither is true that

(1) If $A \subseteq B$ and $C \subseteq D$ then $\text{cohe}(A, C) \leq \text{cohe}(B, D)$  
nor

(2) If $A \subseteq B$ and $C \subseteq D$ then $\text{cohe}(A, C) \geq \text{cohe}(B, D)$

where, as usual, if $A, B \in \mathcal{P}^c(X)$, $A \subseteq B$ if and only if $A(x) \leq B(x)$, $\forall x \in X$.

**3. Coherence and Ambiguity**

As it is well known, in Fuzzy Sets and Systems Theory the ambiguity measures have been typically used as measures of fuzziness, but formerly they come from the classical Sets Theory. In order to relate coherence measures and ambiguity ones, recall first the definition of ambiguity for classical sets, [5].

**Definition 4.** Let $X$ be any set, and denote $\mathcal{P}(X)$ the set of subsets on $X$. Then

$$\alpha: \mathcal{P}(X) \to [0,1]$$

is an ambiguity measure if and only if the following axioms hold:

A1) $\alpha(\emptyset) = 0$  
A2) $\alpha(A) = \alpha(A^c)$  
A3) $\alpha(A \cup B) + \alpha(A \cap B) \leq \alpha(A) + \alpha(B)$

The extension of this definition to the fuzzy case, i.e., to fuzzy sets, can be straight forwarded by using the usual operations of union, intersection and complementation, [3,9]. Hence,

**Definition 5.** Let $X$ be any referential set. Then

$$\alpha: \mathcal{P}^c(X) \to [0,1]$$

is an ambiguity measure if and only if the following axioms hold

A1. $\alpha(\emptyset) = 0$  
A2. $\alpha(A) = \alpha(A^c)$  
A3. $\alpha(A \cup B) + \alpha(A \cap B) \leq \alpha(A) + \alpha(B)$
where

\[
(A \cup B)(x) = \max(\text{A}(x), \text{B}(x)) \\
(A \cap B)(x) = \min(\text{A}(x), \text{B}(x)) \\
A^c(x) = 1 - \text{A}(x)
\]

In order to avoid misunderstandings, in the following such a measure \( \alpha \) will be referred as an ambiguity measure in Fishburn-Yager sense. The following lemma shows a first link between both coherence and ambiguity measures.

**Lemma 6.** Let \( \text{cohe}:P(A) \times P(X) \rightarrow [0,1] \) be a coherence measure. Then

\[
\alpha:P(A) \rightarrow [0,1]
\]

defined as \( \alpha(\text{A}) = 1 - \text{cohe}(\text{A}, \text{A}) \) is an ambiguity measure in Fishburn-Yager sense if and only if:

\[
\text{cohe}(\text{A}, \text{A}) + \text{cohe}(\text{B}, \text{B}) \leq \text{cohe}(\text{A} \cup \text{B}, \text{A} \cup \text{B}) + \text{cohe}(\text{A} \cap \text{B}, \text{A} \cap \text{B})
\]

Let \( \alpha \) now an ambiguity measure in Fishburn-Yager sense. Does it make sense to extend \( \alpha \) to a coherence measure \( \beta \) such that \( \beta(\text{A}, \text{A}) = 1 - \alpha(\text{A}) \)? The following counterexamples give negative answers to this question.

**Counterexample 1:** Consider \( \alpha(\text{A}) = 0, \forall \text{A} \in P(A) \). It is obvious that \( \alpha \) is an ambiguity measure. If \( \beta(\text{A}, \text{A}) = 1 - \alpha(\text{A}) \) is a coherence, by axiom C2:

\[
\beta(\text{A}, \text{A}^c) = 1 - \beta(\text{A}, \text{A}) = \alpha(\text{A})
\]

hence \( \beta(\text{A}, \text{A}^c) = 0, \forall A \in P(A) \). However, by lemma 2, part c), for \( A^* \) holds \( \beta(\text{A}^*, \text{A}^c) = 0.5 \), what is contradictory with \( A^* = A^{\alpha} \).

**Counterexample 2:** Consider \( \alpha \) defined by \( \alpha(\emptyset) = \alpha(X) = 0 \), and \( \alpha(\text{A}) = 1, \forall \text{A} \in P(A) \), \( A \neq \emptyset, X \). By a parallel reasoning to the previous one, a contradiction can be obtained.

Consequently one can conclude that not always is possible to find such an extension. However, in the following, necessary conditions for the existence of such an extension are shown.

**Lemma 7.** Let \( \alpha : P(A) \rightarrow [0,1] \) be an ambiguity measure. To extend \( \alpha \) to a coherence measure \( \beta : P(A) \times P(A) \rightarrow [0,1] \) such that \( \beta(\text{A}, \text{A}) = 1 - \alpha(\text{A}) \), it is necessary that \( \alpha(\text{A}^*) = 0.5 \), where \( \text{A}^*(\text{A}) = 0.5, \forall \text{A} \).

**Lemma 8.** Let \( X \) be a Lebesgue measurable set, with a finite measure \( m \neq 0 \), \( A, B \in P(A) \) and \( \beta : P(A) \times P(A) \rightarrow [0,1] \) a measure defined by:

\[
\beta(\text{A}, \text{B}) = \int_X f(\text{A}(x), \text{B}(x))dx
\]

Then \( \beta \) is a coherence measure if and only if the function \( f:[0,1]^2 \rightarrow [0,1] \) verifies:

---

1 For the sake of brevity, in the remaining we will use "ambiguity" instead of "ambiguity measure in the Fishburn-Yager sense" and "coherence" instead of "coherence measure".
a) \( f(x,y) = f(y,x) \)

b) \( f(x, 1-y) = (1/m) - f(x,y) \)

c) \( f(1-x,y) = (1/m) - f(x,y) \)

d) \( f(0,1) = 0 \)

**Proof.** If \( \beta \) is a coherence measure then \( \beta(A,B) = \beta(B,A) \), hence, concretely let us consider the case that \( A, B \) are constant \( A(x) = a \) and \( B(x) = b \) with \( a, b \in [0,1] \) \( \forall x \in X \), it follows

\[
\beta(A,B) = \int_X f(a,b)dx = f(a,b)\int_X dx = f(a,b)m
\]

\[
\beta(B,A) = \int_X f(b,a)dx = f(b,a)\int_X dx = f(b,a)m
\]

and we can conclude that \( f(a,b) = f(b,a), \forall a,b \in [0,1] \)

Now, by the previous lemma if \( A^*(x) = 0.5, \forall x \in X \), then \( \beta(A^*,A^*) = 0.5 \), and

\[
\beta(A^*, A^*) = \int_X f(0.5, 0.5)dx = f(0.5, 0.5)\int_X dx = f(0.5, 0.5)m = 0.5
\]

consequently \( f(0.5, 0.5) = 1/(2m) \).

Let \( I \subset X \) be, \( l(I) = i < m = l(X) \), an interval included in \( X \). We define \( A(x) \) and \( B(x) \) as:

\[
A(x) = \begin{cases} 
  a & x \in I \\
  0.5 & x \in X - I 
\end{cases}
\]

\[
B(x) = \begin{cases} 
  b & x \in I \\
  0.5 & x \in X - I 
\end{cases}
\]

with \( a,b \in [0,1] \)

\[
\beta(A,B^c) = \int_if(a,1-b)dx + \int_{x-I} \frac{1}{2 \cdot m}dx = i \cdot f(a,1-b) + \frac{m-i}{2 \cdot m}
\]

\[
\beta(A,B) = \int_if(a,b)dx + \int_{x-I} \frac{1}{2 \cdot m}dx = i \cdot f(a,b) + \frac{m-i}{2 \cdot m}
\]

Then,

As \( \beta \) is a coherence measure then \( \beta(A,B^c) = 1 - \beta(A,B) \), and thus

\[
f(a,1-b) + \frac{m-i}{2 \cdot m} = 1 - if(a,b) - \frac{m-i}{2 \cdot m}
\]

\[
f(a,1-b) = -f(a,b) + \frac{1}{i} - 2 \frac{m-i}{2 \cdot m \cdot i}
\]

\[
f(a,1-b) = \frac{1}{m} - f(a,b)
\]
This result holds for $a,b \in [0,1]$. The proof of part c) is similar.

Finally, as $\beta$ is a coherence measure then $\beta(\emptyset, X) = 0$, and

$$\beta(\emptyset, X) = \int_X f(0, 1) \, dx = f(0, 1) \int_X dx = f(0, 1)m = 0$$

Therefore $f(0,1) = 0$.

Reciprocally, if a) to d) are verified, then $\beta$ is a coherence measure. In fact, from a) it is straightforward that $\beta$ is symmetric.

Now, from b) we have $f(x, 1-y) = (1/m) - f(x,y)$ and

$$\beta(A, B^c) = \int_X f(A(x), 1-B(x))\, dx = \int_X [(1/m) - f(A(x), B(x))] \, dx =$$

$$= (1/m) \int_X dx - \int_X f(A(x), B(x))\, dx = 1 - \beta(A, B)$$

As $f(0,1) = 0$ because of d), then

$$\beta(\emptyset, X) = \int_X f(0, 1)\, dx = 0$$

what completes the proof.

The following technical result has a practical importance.

**Note.** Under the same conditions of the above Lemma 8, the following properties hold:

- a) If $f$ is monotonic on $[0, 0.5]^2$, then is decreasing on both arguments
- b) In the interval $[0.5, 1]^2$ $f$ is increasing on both arguments,
- c) in the interval $[0.5, 1] \times [0, 0.5]$ $f$ is decreasing on $x$ and increasing on $y$, and
d) in the interval $[0.0, 5] \times [0.5, 1]$ $f$ is increasing on $x$ and decreasing on $y$

The following table 1 summarizes this property:

<table>
<thead>
<tr>
<th>Table 1. Monotony of $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \uparrow$</td>
</tr>
<tr>
<td>$y \downarrow$</td>
</tr>
<tr>
<td>$x \uparrow$</td>
</tr>
<tr>
<td>$y \uparrow$</td>
</tr>
</tbody>
</table>

The following lemma will give conditions for a desirable and intuitive property of the coherence measures: The nearest from $\emptyset$ or $X$ respectively are two elements $A$ and $B$, the greater coherence measure is.

**Lemma 9.** Let $X$ be a measurable set, $m = l(X) \neq 0$, and $A, B \in P(X)$. Consider $A^*(x) = 0.5$, $\forall x \in X$, and $\beta: P(X) \times P(X) \to [0, 1]$ a coherence measure defined by:

$$\beta(A, B) = \int_X f(A(x), B(x))\, dx$$

If $f$ is monotonic in the interval $[0, 0.5]^2$ then:
1) \( A \subseteq B \subseteq A^*, A' \subseteq B' \subseteq A^* \Rightarrow \beta(A,A') \geq \beta(B,B') \)

2) \( A^* \subseteq B \subseteq A, A^* \subseteq B' \subseteq A' \Rightarrow \beta(A,A') \geq \beta(B,B') \)

**Proof.** Let suppose \( A \subseteq B \subseteq A^* \) and \( A' \subseteq B' \subseteq A^* \). Then, \( \forall x \in X: \)

\[
A(x) \leq B(x) \leq 0.5
\]

and

\[
A'(x) \leq B'(x) \leq 0.5
\]

As \( f \) is decreasing on \([0,0.5]^2\), then

\[
f(A(x), A'(x)) \geq f(B(x), A'(x))
\]

and

\[
f(B(x), A'(x)) \geq f(B(x), B'(x))
\]

and consequently

\[
f(A(x), A'(x)) \geq f(B(x), B'(x)) \Rightarrow \int_X f(A(x), A'(x))dx \geq \int_X f(B(x), B'(x))dx \Rightarrow
\]

\[
\Rightarrow \beta(A,A') \geq \beta(B,B')
\]

Now, let us suppose \( A^* \subseteq B \subseteq A \) and \( A^* \subseteq B' \subseteq A' \), then

\[
0.5 \leq B(x) \leq A(x)
\]

and

\[
0.5 \leq B'(x) \leq A'(x)
\]

but as \( f \) is increasing in the interval \([0.5,1]^2\),

\[
f(A(x), A'(x)) \geq f(B(x), A'(x))
\]

\[
f(B(x), A'(x)) \geq f(B(x), B'(x))
\]

and the proof follows as before.

It is patent therefore that a proximity is more meaningful between two very clear evaluations than between two ambiguous evaluations.

The next result will show a) the way in which coherence measures can be constructed from metrics on \( P^f(X) \), and b) the existence of such measures.

**Lemma 10.** Let \( X \) be a measurable set with a measure \( m \neq 0 \). Let

\[
d:P^f(X) \times P^f(X) \rightarrow [0,1]
\]

be a bounded metric defined by:

\[
d(A,B) = (\int_X h(A(x), B(x))dx)^{1/r}, \ r \geq 1
\]

Then a coherence measure defined as
\[ \beta(A, B) = \left[1 + d(A, B^c) - d(A, B)\right]/2 \]

can be constructed if and only if:

a) \( h(0, 1) = (1/m) \)

b) \( h(a, 1-b) = h(1-a, b) \text{ } \forall a,b \in [0,1] \)

**Proof.** Let us suppose that \( \beta \) is a coherence measure.

a) As \( \beta(\emptyset, X) = 0 \), then \( 1 + d(\emptyset, X^c) - d(\emptyset, X) = 0 \). As \( X^c = \emptyset \), then \( d(\emptyset, X) = 1 \), and from the definition of \( d \),

\[ \left( \int_X h(0, 1) \right)^{1/2} = 1 \]

Thus \( h(0, 1) = (1/m) \).

b) \( \beta(A, B) = \beta(B, A) \). Let us suppose

\[
A(x) = \begin{cases} 
  a & x \in I \\
  0.5 & x \in X - I 
\end{cases}
\]

\[
B(x) = \begin{cases} 
  b & x \in I \\
  0.5 & x \in X - I 
\end{cases}
\]

with \( a, b \in [0, 1] \) whatever,

\[
[(1+d(A, B^c) + d(A, B))/2 = [(1+d(B, A^c) + d(B, A))/2 \Rightarrow d(A, B^c) = d(B, A^c) \Rightarrow
\]

\[
\left( \int_X h(a, 1-b)dx + \int_X h(0.5, 0.5)dx \right)^{1/2} = \left( \int_X h(b, 1-a)dx + \int_X h(0.5, 0.5)dx \right)^{1/2} \Rightarrow
\]

\[
\Rightarrow h(a, 1-b) = h(b, 1-a)
\]

As the function \( d \) is symmetric, also \( h \) it is, and because of this one obtains :

\[ h(a, 1-b) = h(1-a, b) \text{ } \forall a,b \in [0,1]. \]

Taking this into account, now one has to show C1-C3.

C1. What one has to show is the symmetry of \( \beta \) and using b),

\[
\beta(A, B) = \frac{1}{2} \cdot (1 + d(A, B^c) - d(A, B)) = \frac{1}{2} \cdot (1 + (\int_X h(A(x), 1-B(x))^{1/2} - d(A, B)) =
\]

\[
= \frac{1}{2} \cdot (1 + (\int_X h(1-A(x), B(x))^{1/2} - d(A, B)) = \frac{1}{2} \cdot (1 + d(A^c, B) - d(A, B)) = \frac{1}{2} \cdot (1 + d(B, A^c) - d(B, A)) =
\]

\[ = \beta(B, A) \]

C2. One has to show that \( \text{cohe}(A, B^c) = 1 - \text{cohe}(A, B) \). But,

\[
\beta(A, B^c) = \frac{1}{2} \cdot (1 + d(A, B) - d(A, B^c)) \Rightarrow \frac{1}{2} \cdot (1 + d(A, B^c) - d(A, B)) = 1 - \beta(A, B)
\]

C3. Finally, taking account a)

\[
\beta(\emptyset, X) = \frac{1}{2} \cdot (1 - d(\emptyset, X)) = \frac{1}{2} \cdot (1 - (\int_X h(0, 1))^{1/2} = \frac{1}{2} \cdot (1 - \frac{m}{m}) = 0
\]
what show that $\beta$ is a coherence measure.

The application of this lemma to different metrics will permit to obtain a number of coherence measures which, eventually, could be compared. This is the aim of the following examples

**Example 1.** Case of $r$-metrics ($r \geq 1$) on $P^f(X)$. As it is known,

$$d(A,B) = [(1/m) \int_X |A(x) - B(x)|^r dx]^{1/r}$$

It is evident that a) and b) above are verified. Hence from this metric several coherence measures, all of them depending on a parameter $r$, can be defined.

**Example 2.** Consider the metrics that can be constructed from $h:[0,1] \rightarrow [0,1]$

$$h(a,a) = 0 \quad \forall a \in [0,1],$$
$$h(a,b) = 1/m \quad \forall a,b \in [0,1], a \neq b$$

If $G = \{x \in X/ A(x) \neq B(x)\}$, and $g$ is the measure of $G$, then in this case:

$$d(A,B) = (g/m)^{1/r}, \forall r \geq 1$$

Parallel to the above lemma, but now concerning the construction of ambiguity measures, the next result shows a production tool for these measures.

**Lemma 11:** Let $X$ be a measurable set with $m = l(X) \neq 0$, and $\alpha:P^f(X) \rightarrow [0,1]$ defined by

$$\alpha(A) = \int_X g(A(x)) \, dx$$

Then $\alpha$ is an ambiguity measure (in Fishburn-Yager sense) if and only if $g:[0,1] \rightarrow [0,1]$ verifies:

a) $g(0) = 0$

b) $g(a) = g(1-a)$

**Proof:** Let $\alpha$ be an ambiguity measure:

a) $\alpha(\emptyset) = \int_X g(0)dx = m \cdot g(0)$

As $\alpha(\emptyset) = 0$, then $m \cdot g(0) = 0$, and $g(0) = 0$

b) Let us suppose $A$ defined as in Lemma 10 c). Then:

$$\alpha(A) = \int_X g(A(x)) \, dx = \int_I g(a) \, dx + \int_{x-I} g(0.5) \, dx$$

$$\alpha(A^c) = \int_X g(1 - A(x)) \, dx = \int_I g(1-a) \, dx + \int_{x-I} g(0.5) \, dx$$

But $\alpha(A) = \alpha(A^c)$, and then $g(a) = g(1-a)$.

Reciprocally, let us suppose now that a) and b) holds, then $\alpha$ is an ambiguity measure. In fact,

1. $\alpha(\emptyset) = 0$ directly from a)
2. Similarly, from b),
\[ \alpha(A) = \int_A g(a) \, dx = \int_A g(1-a) \, dx = \alpha(A^c) \]
It is patent this lemma suggests to think of a direct way for extending an ambiguity measure to a coherence one. The following result however shows how that direct extension may be in general not trivial.

**Lemma 12.** Let \( X \) be a measurable set with \( m = \lambda(X) \neq 0 \) and \( \alpha: P^f(X) \rightarrow [0,1] \), defined by
\[ \alpha(A) = \int_X g(A(x)) \, dx \]
Then, there is no function \( k:[0,1] \rightarrow [0,1] \) such that:
\[ \beta(A,B) = \int_X k \cdot g(A(x), B(x)) \, dx \]
is a coherence measure extending \( \alpha \), that is to say, such that: \( \alpha(A) = 1 - \beta(A,A) \).

**Proof.** Similar proof to finite case [6]

General conditions under which an ambiguity measure can be extended to a coherence measure are shown in the following theorem which, because of this reason, is called of extension. This theorem concretely enables two (similar and parallel) ways to perform that extension according to the use of either the max or the min operator, and therefore it shows as from each ambiguity measure two coherence measures can be obtained.

**Theorem of Extension 13.** Let \( X \) be a measurable set with \( m = \lambda(X) \neq 0 \) and \( \alpha: P^f(X) \rightarrow [0,1] \), defined by
\[ \alpha(A) = \int_X g(A(x)) \, dx \]
with \( \alpha(A^c) = 0.5 \). Then one can define coherence measures
\[ \beta: P^f(X) \times P^f(X) \rightarrow [0,1] \]
in such a way that \( \beta(A,A) = 1 - \alpha(A) \).

**Proof.** The proof is based upon the consideration of the functions:
\[ \beta(A,B) = 1 - \alpha(A \cup B) \quad \text{and} \quad \beta'(A,B) = 1 - \alpha(A \cap B) \]
both defined on appropriate sets. To this end, we consider first the following partition on \( X \):
\[
\begin{align*}
J_{11} &= \{ x \in X / A(x) \in [0,0.5) \text{ and } B(x) \in [0,0.5) \} \\
J_{12} &= \{ x \in X / A(x) \in [0,0.5) \text{ and } B(x) \in [0.5,1) \} \\
J_{21} &= \{ x \in X / A(x) \in [0.5,1] \text{ and } B(x) \in [0,0.5) \} \\
J_{22} &= \{ x \in X / A(x) \in [0.5,1] \text{ and } B(x) \in [0.5,1) \}
\end{align*}
\]
Let \( \beta \) and \( \beta' \) be two functions, defined on to \( J_{11} \), by
\[ \beta(A,B) = 1 - \alpha(A \cup B) \]
On the Definition of Coherence Measure for Fuzzy Sets

\[ \beta'(A,B) = 1 - \alpha(A \cap B) \]

Then,

\[ \beta(A, B) = \int_{J_{11}} \left( \frac{1}{m} - g(\max(A(x), B(x))) \right) dx = \int_{J_{11}} f(A(x), B(x)) dx \]

\[ \beta'(A, B) = \int_{J_{11}} \left( \frac{1}{m} - g(\min(A(x), B(x))) \right) dx = \int_{J_{11}} f'(A(x), B(x)) dx \]

where \( f \) and \( f' \) are functions from \([0,1]^2\) to \([0,1]\) defined, obviously, by:

\[ f(a,b) = \frac{1}{m} - g(\max(a,b)) \]

\[ f'(a,b) = \frac{1}{m} - g(\min(a,b)) \]

In the following we will extend \( \beta \) and \( \beta' \) to \( P^f(X) \times P^f(X) \) and will show that they are coherence measures. To this end, we consider the following expression for \( \beta \):

\[ \beta(A, B) = \int_{J_{11}} \left( \frac{1}{m} - g(\max(A(x), B(x))) \right) dx + \int_{J_{12}} g(\max(A(x), 1-B(x))) dx + \]

\[ + \int_{J_{21}} g(\max(1-A(x), B(x))) + \int_{J_{22}} \left( \frac{1}{m} - g(\max(1-A(x), 1-B(x))) \right) dx \]

(as it is clear, the corresponding expression for \( \beta' \) can be obtained by changing the operator \( \max \) by the \( \min \) in the previous one).

Therefore, what we want to show is

\[ \beta(A,A) = \beta'(A,A) = 1 - \alpha(A). \]

(it is evident that the proof for \( \beta' \) will be identical).

Let consider \( A \in P^f(X) \). As \( \alpha \) is an ambiguity measure, then \( g(a) = g(1-a) \). Then,

\[ \beta(A, A) = \int_{J_{11}} \left( \frac{1}{m} - g(A(x)) \right) dx + \int_{J_{12}} \left( \frac{1}{m} - g(A(x)) \right) dx = 1 - \int_A g(A(x)) dx = 1 - \alpha(A) \]

Let now show that \( \beta \) is a coherence measure (the proof for \( \beta' \) is identical). The three properties defining a coherence measure are shown as follows.

1. \( \beta(A,B) = \beta(B,A) \) as it is symmetric by construction.
2. By definition we have:

\[ \beta(A, B^c) = \int_{J_{11}} f(A(x), 1-B(x)) dx + \int_{J_{12}} \left( \frac{1}{m} - f(A(x), B(x)) \right) dx + \]

\[ + \int_{J_{21}} \left( \frac{1}{m} - f(1-A(x), 1-B(x)) \right) + \int_{J_{22}} f(1-A(x), B(x)) dx \]
3. Thus, if $x \in J_{11}(A,B^c)$ then $x \in J_{12}(A,B)$. If $x \in J_{12}(A,B^c)$ then $x \in J_{11}(A,B)$, If $x \in J_{21}(A,B^c)$ then $x \in J_{22}(A,B)$, and if $x \in J_{22}(A,B^c)$ then $x \in J_{21}(A,B)$.

Moreover:

$$\beta(A,B) = \int_{J_{11}} f(A(x), B(x)) \, dx + \int_{J_{12}} \frac{1}{m} - f(A(x), 1 - B(x)) \, dx + \int_{J_{21}} \frac{1}{m} - f(1 - A(x), B(x)) + \int_{J_{22}} f(1 - A(x), 1 - B(x)) \, dx$$

Therefore, by adding, one has

$$\beta(A, B^c) + \beta(A, B) = \int_{J_{11}} \frac{1}{m} \, dx + \int_{J_{12}} \frac{1}{m} \, dx + \int_{J_{21}} \frac{1}{m} \, dx + \int_{J_{22}} \frac{1}{m} \, dx = \frac{1}{m} = 1$$

3. $\beta(\varnothing, \varnothing) = 1 - \beta(\varnothing, \varnothing) = 1 - (1 - \alpha(\varnothing)) = \alpha(\varnothing) = 0$.

4. Theorem of Extension Applications

In this section we will focus on the different ways in which the above Theorem of Extension can be applied. Four different examples will be shown.

**Example 1.** Let consider the ambiguity measure defined in [9], and suppose that is normalised so that it can be extended to a coherence measure,

$$\text{Fuzz}(A) = k \cdot \sum_{i=1...m} D(a_i)$$

$$D(a_i) = \min(a_i, 1 - a_i)$$

From Lemma 7 above, if $A^*(x) = 0.5 \ \forall x$, in order to $\text{Fuzz}(\cdot)$ can be extended, it is necessary that $\text{Fuzz}(A^*) = 0.5$. Hence $k = \frac{1}{m}$, and by applying the Theorem of Extension, the following measure of coherence is obtained

$$\beta_{\text{max}}(A,B) = \int_{J_{11}} [(1/m) - (1/m) \cdot \max(A(x), B(x))] \, dx + \int_{J_{12}} (1/m) \cdot \max(A(x), 1 - B(x)) \, dx + \int_{J_{21}} (1/m) \cdot \max(1 - A(x), B(x)) \, dx + \int_{J_{22}} (1/m) \cdot \max(1 - A(x), 1 - B(x)) \, dx$$

As it will result evident another coherence measure, $\beta_{\text{min}}$, could be obtained whether $\min$ were used instead of max.

**Example 2.** Let consider now the application of the theorem to the De Luca and Termini’s ambiguity measure, [2],

$$\text{Fuzz}(A) = - (\sum_{i=1...m} a_i \cdot \ln(a_i) + \sum_{i=1...m} (1 - a_i) \cdot \ln(1 - a_i))$$

If it is normalised, so that can be extended to a coherence measure,

$$\text{Fuzz}(A^*) = -2 \cdot m \cdot 0.5 \cdot \ln(0.5) = m \cdot \ln 2$$

Therefore the normalised ambiguity measure is defined by:
Fuzz*(A) = (2·m·ln2)^1/2 Fuzz(A)

The application of the Theorem of Extension to this measure provides two other new coherence measures, which in the following will be referred as $\beta_{D_{\max}}$ and $\beta_{D_{\min}}$. Expressions for these measures could be obtained easily and they are not shown here by its length.

**Example 3.** Consider any function $g:[0,1] \rightarrow [0,1]$ verifying a) $g(0) = 0$, b) $g(a) = g(1-a)$, and c) $g(0.5) = 1/(2\cdot m)$. From lemmas 7 and 11 above, one can construct an ambiguity measure

$$\alpha(A) = \int_X g(A(x)) \, dx$$

with $\alpha(A^*) = 0.5$, to which one can associate two other coherence measures. Concretely the quadratic function $g(\cdot)$ defined as

$$g(a) = (2/m)a(1-a)$$

verifies the above three conditions. Therefore, the application of the Theorem will provide again two new coherence measures, $\beta_{2\max}$ and $\beta_{2\min}$.

In particular, if the operator max is used,

$$\beta_{2\max} = \int_{11} (1/m) \cdot (1 - 2\max(A(x), B(x))) \cdot (1 - \max(A(x), B(x))) \, dx +$$

$$+ \int_{12} (2/m) \cdot \max(A(x), 1 - B(x)) \cdot (1 - \max(A(x), 1 - B(x))) \, dx +$$

$$+ \int_{21} (2/m) \cdot \max(1 - A(x), B(x)) \cdot (1 - \max(1 - A(x), B(x))) \, dx +$$

$$+ \int_{22} (1/m) \cdot (1 - 2\max(1 - A(x), 1 - B(x))) \cdot (1 - \max(1 - A(x), 1 - B(x))) \, dx$$

The following result is of a remarkable importance because it shows as the above measures are monotonous ones in the interval $[0,0.5]^2$.

**Lemma 14.** The coherence measures obtained from the ambiguity measures of Yager, De Luca and Términi, and the above quadratic function $g$, respectively denoted $\beta_{Y_{\max}}, \beta_{Y_{\min}}, \beta_{D_{\max}}, \beta_{D_{\min}}, \beta_{2\max}$ and $\beta_{2\min}$, verify the hypotheses of the lemma 10 above. Concretely, these measures can be generically defined as:

$$\beta(A,B) = \int_X f(A(x), B(x)) \, dx$$

being $f$ a monotonous function on its arguments in $[0,0.5]^2$.

The proof of this lemma can be found in [1]. This proof remain also valid for the non finite case.

**Example 4.** This last example shows two important properties of the measures of coherence here considered: First, they are context-dependent, that is, these measures depend on $X$. Second they are sensitive to the vagueness of the components. Let $A = (a,b,\alpha,\beta)$ be a fuzzy number with a trapezoidal membership function [1,3]. The parameters $a, b, \alpha$ and $\beta$ have the meaning that the next figure 1 illustrates,
Consider, for instance, the linguistic variable "age" defined on \([0,100]\) by the following labels:

- child \((0, 10, 0, 5)\);
- young \((15, 25, 5, 5)\);
- adult \((25, 60, 5, 10)\);
- old \((65, 100, 5, 0)\)

A possible representation of these labels could be the following figure 2:

Then coherence measures for \(\beta_{Y\text{max}}\) (Yager’s ambiguity) and \(\beta_1\) (coherence based on the r-metric for \(r = 1\)) are shown in the following table 2.

<table>
<thead>
<tr>
<th></th>
<th>(X=[0,100])</th>
<th>(X=[0,50])</th>
<th>(X=[15,25])</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\beta_{Y\text{max}}) (\beta_1)</td>
<td>(\beta_{Y\text{max}}) (\beta_1)</td>
<td>(\beta_{Y\text{max}}) (\beta_1)</td>
</tr>
<tr>
<td>Child-Young</td>
<td>.74 .75</td>
<td>.48 .50</td>
<td>0 .0</td>
</tr>
<tr>
<td>Adult-Young</td>
<td>.63 .55</td>
<td>.38 .33</td>
<td>.31 .25</td>
</tr>
<tr>
<td>Adult-Old</td>
<td>.25 .55</td>
<td>.45 .45</td>
<td>.75 .75</td>
</tr>
<tr>
<td>Child-Old</td>
<td>.50 .63</td>
<td>.74 .75</td>
<td>1 .1</td>
</tr>
</tbody>
</table>
Then, on the one hand, as it can be seen, the coherences between the labels Child and Old are greater on $X = [0, 50]$ than on $X = [0, 100]$. This is so because in the interval $[15, 60]$ both labels coincide. It is also because of this reason that the coherences between Child and Old are equal to 1 at third column. On the other hand, the coherence on Child and Young has a high value on $X = [0, 100]$, but it is lower when only the first part of the life is considered. It is clear, finally, that in the interval $X = [15, 25]$, where Child and Young may be understood as having opposite meanings, the coherences are equal to 0.

From this example the interval dependence of these measures is evident. This fact is important if we want to reduce the semantic of a label without to change its definition only taking into account the interval being considered.

5. Conclusions
An extension of the concept of coherence measures from finite sets to measurable Lebesgue fuzzy sets has been presented. We have shown properties of the coherence measures as well as on the methods to construct them. Relations between ambiguity measures and coherence measures have been also shown. Furthermore a way to obtain different coherence measures from ambiguity measures has been provided. Finally, the interval dependence of the coherence measures was illustrated by means of a numerical example.

Acknowledgements
Authors thank the reviewers for their very useful comments that helped to improve the quality of this paper

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