Long–time Asymptotics for Semiconductor Crystals

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Abstract

This paper is devoted to describe the long–time asymptotics of electrons moving in a semiconductor crystal lattice. We consider a quantum model in the Schrödinger formalism for the semiconductor crystal with a self–consistent Coulomb interaction and an external electric field and use the semiclassical approach to obtain a kinetic transport equation of Vlasov type for the semiconductor. Then, by changing the time scale we analyze the long time behaviour of the electron ensemble in the semiconductor crystal.

1 Introduction

In this paper we investigate the long–time asymptotics of electrons moving in a 3D periodic crystal lattice in a one–band semiclassical approximation. In the model we deal with, the electrons are subjected to periodic forces due to the positive ions, the Coulomb interaction (which verifies the Poisson equation) and to an external electric field. The main difficulty for the mathematical analysis of such a model comes from the fact that the forces applied to the electrons are all of the same order of magnitude. Also, the classical tools of

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homogenization of periodic structures (Bloch decomposition) do not work in this case due to the presence of the external electric field and the nonlinear Coulomb interaction. However, a semiclassical approach to our quantum model can be obtained in terms of the semiconductor Vlasov equation, if we suppose that the initial data (density of electrons) is concentrated in an energy band which is isolated from the others.

It is well-known that in space dimensions strictly greater than one the energy bands generally cross, but in the framework of semiconductor devices there also exist energy bands which do not cross with the others. This is in particular the case of the conduction band, which is of interest for semiconductor technology. In this case we shall prove that the interaction with the other bands via the external electric field or the self-consistent potential remains weak and disappears when we pass to the semiclassical limit as the Planck constant $\hbar \to 0$ (in the sense of the rescaled equation after adimensionalization). We consider here the quantum mixed state description of the electrons in the semiconductor crystal. The dynamics of the electron ensemble is described by the following class of rescaled Schrödinger equations:

\[
\begin{align*}
\frac{i\alpha}{\partial t} \psi_j^\alpha(x, t) &= \left[-\alpha^2 \Delta_x + U(x) + V_{\text{coul}}^{\alpha}(x, t) + V_{\text{ext}}^\alpha(x, t)\right] \psi_j^\alpha(x, t), \\
\psi_j^\alpha(x, t = 0) &= \psi_j^\alpha(x), \\
\Delta_x V_{\text{coul}}^\alpha(x, t) + n^\alpha(x, t) &= 0,
\end{align*}
\]

with $j \in \mathbb{N}, x \in \mathbb{R}^3$ and $t \in \mathbb{R}$, where $\psi_j(x, t)$ is the wave function associated with the $j^{th}$ state of the electron, $U(x)$ is a periodic potential which verifies $U(x + \gamma) = U(x) \; \forall \gamma \in \Gamma$, where we have defined a periodic lattice $\Gamma = \{\gamma = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3, (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}^3\}$ and where $(a_1, a_2, a_3)$ is a vector basis of the lattice. On the other hand, $V_{\text{ext}}^\alpha$ is a given external potential and the potential $V_{\text{coul}}^\alpha(x, t)$ solves the Poisson equation (3), where $n^\alpha$ is the electron density $n^\alpha(x, t) = \sum_{j \in \mathbb{N}} \lambda_j |\psi_j^\alpha(x, t)|^2$. Here, the $\lambda_j \geq 0$ are the mixed state occupation probabilities of the state $j$ and of course verify $\sum_{j \in \mathbb{N}} \lambda_j = 1$.

In a semiclassical level of description, the dynamics of the particle system is represented by a kinetic equation for the electrons. More precisely, the motion of the electrons is described by a density function $f(x, k, t)$, where $x \in \mathbb{R}^3$ and $k \in B$, with $B$ denoting the Brillouin zone. In the $p^{th}$ energy band, the electron is transported with velocity $v_p(k) = \nabla_k \mathcal{E}_p(k)$, which depends on the energy level. The function $f$ satisfies the semiconductor Vlasov equation

\[
\frac{\partial f}{\partial t} + \nabla_k \mathcal{E}_p(k) \cdot \nabla_x f - \nabla_x \left( V_{\text{coul}} + V_{\text{ext}} \right) \cdot \nabla_k f = 0,
\]

where the position density is given by $n(x, t) = \frac{1}{|\Omega|} \int_B f(x, k, t) \, dk$. The transition from the quantum to the classical model is represented by the semiclassical
limit, which is performed via Wigner and Wigner–Bloch series techniques (see [1], [2], [3], [7]). In the following we shall recall the main ingredients concerning the semiclassical limit. The following a priori estimates (proven in [2]) are shown to hold by using the Lieb–Thirring inequalities (see [4]) and will be useful in the next section.

\[ \| n^\alpha(\cdot, t) \|_{L^p(\mathbb{R}^3)} \leq C, \quad p \in [1, \frac{5}{3}], \ t \geq 0, \]
\[ \| J^\alpha(\cdot, t) \|_{L^q(\mathbb{R}^3)} \leq C, \quad q \in [1, \frac{5}{4}], \ t \geq 0, \]
\[ \| V_{coul}^\alpha(\cdot, t) \|_{L^r(\mathbb{R}^3)} \leq C, \quad r \in [3, \infty), \ t \geq 0, \]
\[ \| \nabla V_{coul}^\alpha(\cdot, t) \|_{L^s(\mathbb{R}^3)} \leq C, \quad s \in [\frac{15}{4}, \frac{15}{2}), \ t \geq 0. \]

In addition, \( V_{coul}^\alpha(x, t) \) lies in a compact subset of the space \( C([0, T]; H^1_{loc}(\mathbb{R}^3)) \).

For \( \varphi, \psi \in L^2(\mathbb{R}^3) \) a Wigner series (cf. [7]) is defined as

\[ W_s^\alpha(\varphi, \psi)(x, k) = \sum_{\gamma \in \Gamma} \varphi(x - \frac{\alpha \gamma}{2}) \overline{\psi}(x + \frac{\alpha \gamma}{2}) e^{ik \cdot \gamma}, \quad x \in \mathbb{R}^3, \ k \in B. \quad (5) \]

Also, the following spaces of test functions (see [2]) are considered:

\[ \mathcal{D}_t = \{ \varphi \in C^\infty(\mathbb{R}^6_{x,k}) \text{ with compact support in } x, \Gamma^* - \text{periodic in } k \}, \]
\[ \mathcal{D}_{t,\mathbb{R}} = \{ \varphi(t) \in C^\infty(\mathbb{R}^6_{x,k}) \text{ with compact support in } (x, t), \Gamma^* - \text{periodic in } k \}. \]

It was proved in [2] that the sequence \( W_s^\alpha(\varphi^\alpha, \varphi^\alpha) \) converges (up to a subsequence) to a positive measure in \( \mathcal{D}_t^\prime \)

\[ W_s^\alpha(\varphi^\alpha, \varphi^\alpha)(x, k) \rightarrow W_0^\alpha(\varphi^\alpha, \varphi^\alpha)(x, k) \geq 0 \]

for all sequences of functions \( \{ \varphi^\alpha \} \) bounded in \( L^2 \), and the following convergence property holds for the Coulomb potential:

\[ V^\alpha_{coul}(x, t) \rightarrow V^0_{coul}(x, t) \text{ in } C(\{0, T\}; H^1_{loc}(\mathbb{R}^3)). \]

Also, if \( \psi^\alpha_j \) satisfy the system (1)–(3) with \( \epsilon \)-oscillatory initial data \( \psi^\alpha_{j,\epsilon} \) concentrated in the \( p^\text{th} \) energy band, the initial density \( n^\alpha_{pl}(x) = \sum_{j \in \mathbb{N}} \lambda_j |\Pi_p \psi^\alpha_{j,\epsilon}(x)|^2 \) is compact at infinity and \( V_{ext} \in W^{1,\infty}(\mathbb{R}^4) \), then

\[ W_s^\alpha(\sum_{j \in \mathbb{N}} \lambda_j \Pi_p \psi^\alpha_{j,\epsilon}, \Pi_p \psi^\alpha_{j,\epsilon}) \rightarrow f_{pl} \geq 0 \quad \text{weakly in } L^2(\mathbb{R}^3 \times B), \]
\[ W_s^\alpha(\sum_{j \in \mathbb{N}} \lambda_j \Pi_p \psi^\alpha_{j,\epsilon}, \Pi_p \psi^\alpha_{j,\epsilon}) \rightarrow f_p \geq 0 \quad \text{in } C([0, T]; L^2(\mathbb{R}^3 \times B) \text{ weak }) \]
(up to a subsequence) for all $T > 0$, where the nonnegative distribution function $f_p = f_p(x, k, t)$ verifies the following Vlasov equation in the sense of distributions:

$$
\frac{\partial f_p}{\partial t} + \nabla_k E_p(k) \cdot \nabla_x f_p - \nabla_x \left( V_{\text{coul}}^0(x, t) + V_{\text{ext}}(x, t) \right) \cdot \nabla_k f_p = 0,
$$

with initial data $f_p(x, k, t = 0) = f_{ptl}(x, k)$, where $E_p(k)$ denote the eigenvalues of the Hamiltonian $H^\alpha \Phi = E \Phi$ (isolated energy bands) and $\Pi_p^\alpha$ are the corresponding projectors on the $p^{th}$ band subspaces $V_p$.

Concerning the long time asymptotics, our methods are based on the rescaling techniques employed in [5], [6]. Just in order to compare with the tools developed in the next section, we shall give here a brief survey of these techniques. In a first approach, the authors considered the scale groups $\psi(x, t) = t^{\frac{3}{2}} \psi(tx, t)$, $\tilde{V}_{\text{coul}} = tV_{\text{coul}}(tx, t)$ (see also [8]) and showed that, at the long time $t \to \infty$,

$$
\tilde{\psi}(x, t) \longrightarrow \psi_\infty(x) \quad \text{weakly in } L^2(\mathbb{R}^3),
$$

$$
\tilde{V}(x, t) \longrightarrow \frac{1}{4\pi|x|} \quad \text{strongly in } L^p(\Omega), 1 \leq p \leq 3,
$$

where $\Omega$ is a compact set of $\mathbb{R}^3$ and $\psi_\infty$ satisfies the linear convection equation

$$
\frac{3}{2} \psi_\infty + i(x \cdot \nabla_x) \psi_\infty + \frac{1}{4\pi|x|} \psi_\infty = 0
$$

in an approximation of order $\frac{1}{t}$. The second approach corresponds to the rescaling $\psi_\epsilon(x, t) = \psi(x, \frac{1}{\epsilon}, t)$, $V_{\text{coul}, \epsilon} = V_{\text{coul}}(x, \frac{1}{\epsilon}, t)$ and the analysis of the (long time) limiting behaviour $\epsilon \to 0$. This analysis is rather more complicated because of an obvious lack of compactness with respect to the time variable. To overcome this difficulty, the following generalized space–time Wigner transform was defined in [6]:

$$
W_\epsilon(x, \xi, t, \tau) = \frac{1}{(2\pi)^3} \frac{1}{4t} \int_{\mathbb{R}^3} \int_{-2t}^{2t} \overline{\psi_\epsilon(x + \frac{\eta}{2}, t + \frac{s}{2})} \psi_\epsilon(x - \frac{\eta}{2}, t - \frac{s}{2}) e^{-i\eta \xi} e^{is\tau} ds \, d\eta,
$$

which is reduced to the usual Wigner formulation after integration against $\tau$, that is

$$
w_\epsilon(x, \xi, t) = \frac{2t}{\pi} \int_{\mathbb{R}} W_\epsilon(x, \xi, t, \tau) \, d\tau,
$$

$w$ denoting the Wigner transform associated with the usual density matrix

$$
w(x, \xi, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \overline{\psi(x + \frac{\eta}{2}, t)} \psi(x - \frac{\eta}{2}, t) e^{-i\eta \xi} \, d\eta.
Here, the new variable \( \tau \) represents the frequency of the oscillations in time. Then, the long time behaviour of the Schrödinger–Poisson system is determined through the solutions to the following Wigner–Poisson–type nonlinear limiting problem: 

\[
W^\ast_{\infty}(x, \xi) = w(x, \xi, t = 0) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}_\xi^3} \overline{\psi_1}(x + \frac{\eta}{2})\psi_1(x - \frac{\eta}{2})e^{-in\xi} \, d\eta,
\]

where 

\[
\theta[V_{\text{coul}}, \infty]W_{\infty}(x, \xi, t, \tau) = \frac{i}{(2\pi)^4} \int \int \int \int_{\mathbb{R}_\eta^3 \mathbb{R}_\xi^3 \mathbb{R}_r^1} \left( V_{\text{coul}, \infty}(x + \frac{\eta}{2}, t + \frac{s}{2}) - V_{\text{coul}, \infty}(x - \frac{\eta}{2}, t - \frac{s}{2}) \right)
\]

\[
\times W_{\infty}(x, \xi', t, \tau')e^{-i(\xi - \xi')s}e^{i(\tau - \tau')r} \, ds \, dr \, d\xi' \, d\eta,
\]

\[
V_{\text{coul}, \infty}(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}_\eta^3} \frac{n_{\infty}(y, t)}{|x - y|} \, dy = \frac{t}{2\pi^2} \int \int \int_{\mathbb{R}_\eta^3 \mathbb{R}_\xi^3 \mathbb{R}_r} \frac{W_{\infty}(y, \xi, t, \tau)}{|x - y|} \, d\tau \, d\xi \, dy,
\]

and the limiting density \( n_{\infty} \) is a nonnegative measure given by

\[
n_{\infty}(x, t) = \frac{2t}{\pi} \int_{\mathbb{R}_\xi^3} \int \frac{W_{\infty}(x, \xi, t, \tau)}{\eta} \, d\tau \, d\xi.
\]

The rest of the paper is devoted to the analysis of the large times asymptotics for the (limiting) semiconductor Vlasov equation, which makes use of similar tools to those reported above. In particular, a slightly modified spatio-temporal Wigner–type transformation, acting on the square root of the Vlasov distribution function, will be introduced. This transformation, which includes time delocalization in the whole real line (due to the time reversibility of the Vlasov equation) will play a central role in our proofs.

2 Long time asymptotics

This section is concerned with the analysis of the long time approach \( \epsilon \to 0 \) for the limiting semiconductor Vlasov equation (6), which constitutes our main task in this paper. Denoting \( V_\epsilon(x, t) = V(x, \frac{t}{\epsilon}) \) (for \( V = V_{\text{coul}}^0 \) or \( V = V_{\text{ext}} \)), \( f_\epsilon^s(x, k, t) = f_\epsilon(x, k, \frac{t}{\epsilon}) \) and \( g_\epsilon^s = \sqrt{f_\epsilon^s} \), it is a simple matter to check that \( g_\epsilon^s \) also satisfies the Vlasov equation (6) in \( \mathcal{D}_{t,s}' \):

\[
\epsilon \frac{\partial g_\epsilon^s}{\partial t} + \nabla_k E_\epsilon(k) \cdot \nabla_x g_\epsilon^s - \nabla_x \left( V_{\text{coul}, \epsilon}(x, t) + V_{\text{ext}, \epsilon}(x, t) \right) \cdot \nabla_k g_\epsilon^s = 0. \tag{7}
\]
We now introduce the following temporal Wigner transformation (see also [6]):

\[
W(g_p^\varepsilon, \overline{g_p^\varepsilon})(x, k, t, \tau) = \frac{1}{2\pi} \int_{\mathbb{R}} g_p^\varepsilon(x, k, t + \frac{s}{2})g_p^\varepsilon(x, k, t - \frac{s}{2})e^{-is\tau} \, ds
\]

\[= \mathcal{F}^{-1}_{s \rightarrow \tau}\left(g_p^\varepsilon(x, k, t + \frac{s}{2})g_p^\varepsilon(x, k, t - \frac{s}{2})\right),\]  

(8)

with \(x \in \mathbb{R}^3, k \in B, t \in \mathbb{R}, \tau \in \mathbb{R}\), where we denoted the inverse Fourier transform

\[\mathcal{F}^{-1}_{x \rightarrow y}\varphi(y) = (2\pi)^{-N} \int_{\mathbb{R}^N} \varphi(x)e^{-ix \cdot y} \, dx.\]

Note that we have already taken into account the fact that \(g_p^\varepsilon\) is a real function.

Denote \(W_p^\varepsilon = W(g_p^\varepsilon, \overline{g_p^\varepsilon})\). Some important properties of this transformation are collected in the following

**Lemma 2.1** The following assertions hold true:

(i) \(W_p^\varepsilon\) is a real-valued distribution function.

(ii) \(W_p^\varepsilon(x, k, t, \tau) = \varepsilon W_p(x, k, \frac{t}{\varepsilon}, \varepsilon \tau)\).

(iii) \(W_p^\varepsilon\) is reduced to the semiconductor Vlasov distribution by \(\tau\)-averaging, that is

\[\int_{\mathbb{R}_\tau} W_p^\varepsilon(x, k, t, \tau) \, d\tau = f_p^\varepsilon.\]

(iv) Given \(f \in L^\infty(\mathbb{R})\) and \(g \in \mathcal{S}\), we have \(\int_{\mathbb{R}^2} W(\varepsilon, f)W^\varepsilon(g, g) \, d\tau \, dt \geq 0\).

**Proof.** (i) follows straightforwardly from the fact that \(W(g_p^\varepsilon, \overline{g_p^\varepsilon}) = \overline{W(g_p^\varepsilon, \overline{g_p^\varepsilon})}\), (ii) is an immediate consequence of a change of variables and (iii), (iv) follow from direct calculations.

**Remark 2.1** Assertion (ii) allows to simultaneously interpret our scaling limit \(\varepsilon \to 0\) as a large times (\(\frac{t}{\varepsilon}\)), low frequency (\(\varepsilon \tau\)) limit.

Then, by applying the transformation (8) to the Vlasov equation (7) we find the following equivalent Wigner-type system.

**Proposition 2.1** Let \((g_p^\varepsilon, V_{\text{coul}, \varepsilon}, V_{\text{ext}, \varepsilon})\) be a weak solution (in the sense of \(D'_t\)) of the rescaled Vlasov equation (7) subjected to the initial data \(g_p^\varepsilon(x, k, t = 0) = g_{p1}(x, k)\). Then, for any \(\varepsilon > 0\) fixed, \((W_p^\varepsilon, V_{\text{coul}, \varepsilon}, V_{\text{ext}, \varepsilon})\) is a weak solution of the problem

\[
\varepsilon \frac{\partial}{\partial t} W_p^\varepsilon + \nabla_k E_p(k) \cdot \nabla_x W_p^\varepsilon - \Theta[\nabla_x (V_{\text{coul}, \varepsilon} + V_{\text{ext}, \varepsilon})] \nabla_k W_p^\varepsilon = 0
\]

(9)
in $\mathcal{D}_{t,s}'$, with initial condition

$$W_{p}^{\epsilon}(x, k, \tau) = \frac{1}{2\pi} \int_{\mathbb{R}} g_{p}^{\epsilon}(x, k, \frac{s}{2}) g_{p}^{\epsilon}(x, k, -\frac{s}{2}) e^{-is\tau} \, ds,$$

where $\Theta$ is a pseudo-differential operator with symbol

$$\text{sym}(\Theta)(x, t, s) = \nabla_{x}(V_{\text{coul}, \epsilon} + V_{\text{ext}, \epsilon})(x, t + \frac{s}{2}) + \nabla_{x}(V_{\text{coul}, \epsilon} + V_{\text{ext}, \epsilon})(x, t - \frac{s}{2}).$$

**Proof.** The two first terms of Equ. (9) come out from straightforward calculations. On the other hand, the pseudo-differential term is obtained from

$$\Theta[\nabla_{x}(V_{\text{coul}, \epsilon}(x, t) + V_{\text{ext}, \epsilon}(x, t))]|_{\mathcal{F}^{-1}_{s\to\tau}} \nabla_{k} W_{p}^{\epsilon}(x, k, t, \tau) =$$

$$\mathcal{F}^{-1}_{s\to\tau} \left( \text{sym}(\Theta)(x, t, s) \mathcal{F}_{\tau\to s} \nabla_{k} W_{p}^{\epsilon}(x, k, t, s) \right).$$

Let us now consider the spaces of test functions

$$\mathcal{A} = \{ \Phi \in C^\infty(\mathbb{R}^{6}_{(x,k)} \times \mathbb{R}_{\tau}) \text{ with compact support in } x, \tau, \text{ and periodic in } k \text{ s.t. }$$

$$\Gamma^* - \text{ periodic in } k \text{ s.t. } \int_{-\infty}^{+\infty} \|\mathcal{F}^{-1}_{\tau\to s} \Phi(s)\|_{L^\infty(\mathbb{R}^{3}_{k}; L^2(\mathbb{R}^{3}_{\tau}))} \, ds < +\infty \},$$

$$\mathcal{A}_{t} = \{ \Phi \in C^\infty(\mathbb{R}^{6}_{(x,k)} \times \mathbb{R}^{2}_{(t,\tau)}) \text{ with compact support in } x, t, \tau, \text{ and periodic in } k \text{ s.t. }$$

$$\Gamma^* - \text{ periodic in } k \text{ s.t. } \int_{-\infty}^{+\infty} \|\mathcal{F}^{-1}_{\tau\to s} \Phi(s)\|_{L^1(\mathbb{R}^{3}_{t}; L^\infty(\mathbb{R}^{3}_{k}; L^2(\mathbb{R}^{3}_{\tau})))} \, ds < +\infty \},$$

equipped with the norms

$$\|\Phi\|_{\mathcal{A}} = \int_{-\infty}^{+\infty} \|\mathcal{F}^{-1}_{\tau\to s} \Phi(s)\|_{L^\infty(\mathbb{R}^{3}_{k}; L^2(\mathbb{R}^{3}_{\tau})))} \, ds,$$

$$\|\Phi\|_{\mathcal{A}_{t}} = \int_{-\infty}^{+\infty} \|\mathcal{F}^{-1}_{\tau\to s} \Phi(s)\|_{L^1(\mathbb{R}^{3}_{t}; L^\infty(\mathbb{R}^{3}_{k}; L^2(\mathbb{R}^{3}_{\tau})))} \, ds,$$

respectively. Then, it is a simple matter to check that the sequence $W_{p}^{\epsilon}$ is bounded in $L^\infty(\mathbb{R}_{t}; \mathcal{A}') \cap \mathcal{A}_{t}'$. Moreover, the following regularity (hence compactness) property in time holds.
Proposition 2.2 The sequence of rescaled Wigner functions \( \{ W_p^\epsilon(x, k, t, \tau) \} \) is Lipschitz continuous in time in the sense of \( \mathcal{A}_t \) w.r.t. the variables \( x, k, \tau \). More precisely, we have

\[
|\langle \partial_t W_p^\epsilon, \Phi \rangle_{\mathcal{A}_t, \mathcal{A}}| \leq C_1 \| g_p \|_{L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^3_x \times B))} \| \nabla_k E_p \|_{L^\infty(B)} \| \nabla_x \mathcal{F}_F^{-1} \Phi \|_{\mathcal{A}} \\
+ C_2 \| g_p \|_{L^\infty(\mathbb{R}_t; L^4(\mathbb{R}^3_x \times B))} \| \nabla_k F^{-1} \Phi \|_{\mathcal{A}} \\
\times \left( \| \nabla_x V_{\text{coul}}^0 \|_{L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^3_x))} + \| \nabla_x V_{\text{ext}} \|_{L^\infty(\mathbb{R}_t \times \mathbb{R}^3_x)} \right),
\]

where \( C_1, C_2 \) are positive constants independent of \( \epsilon \).

Proof. The result follows from the rescaled equation (9), which yields

\[
|\langle \partial_t W_p^\epsilon, \Phi \rangle_{\mathcal{A}_t, \mathcal{A}}| \leq \frac{1}{\epsilon} \left( |\langle W_p^\epsilon, \nabla_k E_p(k) \cdot \nabla_x \Phi \rangle_{\mathcal{A}_t, \mathcal{A}}| \\
+ |\Theta[\nabla_x (V_{\text{coul}}^0(x, t) + V_{\text{ext}}(x, t)) W_p^\epsilon, \nabla_k \Phi \rangle_{\mathcal{A}_t, \mathcal{A}}| \right),
\]

after applying the rescaling property stated in Lemma 2.1(ii). We remark that the bound for the gradient of the Coulomb potential in \( L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^3_x)) \) is available according to the compactness of \( V_m^\alpha \) and \( V_{\text{coul}}^\alpha \). Also, \( g_p \) belongs to \( L^\infty(\mathbb{R}_t; L^2 \cap L^4(\mathbb{R}^3_x \times B)) \) because \( f_p \) belongs to \( L^\infty(\mathbb{R}_t; L^1 \cap L^2(\mathbb{R}^3_x \times B)) \) for initial data \( f_{pl} \in L^1 \cap L^2(\mathbb{R}^3_x \times B) \).

We now introduce the following regularization by Gaussians of the Wigner functions: \( \hat{W}_p^\epsilon(x, k, t, \tau) = W_p^\epsilon(x, k, t, \tau) \ast_t \frac{1}{\sqrt{\epsilon}} \Gamma(t) \ast_\tau \Gamma_\epsilon(\tau) \), with \( \Gamma_\epsilon(x) = (2\pi)^{-\frac{d}{2}} \frac{1}{\epsilon} e^{-\frac{|x|^2}{\epsilon}} \). The main properties of this transformation are listed below.

Proposition 2.3 The following assertions hold true:

(i) \( \hat{W}_p^\epsilon \geq 0 \).

(ii) The sequence \( \hat{W}_p^\epsilon \) is bounded in \( \mathcal{A}_t \) and the following estimate

\[
|\langle \hat{W}_p^\epsilon, \Phi \rangle_{\mathcal{A}_t, \mathcal{A}}| \leq C \| g_p \|_{L^\infty(\mathbb{R}_t; L^4(\mathbb{R}^3_x \times B))} \| \Phi \|_{L^1(\mathbb{R}_t \times \mathbb{R}_x; L^\infty(\mathbb{R}^3_x; L^2(B)))}
\]

is fulfilled, where \( C \) is a positive constant independent of \( \epsilon \).

Proof. (i) The positivity of \( \hat{W}_p^\epsilon \) results from (iv) of Lemma 2.1 applied to \( W_p^\epsilon \) and \( W(h^\epsilon) \), defined by (8), where we have chosen

\[
h^\epsilon(t) = \sqrt{\frac{\pi}{\epsilon}} e^{-\frac{|t-t_0|^2}{2\epsilon}} e^{i\tilde{t}t_0}.
\]

To prove (ii) we use the Hölder inequality in \( x, k \) and estimate

\[
|\langle \hat{W}_p^\epsilon, \Phi \rangle_{\mathcal{A}_t, \mathcal{A}}| \leq C \| g_p \|_{L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^3_x \times B))}
\]
\[
\times \int_{\mathbb{R}_t} \frac{1}{\sqrt{\epsilon}} \Gamma_\epsilon(t) \ast_t \left( \int_{-\infty}^{+\infty} e^{-s^2} \| \mathcal{F}_{\tau \mapsto s}^{-1} \Phi(\cdot, \cdot, t) + 2\sqrt{\epsilon} \|_{L^\infty(\mathbb{R}^3_x; L^2(B))} ds \right) dt.
\]

The proof concludes by an application of the Hausdorff–Young theorem and the Young inequality for convolutions.

Notice that we also have

\[ W^\epsilon_p \longrightarrow W_\infty \quad \text{in} \quad L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^3_x \times B; \mathcal{A}_\tau \text{weak-*})), \quad (10) \]

where \( \mathcal{A}_\tau \) is the space of test functions defined by

\[ \mathcal{A}_\tau = \{ \Phi = \Phi(\tau) \in C_0(\mathbb{R}_\tau) \text{ such that } \| \mathcal{F}_{\tau \mapsto \tau}^{-1} \Phi \|_{L^1(\mathbb{R})} < +\infty \}, \]

equipped with the norm \( \| \Phi \|_{\mathcal{A}_\tau} = \| \mathcal{F}_{\tau \mapsto \tau}^{-1} \Phi \|_{L^1(\mathbb{R})} \).

It is clear that (10) yields a stronger convergence for \( W^\epsilon_p \) that in \( \mathcal{A}' \) or \( \mathcal{A}'_t \), since only assumes weak-* convergence as measures in the variable \( \tau \). Moreover, Proposition 2.2 implies

\[ W^\epsilon_p \longrightarrow W_\infty \quad \text{in} \quad C([-T, T]; \mathcal{A}' \text{weak-*}). \quad (11) \]

This (strong convergence in \( t \)) allows to go to the limit \( \epsilon \to 0 \) in the nonlinear pseudo-differential term and we have the following convergence theorem.

**Theorem 2.1 (long–time asymptotics)** Let \( g^\epsilon_p \) be a solution of the rescaled semiconductor Vlasov equation (7) with initial data \( g^\epsilon_p(x, k, t = 0) = g^\epsilon_{p,t} \in L^2 \cap L^4(\mathbb{R}^3_x \times B) \). Then, the convergence property (11) is fulfilled as well as:

\[ n^\epsilon_p(x, t) = \frac{2\pi}{|B|} \int_{\mathbb{R}^3} W^\epsilon_p d\tau dk \longrightarrow n^\infty_p(x, t) = \frac{2\pi}{|B|} \int_{\mathbb{R}^3} W^\infty_p d\tau dk \]

in \( C([-T, T]; \mathcal{A}' \text{ weak-*}) \cap L^\infty(\mathbb{R}_t; L^p(\mathbb{R}^3)) \text{ weak-*} \) with \( 1 \leq p \leq \frac{5}{3}, \)

\[ V_{\text{coul}, \epsilon}(x, t) \longrightarrow V_{\text{coul}, \infty}(x, t) = \frac{2\pi}{|B|} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W^\infty_{\text{coul}}(y, k, t, \tau)}{|x - y|} d\tau dy \]

in \( C([-T, T]; L^q(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}_t; H^1_{\text{loc}}(\mathbb{R}^3)) \) with \( q < 3, \) and

\[ W^\epsilon_p(x, k, t, \tau) \longrightarrow \int_{\mathbb{R}_t} W^\infty_p(x, k, t, \tau) d\tau \geq 0 \quad \text{in} \quad \mathcal{A}'_t. \]

Also assume that the given external potential \( V_{\text{ext}, \epsilon} \) converges in an appropriate space towards some limiting function \( V_{\text{ext}, \infty} \) (for instance, we can suppose that
$V_{\text{ext},\infty}$ belongs to $L^\infty(\mathbb{R}_t; W^{1,\infty}(\mathbb{R}^3))$. Then, the following Wigner–Poisson–type limiting system holds in the sense of distributions:

$$
\nabla_k E_p(k) \cdot \nabla_x W_p^\infty - \Theta[\nabla_x (V_{\text{coul},\infty}^0 + V_{\text{ext},\infty})] \nabla_k W_p^\infty = 0,
$$

(12)

with initial data $W_p^\infty(x, k, \tau) = W_p^\infty(x, k, t = 0, \tau)$.

**Proof.** The strong convergence of $W_p^\epsilon$ in the variable $t$ is deduced from Proposition 2.2 and the application of the generalized Ascoli–Arzela compactness theorem. Also, the convergence of $\nabla_x V_{\text{coul},\epsilon}^0$ to $\nabla_x V_{\text{coul},\infty}$ in $L^\infty(\mathbb{R}_t; L^p_{\text{loc}}(\mathbb{R}^3))$ is guaranteed because of the convergence in $H^1_{\text{loc}}$ of $V_{\text{coul},\epsilon}^0$ (as follows from the a priori estimates of the previous section). For the passage to the limit in the nonlinear term we shall focus on the proof of the convergence

$$
\int \int \int_{\mathbb{R}^3_t \times \mathbb{R}^3_k \times \mathbb{R}} \nabla_x V_{\text{coul},\epsilon}^0(x, t + \frac{s}{2})W_p^\epsilon(x, k, t, \tau) \nabla_k \Phi(x, k, \tau) \, d\tau \, dk \, dx \longrightarrow
$$

$$
\int \int \int_{\mathbb{R}^3_t \times \mathbb{R}^3_k \times \mathbb{R}} \nabla_x V_{\text{coul},\infty}^0(x, t + \frac{s}{2})W_p^\infty(x, k, t, \tau) \nabla_k \Phi(x, k, \tau) \, d\tau \, dk \, dx.
$$

(13)

The rest of the terms occurring in $\Theta[\nabla_x (V_{\text{coul},\epsilon}^0(x, t) + V_{\text{ext},\epsilon}(x, t))]$, involving $\nabla_x V_{\text{coul},\epsilon}(x, t - \frac{s}{2})$ and $V_{\text{ext},\epsilon}$, are dealt with similarly.

Consider a sequence $(\nabla_x V_{\text{coul},\epsilon}^0)_n \in C^1_0(\mathbb{R}^3)$ approximating $\nabla_x V_{\text{coul},\epsilon}^0$ in $H^1$. Then, we have

$$
\int \int \int_{\mathbb{R}^3_t \times \mathbb{R}^3_k \times \mathbb{R}} \nabla_x V_{\text{coul},\epsilon}^0(x, t + \frac{s}{2})(W_p^\epsilon - W_p^\infty)(x, k, t, \tau) \nabla_k \Phi(x, k, t, \tau) \, d\tau \, dk \, dx \leq
$$

$$
\| (\nabla_x V_{\text{coul},\epsilon}^0)_n - \nabla_x V_{\text{coul},\epsilon}^0 \|_{L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^3_k))} \| g \|_{L^2(\mathbb{R}_t; L^4(\mathbb{R}^2_k \times \mathbb{R}^2_k))} \| \nabla_k \Phi \|_{A} + \| (W_p^\epsilon - W_p^\infty, \nabla_k \cdot (\Phi(\nabla_x V_{\text{coul},\epsilon}^0)_n))_{A', A} \|,
$$

where we have used that the function $\nabla_k \cdot (\Phi(\nabla_x V_{\text{coul},\epsilon}^0)_n)$ belongs to $A$. Indeed,

$$
\| \nabla_k \cdot (\Phi(\nabla_x V_{\text{coul},\epsilon}^0)_n) \|_A \leq \| (\nabla_x V_{\text{coul},\epsilon}^0)_n \|_{L^\infty(\mathbb{R}_t \times \mathbb{R}^3_k)} \| \nabla_k \Phi \|_A.
$$

Now, passing to the limit first as $n \to \infty$ and then as $\epsilon \to 0$ (after a diagonal extraction) we get (13). The convergence of $W_p^\epsilon$ can be examined by considering the following splitting:

$$
\int \int \int \int_{\mathbb{R}^3_t \times \mathbb{R}^3_k \times \mathbb{R} \times \mathbb{R}} \left( W_p^\epsilon - \int_{\mathbb{R}_t} W_p^\infty \, d\tau \right) \Phi \, d\tau \, dt \, dk \, dx =
$$
\[
\int \int \int \int (W_p^\epsilon - W_p^\infty)(\Phi \ast_{t, \Gamma}(t) \ast_{k, \Gamma}(s)) \, \, dt \, dk \, dx + \\
\int \int \int \int W_p^\infty [\Phi \ast_{t, \Gamma}(t) \ast_{t, \Gamma}(t) - \delta(t)] \, \, dt \, dk \, dx + \\
\int \int \int \int W_p^\infty [\Phi \ast_{t, \Gamma}(t) - 1] \, \, dt \, dk \, dx.
\]

Then, it follows from direct calculations (by using the dominated convergence theorem) that both $\Phi \ast_{t, \Gamma}(t)$ and $\Phi \ast_{t, \Gamma}$ converge in $A_t$ to the same multiple of $||F_{r_{r+\delta}}(s = 0)||_{L^1(\mathbb{R}_t; L^\infty(\mathbb{R}_k; L^2(B)))}$ and the same occurs for the sequences $\Phi \ast_{t, \Gamma}(t) \ast_t \Gamma_c(t)$ and $\Phi \ast_{t, \Gamma}(t) \ast_t \delta(t)$. Here, we have also used that $||\Phi \ast_{t, \Gamma}(t) \ast_t \Gamma_c(t)||_{A_t} \leq C||\Phi||_{L^1(\mathbb{R}_t \times \mathbb{R}_r; L^\infty(\mathbb{R}_k; L^2(B)))}$, where $C$ is a positive constant independent of $\epsilon$.

There only remains to identify the density and the Coulomb potential at the (long time) limit. The former is an easy consequence of the convergence of $W_p^\epsilon$ towards $W_p^\infty$ in $A_t$, by just writing

\[
\int n_p^\epsilon(x, t) \Phi(x) \, dx = \\
\frac{2\pi}{|B|} \int \int \int (W_p^\epsilon - W_p^\infty)(x, k, t, \tau) \Phi(x) \theta_R(\tau) \, dt \, dk \, dx + \\
\frac{2\pi}{|B|} \int \int \int (W_p^\epsilon - W_p^\infty)(x, k, t, \tau) \Phi(x) (1 - \theta_R(\tau)) \, dt \, dk \, dx + \\
\frac{2\pi}{|B|} \int \int \int W_p^\infty(x, k, t, \tau) \Phi(x) \, dt \, dk \, dx,
\]

where $\theta_R(\tau) = \theta(\frac{\tau}{R})$ with $\theta \in C_0(\mathbb{R}_r)$, $\theta = 1$ in $B_1$ and $\theta = 0$ in $B^c_2$. Here, we have denoted $B_a$ the ball centered at zero with radius $a$ and $B^c_a$ its complementary set. We also remark that

\[
||\Phi \theta_R||_A = \sqrt{|B|} ||\Phi||_L^\infty(\mathbb{R}_k) \int_{-\infty}^{+\infty} F_{r+\delta_r}^{-1} \theta_R(s) \, ds
\]

is bounded. Thus, letting $\epsilon \rightarrow 0$ and $R \rightarrow \infty$ (after a diagonal extraction) we obtain

\[
n_p^\infty(x, t) = \frac{2\pi}{|B|} \int \int W_p^\infty \, dt \, dk.
\]

As consequence, the limiting Coulomb potential can be also identified in terms of $W_p^\infty$ through the Poisson equation $\Delta x V_{coul, \infty}^0 + n_p^\infty = 0$ and we are done.
with the proof.

**Remark 2.2** The convergence property for $W^\epsilon_p$ established in (11) might be interpreted in terms of a long time limit via the identification $t = 1$ and $\epsilon = \frac{1}{i}$, i.e.

$$W^\epsilon_p(x, k, t, \frac{T}{\ell}) \to W^\infty_p(x, k, 1, \tau)$$

in $\mathcal{A}'$ as $t \to \infty$. We can thus conclude that the long time limiting equation (12) incorporates a memory process, in the sense that still depends on time and "remembers" the initial data, besides preserving the fully nonlinear character of the original equation. The analysis of this limiting system will be carried over in a forthcoming paper.

References


