$L^\infty$ Stability for Weak Solutions of the Navier–Stokes Equations in $\mathbb{R}^3$ with Singular Initial Data in Morrey Spaces

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In this paper we study the $L^\infty$ stability of weak solutions of the Navier–Stokes equations when the initial data is a measure belonging to a suitable Morrey space. It is also assumed that these measures are sufficiently small in the corresponding norms. The physically relevant cases of vortex filament, vortex ring and vortex sheet structure of vorticity are included in this analysis.

1. Introduction

Consider the initial value problem for the nonstationary Navier–Stokes equations in the whole space which are written in vorticity–velocity formulation as follows:

\[
\frac{\partial \omega}{\partial t} + (u \cdot \nabla) \omega - (\omega \cdot \nabla)u - \nu \Delta \omega = 0, \quad t > 0, \quad x \in \mathbb{R}^3, \quad (1.1)
\]

\[
\omega(x, 0) = \omega_0, \quad (1.2)
\]

\[
\begin{align*}
\text{div } u &= 0, \\
\omega &= \text{rot } u,
\end{align*} \quad (1.3)
\]

\[
u(x, t) \to 0 \text{ as } |x| \to \infty. \quad (1.4)
\]

The purpose of this paper is the study of the asymptotic stability in $L^\infty$ of weak solutions of the problem (1.1)–(1.4), with respect to perturbations of the initial data. This problem was studied first by Secchi in [7]. Secchi
assumes the existence of global strong solution $v(x, t)$ of the Navier–Stokes equations with initial data $v_0$. He proved that for any $u_0 \in H$ there is a so-called suitable weak solution $u \in L^2(0, T; H) \cap L^2(0, T; V)$ such that $u(\cdot, t) - v(\cdot, t)$ goes to zero strongly in $L^2$ as $t \to \infty$ (where $H$ and $V$ denote the $L^2$-closure and the $H^1$-closure, respectively, of the set of all smooth functions with compact support and free divergence); see Caffarelli et al. [1] for more details about the concept of suitable weak solutions. Hence, this result gives a global stability for global strong solutions in $L^2$ sense.

By a weak solution of problem (1.1)–(1.4), we mean a pair $(u, \omega)$ such that $(\nu t)^{1/2}u \in L^\infty[0, T; L^\infty(\mathbb{R}^3)^3]$ and $\omega \in L^2(0, T; L^2_{3/2}(\mathbb{R}^3)^3)$, for all $T > 0$, $u = k \ast \omega$ where the matrix kernel $k$ is defined by $k(z) = (4\pi|z|^3)^{-1}z \times x$, $x$ being the matrix product, and the identity below holds for all $\varphi \in C^\infty_c(\mathbb{R}^3\times[0, T])$,

$$\int_0^T \left\{ - \left< \omega_j, \frac{\partial \varphi_i}{\partial t} + u_j \frac{\partial \varphi_i}{\partial x_j} + \nu \Delta \varphi_i \right> + \left< \omega_j, u_i \frac{\partial \varphi_i}{\partial x_j} \right> \right\} dt = \left< \omega_0, \varphi(x, 0) \right>,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between spaces, $L_{3/2}(\mathbb{R}^3)$ is the Morrey space of order $3/2$ (see Definition 2.2 below), and $\omega_0 \in L_{3/2}(\mathbb{R}^3)^3$ with $\text{div } \omega_0 = 0$. As we will see below, the cases in which the initial vorticity is a vortex filament or a vortex ring are contained in the general situation of a vorticity $\omega_0 \in L_{3/2}(\mathbb{R}^3)^3$.

The main result of this paper is the following.

**Theorem A.** Assume that the pair $(u, \omega)$ is a weak solution of problem (1.1)–(1.4) with initial data $(u_0, \omega_0)$, $\omega_0 \in L_{3/2}(\mathbb{R}^3)^3$ and $\text{div } \omega_0 = 0$. Also, we assume that the initial data $\omega_0$ is sufficiently small in the corresponding norm. Then for every $\omega_0 - \hat{\omega}_0 \in L_{3/2}(\mathbb{R}^3)^3$, sufficiently small, and $\text{div } (\omega_0 - \hat{\omega}_0) = 0$ the inequalities below hold,

$$||(u - \hat{u})(\cdot, t)||_{L^\infty(\mathbb{R}^3)^3} \leq c(\nu t)^{-1/2}||\omega_0 - \hat{\omega}_0||_{L_{3/2}(\mathbb{R}^3)^3}, \quad (1.5)$$

$$||(\omega - \hat{\omega})(\cdot, t)||_{L_{3/2}(\mathbb{R}^3)^3} \leq c||\omega_0 - \hat{\omega}_0||_{L_{3/2}(\mathbb{R}^3)^3}, \quad (1.6)$$

where $(\hat{u}, \hat{\omega})$ is the solution associated to the initial data $\hat{\omega}_0$ and $c$ is a positive constant which is independent of $\omega_0$ and $\hat{\omega}_0$.

Note that the hypothesis on the initial data $\hat{\omega}_0$ allows us to consider regular or singular perturbations of $\omega_0$, and that in any case the difference between velocities tends asymptotically to zero in $L^\infty$ norm as the time increases.

Questions concerning existence and uniqueness of weak solutions for these equations have been recently studied. In [2], Cottet and Soler proved the existence, uniqueness, and decay of solutions of the Navier–
Stokes equations when the initial vorticity is a singular filament measure or, in general, a measure located on a curve. The case of vortex sheet is also included in this analysis. Later in [3], Giga and Miyakawa studied the case in which the vorticity is a measure belonging to an appropriate Morrey space. These results include, in particular, the case of vortex filaments and vortex rings measures.

Let us give a few comments about the techniques employed in the proof of Theorem A as well as some relationships with the work of [2, 3]. In the same context introduced in [3], we assume that our initial data is in the Morrey space $\tilde{L}^{3/2}$ and has free divergence. Regularization and integral techniques (inspired in [2]) and a retardation of the nonlinear term allows us to obtain (1.5) and (1.6). Some estimations in Morrey spaces given in [3] concerning the convolution between the heat kernel and functions in Morrey spaces are also used. An alternative method to obtain (1.5) and (1.6) is an extension of the arguments which are used by Kato [4] to study strong $L^p$ solutions for this kind of weak solution. This is the idea of [3], to prove the existence of weak solutions.

The paper is structured as follows: In Section 2, we give some results concerning the regularity of the velocity when $\omega$ is a vortex filament or a vortex ring. These results are made in function of the regularity of the vortex filament or of the vortex ring strength. Also, some properties of smoothness and convolution between functions of Marcinkiewicz and Morrey spaces and its relations with the heat and Biot–Savart kernels are given. These results are related to the results of Giga and Miyakawa [3]. In Section 3, we obtain some a priori estimates which lead us to deduce (1.5) and (1.6).

2. VORTICITY AND MARCINKIEWICZ AND MORREY SPACES

We are interested in studying the stability of weak solutions when the initial vorticity is a vortex filament or a vortex ring measure.

**Definition 2.1.** We shall say that the initial vorticity has a structure of vortex filament if it is a vector-valued measure which can be represented in the form

$$\omega_0 = \alpha \mu_F,$$

(2.1)

where $\alpha$ is a constant and $F$ is a curve with no end points in $\mathbb{R}^3$ parametrized by a piecewise $C^1$ function $\gamma: I \rightarrow \mathbb{R}^3$ being $I$ an open real interval and such that

$$\langle \omega_0, \varphi \rangle = \sum_{i=1}^3 \int_I \alpha \varphi(\gamma(\xi)) \frac{\partial \gamma_i(\xi)}{\partial \xi} \, d\xi, \quad \forall \varphi \in C_0(\mathbb{R}^3)^3.$$
Since $\omega_0$ must be a divergence-free vector, the function $\alpha$ which represents the circulation of the filament must be constant. This fact agrees with the physical property given by Helmholtz's theorem.

In the same way, we shall say that the initial vorticity has a structure of vortex ring if it is represented by (2.1), and the interval $I$ must then be replaced by the circle $S^1$. In any case, we consider measures located on lines in the form

$$\langle \omega_0, \varphi \rangle = \int_F \varphi(\gamma) \cdot (\alpha \tau)(\gamma) \, d\sigma(F), \quad (2.2)$$

where $\tau$ is the unit vector tangent to $F$, $\cdot$ is the inner product, and $\sigma(F)$ is the measure on the line $F$.

In order to inscribe the vortex filament measures in appropriate spaces, let us consider the Morrey spaces. These spaces allow us to study more general initial data of which vortex filaments and vortex rings can be considered as a particular case. An introduction to these spaces can be seen in Kufner et al. [5] and Giga and Miyakawa [3].

Let $L_p(\mathbb{R}^3)$ be the Morrey space defined as a set of all measures $\mu$ such that

$$TV_{B(x,r)}(\mu) \leq cr^{3/p^*}, \quad 1/p + 1/p^* = 1,$$

$c$ being a positive constant independent of $x \in \mathbb{R}^3$ and $r > 0$, where $TV_{B(x,r)}(\mu)$ is the total variation of $\mu$ in the open ball centered at $x$ with radius $r$. We also define the Morrey space $L_{\infty}(\mathbb{R}^3)$ as $L_p(\mathbb{R}^3) \cap L_{\infty}(\mathbb{R}^3)$.

Endowed with the norm

$$\|\mu\|_{L_p} = \sup_{x \in \mathbb{R}^3, \ r > 0} r^{-3/p^*} TV_{B(x,r)}(\mu),$$

the space $L_p(\mathbb{R}^3)$ is a Banach space. The Morrey spaces verify also the following properties: (a) $L_1(\mathbb{R}^3)$ is the space of finite Radon measures $\mathcal{M}(\mathbb{R}^3)$ and $L_1(\mathbb{R}^3) = L^1(\mathbb{R}^3)$. (b) We have the following identities: $L_p(\mathbb{R}^3) = L_\infty(\mathbb{R}^3) = L^\infty(\mathbb{R}^3)$. (c) The following inclusions are verified: $L_p(\mathbb{R}^3) \subset M^\omega(\mathbb{R}^3) \subset L_p(\mathbb{R}^3)$ for $1 < p < \infty$, with continuous injections, where $M^\omega$ is the Marcinkiewicz space of order $\omega$ (see [5, 9]).

The interest in introducing the Morrey spaces is given by the following proposition. The proof is a simple and straightforward computation using the definition of Morrey spaces.

**Proposition 2.1.** Assume that the vorticity is a vortex filament or a vortex ring in the form (2.1). If $\alpha \tau$ is a $L^q$ vector function on the curve $F$ with $q \geq 1$, where $\tau$ is the unit tangent vector to $F$, the vorticity lies in $L_p(\mathbb{R}^3)$ with $p = 3q^*(3q^* - 1)^{-1}$ and $1/q + 1/q^* = 1$. 
Once we have specified the kind of initial conditions to be studied, let us define the property of decay at infinity of the velocity in the context of weak solutions. We will say that \( u(x, t) \) decay weakly at infinity if
\[
\lim_{n \to \infty} n^{-3} \int_{|y| < 2n} |u(y, t)| \, dy = 0. \tag{1.4'}
\]

Note that functions belonging to Morrey and Marcinkiewicz spaces verify (1.4').

The velocity and the vorticity satisfy certain relationships given by Eqs. (1.3)–(1.4') which lead to an elliptic equation for the velocity and the vorticity. When the vorticity belongs to the Morrey space \( \Eu(\mathbb{R}^3) \) relation \( u = k * \omega \) is well-defined, where by \( f * g \) we will denote the convolution product. In fact, it can be proved that if \( \omega \in \Eu(\mathbb{R}^3) \) then \( u = k * \omega \in \Eu(\mathbb{R}^3) \) (see [3]) and, in addition, \( u \) belongs to the Marcinkiewicz space \( M^\alpha(\mathbb{R}^3) \) (see [8]), \( 1/q = 1/p - 1/3 \).

Another case of great physical interest is the vortex sheet structure of the initial vorticity. In mechanics, see [6], a surface across which the tangential components of velocity are discontinuous is called a vortex sheet. The discontinuity in the velocity is the strength of the sheet. Then we can say that the initial data has a vortex sheet structure if the vorticity can be written as
\[
\omega_0 = \beta \delta_S, \tag{2.3}
\]
where the strength \( \beta \) is a vector parallel to the surface \( S \) and \( \delta_S \) is the scalar Dirac measure located on the surface \( S \).

In this case, the velocity solution is more regular than in the vortex filament or vortex ring cases. The velocity can be written as a double layer potential of strength \( \beta \),
\[
u = k * \omega = \int_\partial k(x - \eta(\xi))\beta(\eta(\xi)) \, d\sigma(S),
\]
where \( \eta: \partial \to \eta(\xi) \in \mathbb{R}^3 \) is a parametrization of the surface \( S \) and \( \partial \) is an open set of \( \mathbb{R}^2 \).

We complete the study of vorticity and Morrey spaces with the following lemma which will be used several times in the next section. For a proof of this lemma, see [3, Prop. 3.1 and 3.2].

**Lemma 2.2.** (i) If \( 1/p + 2/3 < 1 < 1/q + 2/3 \) and \( f \in \Eu(\mathbb{R}^3) \cap \Eu(\mathbb{R}^3) \), then \( u = k * f \in \Eu(\mathbb{R}^3) \) and
\[
\|u\|_{\Eu(\mathbb{R}^3)} \leq c \|f\|_{\Eu(\mathbb{R}^3)} \|f\|_{\Eu(\mathbb{R}^3)},
\]
where \( c \) is independent of \( f \).
(ii) The following estimates hold for \( f \in \dot{L}_q(\mathbb{R}^3)^1, q \leq p \leq \infty \) and \( k = 0, 1, 2, \ldots \):

\[
\|\delta^k(\Gamma(x, t) \ast f)\|_{L_q(\mathbb{R}^3)^1} \leq c(\nu t)^{-k(1-1/q-1/p)/2} \|f\|_{L_p(\mathbb{R}^3)^1}.
\]

Here \( c \) is independent of \( f \) and \( t > 0 \), and \( \Gamma \) is the heat kernel which is defined as \( \Gamma(x, t) = (4\pi \nu t)^{-3/2} \exp(-|x|^2/4\nu t) \).

3. Stability of Weak Solutions

Let us summarise the existence results for weak solutions contained in [3] and in the particular case of a vortex filament in [2] in the following theorem.

**Theorem 3.1.** Let \( \omega_0 \in \dot{L}_{3/2}(\mathbb{R}^3)^3 \) with \( \text{div}(\omega_0) = 0 \). Let \( \pi(\omega_0) \) a constant depending on the norm \( L_{3/2}(\mathbb{R}^3)^3 \) of \( \omega_0 \). Then two positive constants \( c_0 \) and \( c_1 \) exist such that if \( \pi(\omega_0)/\nu \leq c_0 \) the system (1.1)-(1.4) has a unique weak solution \((u, \omega)\) such that

\[
\|u(\cdot, t)\|_{L^2(\mathbb{R}^3)^1} \leq c_1(\nu t)^{-1/2}\|\omega_0\|_{L_{1/2}(\mathbb{R}^3)^1}, \quad \forall t \in [0, \infty], \tag{3.1}
\]

\[
\|\omega(\cdot, t)\|_{L_{1/2}(\mathbb{R}^3)^1} \leq c_1\|\omega_0\|_{L_{1/2}(\mathbb{R}^3)^1}, \quad \forall t \in [0, \infty], \tag{3.2}
\]

where \( c_1 \) is a positive constant independent of \( \omega_0 \).

In order to construct a regularized Navier–Stokes system, let \( \rho(t) \) and \( \psi(x) \) be two \( C^\infty \) positive functions such that

(i) \( \psi(|x|) = \psi(x) \) is a nonincreasing function of \( |x| \),

(ii) \( \int \rho(t) \, dt = 1 \) and \( \int_{\mathbb{R}^3} \psi(x) \, dx = 1 \),

(iii) \( \rho(t) = 0 \) if \( 1 > t \) or \( t > 2 \).

Given \( \epsilon > 0 \), we set \( \rho_\epsilon(t) = e^{-1}\rho(t/\epsilon), \psi_\epsilon(x) = e^{-1}\psi(x/\epsilon) \).

Now, given a function \( f \) defined on \( \mathbb{R}^3 \times \mathbb{R}_+ \), we introduce a time retarded mollification of \( f \), denoted by \( M^\epsilon(f) \), defined by

\[
M^\epsilon(f) = \int_{2\epsilon}^\infty \rho_\epsilon(t - \tau)f(x, \tau) \, d\tau.
\]

If \( X \) is a Marcinkiewicz, a Morrey or an \( L^p \) space, from the definition of \( M^\epsilon(f) \) we obtain

\[
\|M^\epsilon(f)(\cdot, t)\|_X \leq \max_{0 \leq \tau \leq t} \|f(\cdot, \tau)\|_X.
\]
We introduce the regularized Navier–Stokes system
\[
\partial \omega^\varepsilon/\partial t + M^\varepsilon((u^\varepsilon \cdot \nabla)\omega^\varepsilon - (\omega^\varepsilon \cdot \nabla)u^\varepsilon) - \nu \Delta \omega^\varepsilon = 0, \quad t > 0, \quad x \in \mathbb{R}^3, \tag{3.3}
\]
\[
\omega^\varepsilon(\cdot, 0) = \omega^\varepsilon_0 = \omega_0 * \psi, \tag{3.4}
\]
\[
u^\varepsilon = k * \omega^\varepsilon. \tag{3.5}
\]

In [2, Prop. 3.1], it was proved that the regularized system (3.3)–(3.5) admits for each \(\varepsilon > 0\) a unique \(C^\infty\) classical solution. The existence and uniqueness arguments are based on the fact that in each time interval \([ke, (k + 1)e]\), \(k \in \mathbb{N}\), the regularized Euler equations reduce to a linear problem with \(\omega^\varepsilon_0 \in C^\infty\).

Let us expose in the following lemma some properties of the sequence \((u^\varepsilon, \omega^\varepsilon)_{\varepsilon > 0}\) for general initial data \(\omega_0 \in \dot{L}^{3/2}(\mathbb{R}^3)^3\) with \(\text{div} \omega_0 = 0\). The proof of this lemma can be seen in the appendix of this paper.

**Lemma 3.2.** Assume that \(\omega_0 \in \dot{L}^{3/2}(\mathbb{R}^3)^3\) and \(\text{div} \omega_0 = 0\) and that the initial data \(\omega_0\) is sufficiently small, depending on the viscosity, in the corresponding norm. Then the sequence \((u^\varepsilon, \omega^\varepsilon)_{\varepsilon > 0}\) satisfies the following properties:

(i) \(\|u^\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq c(\nu t)^{-1/2}\|\omega_0\|_{L^1(\Omega)}\),

(ii) \(\|\omega^\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq c\|\omega_0\|_{L^2(\Omega)}\),

(iii) The sequences \((\nu t)^{1/2}u^\varepsilon)_{\varepsilon > 0}\) and \((\omega^\varepsilon)_{\varepsilon > 0}\) are Cauchy in \(L^2(\mathbb{R}^3)^3\), and \(\dot{L}^{3/2}(0, T, \dot{L}^{3/2}(\mathbb{R}^3)^3)\), \(\forall T > 0\), respectively, converge to the pair of functions \((u, \omega)\) defined by Theorem 3.1 which is the unique weak solution of the problem (1.1)–(1.4) with initial data \(\omega_0\). Here \(c\) is independent of \(\nu, \omega_0, \) and \(\varepsilon\).

Assume that the distribution \(\tilde{\omega}_0\) is such that \(\tilde{\omega}_0 \in \dot{L}^{3/2}(\mathbb{R}^3)^3\) with \(\text{div} \tilde{\omega}_0 = 0\). Let \((\tilde{u}^\varepsilon, \tilde{\omega}^\varepsilon)\) be the solution of the system (3.3)–(3.5), with initial data \(\tilde{\omega}_0\), which converges according to Lemma 3.2 to the weak solution \((\tilde{u}, \tilde{\omega})\) of problem (1.1)–(1.4) associated with the initial condition \(\tilde{\omega}_0\). We set \(W = \omega - \tilde{\omega}, U = u - \tilde{u}, W_0 = \omega_0 - \tilde{\omega}_0, W^\varepsilon = \omega^\varepsilon - \tilde{\omega}^\varepsilon,\) and \(U^\varepsilon = u^\varepsilon - \tilde{u}^\varepsilon\).

The pair \((W^\varepsilon, U^\varepsilon)\) verifies the following system of equations:
\[
\partial W^\varepsilon/\partial t + F(W^\varepsilon, u^\varepsilon, \tilde{\omega}^\varepsilon) - \nu \Delta W^\varepsilon = 0, \quad t > 0, \quad x \in \mathbb{R}^3, \tag{3.6}
\]
\[
W^\varepsilon(\cdot, 0) = W^\varepsilon_0, \tag{3.7}
\]
\[
U^\varepsilon = k_\varepsilon * W^\varepsilon. \tag{3.8}
\]

Here
\[
F(W^\varepsilon, u^\varepsilon, \tilde{\omega}^\varepsilon) = M^\varepsilon((u^\varepsilon \cdot \nabla)W^\varepsilon - (W^\varepsilon \cdot \nabla)u^\varepsilon + (U^\varepsilon \cdot \nabla)\tilde{\omega}^\varepsilon - (\tilde{\omega}^\varepsilon \cdot \nabla)U^\varepsilon).
\]
Lemma 3.3. Under the hypothesis of Theorem A, the following inequalities hold for \( t \in [0, +\infty[ \):

\[
\| U^\varepsilon(\cdot, t) \|_{L^\infty(\mathbb{R}^d)} \leq c(\varepsilon t)^{-1/2} \| W_0 \|_{L^1(\mathbb{R}^d)}^{1/2},
\]

(3.9)

\[
\| W^\varepsilon(\cdot, t) \|_{L^1(\mathbb{R}^d)} \leq c \| W_0 \|_{L^1(\mathbb{R}^d)}.
\]

(3.10)

Proof. It is a classical matter to check that the solution of the differential equation (3.6) is given by

\[
W^\varepsilon(\cdot, t) = \Gamma(\cdot, t) \ast W_0^\varepsilon + \int_0^t \Gamma(\cdot, t - s) \ast F(W^\varepsilon, u^\varepsilon, \omega^\varepsilon)(\cdot, s) \, ds.
\]

Observe that from the definition of \( M^\varepsilon \) and from the fact that \( W^\varepsilon = \text{rot } U^\varepsilon \), we have the following interesting properties which are used continuously from now on:

\begin{align*}
(1) & \quad F(W^\varepsilon, u^\varepsilon, \omega^\varepsilon) = \nabla M^\varepsilon(u^\varepsilon \otimes W^\varepsilon - W^\varepsilon \otimes u^\varepsilon + U^\varepsilon \otimes \omega^\varepsilon - \omega^\varepsilon \otimes U^\varepsilon), \\
(2) & \quad F(W^\varepsilon, u^\varepsilon, \omega^\varepsilon) = \text{rot } \nabla M^\varepsilon(u^\varepsilon \otimes u^\varepsilon + \hat{u}^\varepsilon \otimes U^\varepsilon).
\end{align*}

Here \( f \otimes g = f_i g_j \) and \( I_2 \) can be derived from the following computations:

\[
M^\varepsilon((u^\varepsilon \cdot \nabla) \omega^\varepsilon - (\omega^\varepsilon \cdot \nabla) u^\varepsilon + (\hat{u}^\varepsilon \cdot \nabla) \omega^\varepsilon - (\omega^\varepsilon \cdot \nabla) \hat{u}^\varepsilon)
= M^\varepsilon[\text{rot }((u^\varepsilon \cdot \nabla) u^\varepsilon - (\hat{u}^\varepsilon \cdot \nabla) \hat{u}^\varepsilon)]
= M^\varepsilon[\text{rot }\nabla(u^\varepsilon \otimes u^\varepsilon - \hat{u}^\varepsilon \otimes \hat{u}^\varepsilon)]
= M^\varepsilon[\text{rot }\nabla(U^\varepsilon \otimes u^\varepsilon + \hat{u}^\varepsilon \otimes U^\varepsilon)]
= \text{rot } \nabla M^\varepsilon(U^\varepsilon \otimes u^\varepsilon + \hat{u}^\varepsilon \otimes U^\varepsilon).
\]

Let us estimate the norm of \( U^\varepsilon \) in \( L^\infty \)

\[
\| U^\varepsilon(\cdot, t) \|_{L^\infty(\mathbb{R}^d)} \leq \| k \ast \Gamma(\cdot, t) \ast W_0^\varepsilon \|_{L^\infty(\mathbb{R}^d)}
\]

\[
+ \int_0^t \| k \ast \Gamma(\cdot, t - s) \ast F(W^\varepsilon, u^\varepsilon, \omega^\varepsilon)(\cdot, s) \|_{L^\infty(\mathbb{R}^d)} \, ds
\]

\[
= J_1 + \int_0^t J_2 \, ds.
\]

Let us analyse each term of the right hand side of this bound. First, using Lemma 2.2, we find for \( q > 3 \)

\[
J_1 \leq c \| \Gamma(\cdot, t) \ast W_0^\varepsilon \|_{L^1(\mathbb{R}^d)}^{1/(2 - 1/q)} \| W_0^\varepsilon \|_{L^q(\mathbb{R}^d)}^{1/(2 - 1/q)}
\]

\[
= ct^{-1/2} \| W_0^\varepsilon \|_{L^1(\mathbb{R}^d)}^{1/(2 - 1/q)} \| W_0^\varepsilon \|_{L^q(\mathbb{R}^d)}^{1/(2 - 1/q)}
\]

\[
= ct^{-1/2} \| W_0^\varepsilon \|_{L^1(\mathbb{R}^d)}
\]
Now, in order to estimate $J_2$, we use Lemma 2.2 to obtain
\[ J_2 \leq c \| \Gamma(\cdot, t - s) * F(\cdot, s) \|_{L^2(\mathbb{R}^3)}^{1/2} \| \Gamma(\cdot, t - s) * F(\cdot, s) \|_{L^2(\mathbb{R}^3)}^{1/2}. \]
To estimate the second term of the right hand side of the above inequality, we can use $I_2$ to write it as follows:
\[ \| \Gamma(\cdot, t - s) * F(\cdot, s) \|_{L^1(\mathbb{R}^3)} = \| \text{rot} \ \nabla \Gamma(\cdot, t - s) * M^\varepsilon(U^\varepsilon \otimes u^\varepsilon + \tilde{u}^\varepsilon \otimes U^\varepsilon)(\cdot, s) \|_{L^1(\mathbb{R}^3)}. \]
Then applying Lemma 2.2, we obtain the following estimate:
\[ \| \text{rot} \ \nabla \Gamma(\cdot, t - s) * M^\varepsilon(U^\varepsilon \otimes u^\varepsilon + \tilde{u}^\varepsilon \otimes U^\varepsilon)(\cdot, s) \|_{L^1(\mathbb{R}^3)} \]
\[ \leq c \nu(t - s)^{-1} \max_{s \leq t \leq s} \{ \| u^\varepsilon(\cdot, \tau) \|_{L^1(\mathbb{R}^3)} \| u^\varepsilon(\cdot, \tau) \|_{L^1(\mathbb{R}^3)} \]
\[ + \| U^\varepsilon(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \| \tilde{u}^\varepsilon(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \}. \]
On the other hand, in order to estimate $\Gamma(\cdot, t - s) * F(\cdot, s)$ in the norm $L^1(\mathbb{R}^3)$, we can write it as follows:
\[ \| \Gamma(\cdot, t - s) * F(\cdot, s) \|_{L^1(\mathbb{R}^3)} \]
\[ = \| \Gamma(\cdot, t - s) * \nabla M^\varepsilon(u^\varepsilon \otimes W^\varepsilon - W^\varepsilon \otimes u^\varepsilon + U^\varepsilon \otimes \tilde{u}^\varepsilon - \tilde{u}^\varepsilon \otimes U^\varepsilon)(\cdot, s) \|_{L^1(\mathbb{R}^3)} \]
\[ = \| \nabla \Gamma(\cdot, t - s) * M^\varepsilon(u^\varepsilon \otimes W^\varepsilon - W^\varepsilon \otimes u^\varepsilon + U^\varepsilon \otimes \tilde{u}^\varepsilon - \tilde{u}^\varepsilon \otimes U^\varepsilon)(\cdot, s) \|_{L^1(\mathbb{R}^3)}. \]
Hence this term can be bounded as
\[ \| \Gamma(\cdot, t - s) * F(\cdot, s) \|_{L^1(\mathbb{R}^3)} \]
\[ \leq c(\nu(t - s))^{-1/2} \max_{s \leq t \leq s} \{ \| u^\varepsilon(\cdot, \tau) \|_{L^1(\mathbb{R}^3)} \| W^\varepsilon(\cdot, \tau) \|_{L^1(\mathbb{R}^3)} \}
\[ + \| U^\varepsilon(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \| \tilde{u}^\varepsilon(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \}. \]
Therefore, combining the above estimates we get the following bound for the norm $L^2$ of $U^\varepsilon$:
\[ \| U^\varepsilon(\cdot, t) \|_{L^2(\mathbb{R}^3)} \leq c t^{-1/2} \| W_0 \|_{L^1(\mathbb{R}^3)} \]
\[ + c \int_0^t (\nu(t - s))^{-1/4} \max_{s \leq t \leq s} \{ \| U^\varepsilon(\cdot, \tau) \|_{L^1(\mathbb{R}^3)} \| \tilde{u}^\varepsilon(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \}
\[ + \| U^\varepsilon(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \| \tilde{u}^\varepsilon(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \| \tilde{u}^\varepsilon(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \| \tilde{u}^\varepsilon(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \}. \]
\[ + \|\tilde{u}^\varepsilon(\cdot, \tau)\|_{\mathcal{L}^2(\mathbb{R}^3)} \right) \right)^{1/2} \|W^\varepsilon(\cdot, \tau)\|_{\mathcal{L}^2(\mathbb{R}^3)} \] \[ + \|U^\varepsilon(\cdot, \tau)\|_{\mathcal{L}^2(\mathbb{R}^3)} \right) \right)^{1/2} \|W^\varepsilon(\cdot, \tau)\|_{\mathcal{L}^2(\mathbb{R}^3)} \] \[ \|U^\varepsilon(\cdot, \tau)\|_{\mathcal{L}^2(\mathbb{R}^3)} \right) \right)^{1/2} \].

In the same way, we obtain

\[ \|W^\varepsilon(\cdot, t)\|_{\mathcal{L}^3(\mathbb{R}^3)} \leq c \|W_0\|_{\mathcal{L}^3(\mathbb{R}^3)} \]

\[ + \int_0^t \|\nabla \Gamma(\cdot, t - s) * M^\varepsilon(u^\varepsilon \otimes W^\varepsilon - W^\varepsilon \otimes u^\varepsilon \]

\[ + U^\varepsilon \otimes \tilde{\omega}^\varepsilon - \tilde{\omega}^\varepsilon \otimes U^\varepsilon(\cdot, s)\|_{\mathcal{L}^3(\mathbb{R}^3)} \, ds \]

\[ \leq c \|W_0\|_{\mathcal{L}^3(\mathbb{R}^3)} \]

\[ + c \int_0^t (\nu(t - s))^{-1/2} \max_{\|\cdot\|_{\mathcal{L}^2(\mathbb{R}^3)}} \{ \|u^\varepsilon(\cdot, \tau)\|_{\mathcal{L}^2(\mathbb{R}^3)} \}

\[ \|W^\varepsilon(\cdot, \tau)\|_{\mathcal{L}^3(\mathbb{R}^3)} \]

\[ + \|U^\varepsilon(\cdot, \tau)\|_{\mathcal{L}^3(\mathbb{R}^3)} \|\tilde{\omega}^\varepsilon(\cdot, \tau)\|_{\mathcal{L}^3(\mathbb{R}^3)} \} \, ds. \]

Using the estimations of Lemma 3.2 and setting

\[ \lambda(t) = \sup_{s \leq t} \{ (\nu s)^{1/2} \|U^\varepsilon(\cdot, s)\|_{\mathcal{L}^2} \|W^\varepsilon(\cdot, s)\|_{\mathcal{L}^3}, \]

\[ (\nu s)^{1/2} \|\tilde{u}^\varepsilon(\cdot, s)\|_{\mathcal{L}^2}, \|\tilde{\omega}^\varepsilon(\cdot, s)\|_{\mathcal{L}^2}, \]

\[ (\nu s)^{1/2} \|u^\varepsilon(\cdot, s)\|_{\mathcal{L}^2}, \|\omega^\varepsilon(\cdot, s)\|_{\mathcal{L}^2} \}, \]

we obtain for the estimation of \( U^\varepsilon \) in norm \( L^\varepsilon(\mathbb{R}^3)^3 \)

\[ \lambda(t) \leq c \|W_0\|_{\mathcal{L}^3(\mathbb{R}^3)} + c\nu^{-1} t^{1/2} \lambda(t)^2 \int_0^t (t - s)^{-3/4} s^{-3/4} ds \]

\[ \leq c \|W_0\|_{\mathcal{L}^3(\mathbb{R}^3)} + c\nu^{-1} \lambda(t)^2. \] (3.11)

In an analogous way, we obtain the same inequality for \( \lambda(t) \) using the estimation of \( W^\varepsilon \) in the norm \( \mathcal{L}^3(\mathbb{R}^3)^3 \).

Then if the discriminant

\[ \sigma = (1 - 4c\nu^{-1} \|W_0\|_{\mathcal{L}^3(\mathbb{R}^3)}) \]
is positive, $\lambda(t)$ satisfies the nonlinear inequality (3.11). Note that this condition implies that the quantity

$$
\nu^{-1/2}||W_0||_{L^{1/2}(\mathbb{R}^3)}
$$

must be small.

Therefore, under this condition we find the announced estimates

$$
||U^\varepsilon(\cdot, t)||_{L^\infty(\mathbb{R}^3)} \leq c(\nu t)^{-1/2}||W_0||_{L^{1/2}(\mathbb{R}^3)}, \quad (3.9)
$$

$$
||W^\varepsilon(\cdot, t)||_{L^{1/2}(\mathbb{R}^3)} \leq c||W_0||_{L^{1/2}(\mathbb{R}^3)}. \quad (3.10)
$$

Now, it is necessary to pass to the limit in the sequence $U^\varepsilon$, $W^\varepsilon$, $\varepsilon > 0$, as $\varepsilon$ tends to zero, which will prove that inequalities (3.9) and (3.10) are also valid for the functions $U(\cdot, t)$ and $W(\cdot, t)$ and therefore the proof of Theorem A will be achieved. Using (3.9) and (3.10) and repeating the same reasoning used in Lemma 3.3 applied to the difference $W^\varepsilon - W^\varepsilon'$ and $U^\varepsilon - U^\varepsilon$, for positive $\varepsilon$ and $\varepsilon'$, it is a simple matter to prove the convergence property in $L^p([0, T] \times \mathbb{R}^3)$, $L^p(0, T, \mathcal{L}_{3/2}(\mathbb{R}^3))$, $\forall T > 0$, of $((\nu t)^{1/2}U^\varepsilon, W^\varepsilon)_{\varepsilon > 0}$ toward $(\nu t)^{1/2}U, W$.

Reference [4] studies the $L^p$ solutions of Navier–Stokes equations in $\mathbb{R}^m$. These arguments can be extended to the case of initial data lying in a Morrey space. This idea can be used to obtain an alternative proof of inequalities (3.10) and (3.11). This is the method used in [3] for the study of the existence of strong solutions with initial conditions satisfying assumptions similar to those studied in this paper.

From these inequalities, we can deduce some properties of weak solutions of the Navier–Stokes equations. In the first place, (3.9) and (3.10) show the uniqueness of weak solutions, for initial data $\omega_0 \in \mathcal{L}_{3/2}(\mathbb{R}^3)$ with $\text{div} \omega_0 = 0$, in the class of functions $(f, g)$ such that

$$
(u - f) \in L^6(0, T; L^6(\mathbb{R}^3)) \quad \text{and} \quad (\omega - g) \in L^6(0, T; \mathcal{L}_{3/2}(\mathbb{R}^3)),
$$

$(u, \omega)$ being the weak solution associated to the Navier–Stokes equations.

Also the inequalities (3.9) and (3.10) allow us to assure the $L^6(\mathbb{R}^3)$ stability of velocities and the $\mathcal{L}_{3/2}(\mathbb{R}^3)$ stability of vorticities which are solutions of the Navier–Stokes equations for perturbations of the initial data such that

$$
\omega_0 - \tilde{\omega}_0 \in \mathcal{L}_{3/2}(\mathbb{R}^3) \quad \text{and} \quad \text{div}(\omega_0 - \tilde{\omega}_0) = 0. \quad (H)
$$

Finally, also remarkable is the fact that the difference between velocities tends to zero in $L^6(\mathbb{R}^3)$-norm for large times.
APPENDIX: Proof of Lemma 3.2

The proof of this lemma basically follows from the arguments given in [2, 3], but since the initial data in [2] is a little more restrictive, let us sketch a proof based in the same ideas that have been developed to prove the results of Lemma 3.3. Then estimates (i) and (ii) can be deduced in an identical way to the (3.9) and (3.10).

In order to prove that the sequences \((\nu t)^{1/2}u^\varepsilon)_{\varepsilon>0}\) and \((\omega^\varepsilon)_{\varepsilon>0}\) are Cauchy in \(L^r([0, T] \times \mathbb{R}^3)\), and \(L^r(0, T, L^3_\delta(\mathbb{R}^3))\), \(\forall T > 0\), respectively, we consider the functions \((u^\varepsilon, \omega^\varepsilon)\) and \((u'^\varepsilon, \omega'^\varepsilon)\), for positive \(\varepsilon\) and \(\varepsilon'\). These pairs of functions are approximated solutions verifying (3.3)–(3.5) for the initial data \(\omega_0 = \omega_0 * \psi_0\) and \(\omega_0' = \omega_0 * \psi_{\varepsilon'}\), respectively. Let us define the functions \(V(x, t) = (u^\varepsilon - u'^\varepsilon)(x, t)\) and \(O(x, t) = (\omega^\varepsilon - \omega'^\varepsilon)(x, t)\). Following the same steps as in Lemma 3.3, it is a simple matter to obtain the estimates

\[
\|V(\cdot, t)\|_{L^r(\mathbb{R}^3)} \leq c(\nu t)^{-1/2}\|\omega_0 - \omega_0'\|_{L^3_\delta(\mathbb{R}^3)}, \quad (3.12)
\]

\[
\|O(\cdot, t)\|_{L^r(\mathbb{R}^3)} \leq c\|\omega_0 - \omega_0'\|_{L^3_\delta(\mathbb{R}^3)}, \quad (3.13)
\]

where the constant \(c\) is independent of \(\omega_0\), \(\varepsilon\) and \(\varepsilon'\). Since \((\omega_0)_{\varepsilon>0}\) is a Cauchy sequence, the sequences \((\nu t)^{1/2}u^\varepsilon)_{\varepsilon>0}\) and \((\omega^\varepsilon)_{\varepsilon>0}\) are also Cauchy.

Finally, the uniqueness property of weak solutions under our assumptions on the initial data can be seen in [3]. (It is also possible to prove it using the same techniques of Lemma 3.3 taking \(\omega_0 = \tilde{\omega}_0\).)

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References


