Parameter Estimation of Exponentially Damped Sinusoids Using a Higher Order Correlation-Based Approach

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Abstract—A very common problem in signal processing is parameter estimation of exponentially damped sinusoids from a finite subset of noisy observations. When the signal is contaminated with colored noise of unknown power spectral density, a cumulant-based approach provides an appropriate solution to this problem. In this paper, we propose a new class of estimator, namely, a covariance-type estimator, which reduces the deterministic errors associated with imperfect estimation of higher order correlations from finite-data length. This estimator allows a higher order correlation sequence to be modeled as a damped exponential model in certain slices of the moments plane. This result shows a useful link with well-known linear-prediction-based methods, such as the minimum-norm principal-eigenvector method of Kumaresan and Tufts (KT), which can be subsequently applied to extracting frequencies and damping coefficients from the 1-D correlation sequence. This paper discusses the slices allowed in the moments plane, the uses and limitations of this estimator using multiple realizations, and a single record in a noisy environment. Monte Carlo simulations applied to standard examples are also performed, and the results are compared with the KT method and the standard biased-estimator-based approach. The comparison shows the effectiveness of the proposed estimator in terms of bias and mean-square error when the signals are contaminated with additive Gaussian noise and a single data record with short data length is available.

I. INTRODUCTION

PARAMETER estimation of exponentially damped sinusoidal signals is a problem that arises in several practical fields, and it has generated an enormous amount of contributions in the literature [1]–[9]. Related problems include pole-zero modeling of empirically generated time-series data since the impulse response of a linear stable rational model is composed of exponentially damped sinusoids. From the start, the different techniques developed for this purpose have been aimed at improving parameter-estimation characteristics, especially when data length is short, the signal-to-noise ratio (SNR) is low, and closely spaced sinusoids are present. Fitting an exponentially damped sinusoidal model to the set of available noisy data is basically a nonlinear problem for the amplitudes and frequencies. Two principal approaches have been previously proposed: 1) parameter estimation based on maximum-likelihood methods (ML) [1]–[3] and 2) linear-prediction methods (LP) combined with lower rank approximation of the data matrix. The ML method is based on the maximization of a highly nonlinear function of the unknown parameters, which requires the use of an $M$-dimensional space ($M$ numbers of sinusoids) [1]. Therefore, although this method gives good estimations when the noise is white Gaussian and the SNR is low, it is computationally involved. The linear-prediction method of Kumaresan and Tufts (KT) [4] is based on singular value decomposition (SVD) of the data-dependent matrix to replace it with a matrix of rank equal to the number of sinusoids present in the signal. Kumaresan and Tufts showed that their method achieves a threshold SNR much below the value achieved in conventional LP methods and that it performs close to ML estimation when the additive noise is white.

Recently, matrix-pencil methods and methods using higher order statistics (HOS) have been presented [6]–[8]. The latter have been the object of growing interest during the last few years as they have the advantage of handling colored Gaussian noise without using its spectral characteristics [11, 12]. The key lies in that all the Gaussian-noise cumulants of order greater than two are equal to zero; therefore, the noise can be suppressed in the cumulant domain [12]. The problem of deterministic-signal-parameter estimation in white or colored noise using finite-data length and a single data record is studied by Papadopoulos and Nikias [7]. They suggest computing the third-order correlations or fourth-order cumulants of noisy data using a biased estimator for a single data record.

This paper is concerned with damped-sinusoidal parameter estimation in noise using multiple correlations. Note that this process is neither ergodic nor stationary; therefore, the statistical properties cannot be determined from a single realization. Up to now, research (e.g., [15] and [16]) has focused on parameter estimation using multiple realizations or a sufficient number of available data [7]. The problem addressed here uses a single-data record and a small number of measured data, which is typical of many practical situations [19]. We show that when only a single realization is available and the amount of data is small, the deterministic errors associated with the use of the standard biased estimator [10]–[14], [17], [18] are considerably higher than stochastic errors due to noise. This paper proposes the use of a covariance-type estimator that overcomes the problem with the biased estimator and maintains the same performance with noisy data (white or...
colored) as the HOS-based method even if the amount of available data is small.

This paper is organized as follows: In Section II, we give a brief review of the previous results concerned with the problem addressed here. In Section III, some preliminary definitions are introduced, and the parameter-identification problem using a slice in the moment plane (extension to fourth-order statistics are provided in Appendix D) is established. This section also includes a study of the different possible useful slices with the biased estimator and introduces a new estimator called “the covariance-type estimator” for the estimation of the third-order moment sequence to overcome the problem of the standard biased estimator when the amount of data is small. In Section IV, the problem of measured noise is discussed when multiple or a single realization of the noisy signal is available. Section V deals the identification issue using a single 1-D slice and summarizes the steps of a consistent algorithm for estimating the signal parameters. Section VI provides simulation results to demonstrate the effectiveness of the covariance-type estimator when short data length are available, including a study of the parameters involved in this estimation problem. Finally, the major results obtained here are summarized in Section VII.

II. PRELIMINARIES AND PREVIOUS RESULTS

Suppose that the signal under consideration can be modeled for any instant \( n \geq 0 \) as \( M \) exponentially damped complex sinusoids of the type

\[
x(n) = \sum_{m=1}^{M} a_m e^{s_m n}
\]

(1)

where the complex numbers \( a_m, m = 1, 2 \cdots M \), are the complex amplitudes, and \( s_m = \sigma_m + j2\pi f_m \), \( m = 1, 2 \cdots M \) are the complex frequencies. \( \sigma_m \) are the damping factors and \( f_m \) the pole frequencies. The problem addressed here deals with the estimation of the frequencies, damping factors, and, when desired, complex amplitudes from a finite amount of observed data \( x(n), n = 1 \cdots N \). The well-known KT method [4] sets up the following linear-prediction equations using complex conjugate data in the backward direction to find the polynomial coefficients \( b(i) \):

\[
\begin{pmatrix}
x^*(0) & x^*(1) & \cdots & x^*(K) \\
x^*(1) & x^*(2) & \cdots & x^*(K+1) \\
x^*(2) & x^*(3) & \cdots & x^*(K+2) \\
\vdots & \vdots & \ddots & \vdots \\
x^*(N-K-1) & x^*(N-K) & \cdots & x^*(N-1)
\end{pmatrix}
\times \begin{pmatrix}
b(1) \\
b(2) \\
\vdots \\
b(K)
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

(2)

where \( K \) is the prediction-error filter order that satisfies \( M \leq K \leq N - M \). Kumaresan and Tufts [5] showed that if the minimum-norm solution to (2) is found, the prediction-error filter polynomial

\[
B(z) = 1 + b(1)z^{-1} + b(2)z^{-2} + \cdots + b(K)z^{-K}
\]

(3)

has \( M \) zeros at \( \exp\{-s_k^*\} \), \( k = 1, 2 \cdots M \), and \( K-M \) extraneous zeros within the unit circle, which allows us to identify frequency and damping coefficients by simply rooting the polynomial and selecting zeros outside the unit circle.

The HOS-based approach proposed by Papadopoulos and Nikias [7] comprises the linear system of equations

\[
R_x b = 0
\]

(4)

and (5), which appears at the bottom of the page, where vector \( b \) in given in (2). In this equation, \( R_x(-\tau, -\tau) = 0, \ldots K \) is the third-order-moments sequence of signal \( x(n) \) along the diagonal slice in the third quadrant [12]. Papadopoulos and Nikias showed [7] via matrix decomposition that if \( R_x \) are the true third-order correlations of signal \( x(n) \) and the slice is full rank, the matrix \( R_x \) is of rank \( M \), and the prediction-error filter polynomial defined in (3) has \( M \) roots at \( \exp\{-s_k^*\} \), \( k = 1, 2 \cdots M \). The extraneous \( K-M \) zeros lie inside the unit circle if \( b \) is the minimum-norm solution of the linear system, and \( K \geq M \). Similarly, these results can be extended to the diagonal-slice of the fourth-order signal correlations.

Although the above results for KT and HOS methods are valid only for the ideal situation of noise-free data and true moments sequence, they also turn out to be valid if a truncated SVD solution is used by setting the smaller \( K-M \) singular values of \( R_x \) or \( X \) to zero, thereby forcing matrices \( R_x \) or \( X \) to be of rank \( M \) and finding the minimum-norm solution of the resulting linear system.

III. PROBLEM STATEMENT AND HIGHER ORDER CORRELATION-BASED APPROACH

Consider the noiseless signal in (1). We are concerned with the estimation of the frequencies and damping factors from a 1-D slice (full-rank slice) of the higher order correlation sequence of energy signal \( x(n) \). In the third-order case, the

\[
R_x = \begin{pmatrix}
R_x^*(0,0) & R_x^*(-1,-1) & R_x^*(-2,-2) & \cdots & R_x^*(-K,-K) \\
R_x^*(-1,-1) & R_x^*(-2,-2) & R_x^*(-3,-3) & \cdots & R_x^*(-K-1,-K-1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
R_x^*(-K,-K) & R_x^*(-K-1,-K-1) & R_x^*(-K-2,-K-2) & \cdots & R_x^*(-2K,-2K)
\end{pmatrix}
\]

(5)
third-order correlation sequence is defined as [18]
\[
R_x(\tau_1, \tau_2) = \sum_{n=-\infty}^{\infty} x(n)x^*(n + \tau_1)x^*(n + \tau_2)
\]
where \( \tau_1, \tau_2 = 0, \pm 1, \pm 2 \cdots \) (6)

One-dimensional slices of third-order correlations can be defined as
\[
r_x(\tau) = R_x(\tau, a\tau + b) \quad \tau = 0, \pm 1, \pm 2 \cdots
\]
where \( a \) and \( b \) are the slope and intercept, respectively, of the line along the moments plane \( \tau_1, \tau_2 \).

The signal parameters could be recovered from a 1-D slice in the moments plane, provided the 1-D slice can be modeled as the sum of \( M \) decaying complex exponentials oscillating with the same frequencies and damping factors as the original signal. In such a case, we can construct a Hankel matrix of type (2) with correlations instead of data. Since the 1-D slice behaves as a damped exponential signal, the results from the previous section for the KT method can be applied; therefore, the frequencies and damping coefficients of the signal can be identified.

A question arises here as to when the 1-D slice can be modeled as the sum of \( M \) damped exponentials. We can distinguish two cases for the deterministic signal.

A. Infinite-Data Length

If we know the energy signal elsewhere, we can construct the true third-order correlation sequence without “deterministic” errors. If the energy signal is defined as in (1) from \( n = 0, 1, \ldots \), the 1-D slice in the plane \( \tau_1, \tau_2 \) becomes
\[
r_x(\tau) = R_x(\tau, a\tau + b) = \sum_{n=-n_0}^{\infty} x(n)x^*(n + \tau)x^*(n + a\tau + b)
\]
where \( n_0 = \max\{0, -\tau, -a\tau - b\} \) (8)

since \( x(n) = 0 \) for \( n < 0 \). Making use of (1), (8) results in
\[
r_x(\tau) = \sum_{i,j,k=1}^{M} a_i a_j \tau^{-k} e^{s_i^* s_j^*} e^{s_i^* (a\tau + b)} e^{(s_i + s_j^*)n}
\]
and by performing the geometric sum, we get
\[
r_x(\tau) = \sum_{i,j,k=1}^{M} a_i a_j \tau^{-k} e^{s_i^* s_j^*} e^{s_i^* (a\tau + b)} e^{(s_i + s_j^*)n} / (1 - e^{s_i + s_j^*} e^{s_i^* (a\tau + b)})
\]

(9)

To build a damped exponential model for \( r_x(\tau) \), we can vary \( \tau \) from 0 to \( \infty \) or from 0 to \( -\infty \) with several cases appearing depending on the quadrant on which the 1-D slice lies in the moments plane. These cases are collected in the following proposition.

Proposition 1: Only horizontal slices in the first and fourth quadrants and diagonal slices in the third quadrant allow the original damped exponential structure of data for the moments sequence to be preserved.

Proof: See Appendix A.

B. Finite Data Length

In the above subsection, we dealt with true higher order correlations of the energy signal \( x(n) \). In practice, we have only a finite subset of data samples, namely, \( n = 0, \ldots, N - 1 \). In this case, there will be “deterministic” errors associated with the imperfect estimation of correlations. These errors can eliminate the fundamental property of higher order correlations, allowing the signal parameters to be recovered, i.e., the moments sequence should be modeled as a sum of exponentially damped sinusoids even for the finite-data case.

These deterministic errors depend on the method used for estimating the correlations. Here, we analyze the standard-biased estimator, which is usually used for estimation of higher order correlations and propose a new estimator we call the covariance-type estimator (cov-type). It allows us to retain the fundamental property described above and to extend the type of possible useful slices to recover signal parameters. These characteristics make the cov-type estimator more reliable than the biased one, especially when the amount of available data is small.

1) Biased Estimator: Suppose we are interested in estimating \( nm \) moments from \( N \) available data \( x(n) \). The standard biased estimator is defined as [12]–[14], [17], [18]
\[
R_x(\tau_1, \tau_2) = \frac{1}{N} \sum_{n=S_1}^{S_2} x(n)x^*(n + \tau_1)x^*(n + \tau_2)
\]
\[
S_1 = \max\{0, -\tau_1, -\tau_2\}
\]
\[
S_2 = \min\{N - 1, N - 1 - \tau_1, N - 1 - \tau_2\}
\]

(11)

We want to use a 1-D slice to estimate signal parameters. Taking into account (1), the estimated sequence will be
\[
\hat{r}_x(\tau) = R_x(\tau, a\tau + b) = \frac{1}{N} \sum_{i,j,k=1}^{M} a_i a_j \tau^{-k} e^{s_i^* s_j^*} e^{s_i^* (a\tau + b)}
\]
\[
\times e^{s_i^* (a\tau + b)} e^{s_i^* (a\tau + b)} e^{s_i^* (a\tau + b)} e^{s_i^* (a\tau + b)}
\]
\[
\times (1 - e^{s_i + s_j^*} e^{s_i^* (a\tau + b)}) S_1
\]
\[
\times (1 - e^{s_i + s_j^*} e^{s_i^* (a\tau + b)}) S_2
\]
\[
\tau = 0, \pm 1, \pm nm
\]

(12)

As can be seen from (12), when this estimator is used, there appears a link among different frequencies since \( S_1 \) and \( S_2 \) are functions of \( \tau \). This situation can invalidate the damped exponential model for \( r_x(\tau) \), thus making it impossible to obtain the parameters. The results for the different cases that arise with this estimator are given in the following proposition:

Proposition 2: If a biased (or unbiased) estimator is used to estimate the moments sequence from \( N \) data, only certain horizontal slices (with high lags) in the first quadrant allow \( r_x(\tau) \) to be modeled as a sum of \( M \) decaying complex exponentials. In this case, the diagonal line in the third quadrant would not be valid.

Proof: See Appendix B.

2) Covariance-Type Estimator: In order to avoid the problems associated with a finite amount of data in the biased estimator, it would be necessary to make the term in brackets in (12) independent of \( \tau \). To accomplish this, it is enough to impose \( S_2 - S_1 + 1 = constant \) so that the third-order sequence
where cumulant sequence \( C_{\gamma'} \) depends on \( n \) since the process is not stationary. Taking into account that the signal is independent of noise and that the third-order moment of a Gaussian process is zero, we get

\[
C_{\gamma'}(n, \tau_1, \tau_2) = x'(n)x'^*(n+\tau_1)x'^*(n+\tau_2).
\]

If the sum over index \( n \) is taken, we obtain the \( \tau_1, \tau_2 \) cumulant sequence \( C_{\gamma'}(\tau_1, \tau_2) \)

\[
C_{\gamma'}(\tau_1, \tau_2) = \frac{1}{N} \sum_{n=N}^{T_2} x'(n)x'^*(n+\tau_1)x'^*(n+\tau_2)
\]

where \( T_1 \) and \( T_2 \) are defined as in (13). In this case, the \( \tau_1, \tau_2 \) cumulant sequence become the estimated third-order correlations of the signal. Using the cov-type estimator ensures that the signal parameters can be obtained from an estimated 1-D slice of moments, as stated in Proposition 3.

When only a single realization is available, which occurs in many practical situations such as radar-target discrimination [19], the \( \tau_1, \tau_2 \) third-order cumulant sequence is the third-order correlation sequence of noisy data that, taking into account (15), becomes

\[
\hat{R}'_{\gamma'}(\tau_1, \tau_2) = \frac{1}{N} \sum_{n=N}^{T_2} y'(n)y'^*(n+\tau_1)y'^*(n+\tau_2)
\]

\[
= \hat{R}'_{\gamma'}(\tau_1, \tau_2) + \frac{1}{N} \sum_{n=N}^{T_2} x'(n)x'^*(n+\tau_1)w'^*(n+\tau_2)
+ \frac{1}{N} \sum_{n=N}^{T_2} x'(n)w'^*(n+\tau_1)x'^*(n+\tau_2)
+ \frac{1}{N} \sum_{n=N}^{T_2} w(n)x'^*(n+\tau_1)x'^*(n+\tau_2)
+ \frac{1}{N} \sum_{n=N}^{T_2} w(n)x'^*(n+\tau_1)w'^*(n+\tau_2)
+ \frac{1}{N} \sum_{n=N}^{T_2} w(n)x'^*(n+\tau_1)w'^*(n+\tau_2)
\]

(19)

As can be seen, the estimated moments of the measurement signal includes the estimated moments of the exponential signal plus the crossmoments between the signal and the noise and the estimated third-order moments of the noise. These terms are undesirable and behave as a “noise” that perturbs the moment sequence. In this case, when only a single realization is available, the influence of the noise on the third-order correlations of the signal cannot be avoided. Combining this with robust techniques such as SVD to solve for signal parameters, however, allows us to obtain better results when
using these techniques with raw data, as demonstrated in Section VI.

In like manner, all the results of these sections can be extended to higher order correlations without changes. Particularly, the extension to fourth-order correlations or cumulants is useful and is provided in Appendix D.

V. HIGHER ORDER CORRELATION MATRICES

Let us suppose we have N samples from a damped exponential signal defined in (1). In accordance with the results of Proposition 2, if the correlation sequence along a permitted 1-D slice is estimated using the cov-type estimator and provided it is a full-rank slice, this sequence retains the damped exponential data model. Since the correlation sequence can be modeled as data, the KT method may be used with the moments sequence to construct the following correlation matrix:

\[
\begin{pmatrix}
\hat{r}_x^*(0) & \hat{r}_x^*(\pm1) & \cdots & \hat{r}_x^*(\pm K) \\
\hat{r}_x^*(\pm1) & \hat{r}_x^*(\pm2) & \cdots & \hat{r}_x^*(\pm(K+1)) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{r}_x^*(\pm(K+1)) & \hat{r}_x^*(\pm(K+1)) & \cdots & \hat{r}_x^*(\pm(N))
\end{pmatrix}
\]  

where the estimated moments sequence using the cov-type estimator (see (13)) is defined as

\[
\hat{r}_x(\tau) = \frac{1}{N} \sum_{n=T_1}^{T_2} x(n)x^*(n+\tau)x^*(n+a\tau+b) \quad a = \{0,1\}
\]

\[
\tau = 0, \pm1, \ldots, \tau_{\text{max}}
\]

where b stands for the specific slice taken for the parameter estimation, and the complex-conjugate sequence of (3) must be used in (2) when a = 0. If a diagonal slice (a = 1) is taken in the third quadrant and \(\tau_{\text{max}} = 2\tau_x\), the result is the kind of matrix used in [7] with the standard (un)biased estimator. Note that, in contrast to [7], this matrix does not need to be square. The prediction-error filter order is therefore not related to the maximum higher order correlation estimated lag.

When only noisy data is available, matrix (20) will be the same one that replaces \(x(n)\) by \(y(n)\). Note also that unbiased estimators can be used in all cases by replacing the term 1/N by the inverse of the number of lag products, i.e., N - \(\tau_{\text{max}} + |b|\).

A. The Identifiability Issue: An interesting question that arises here is whether we can estimate the signal parameters using only a single moment slice. This would be possible if the matrix constructed using one particular slice had a rank of M, i.e., the number of exponentials in the data. In this situation the slice is called a full-rank slice. A given slice will be a full-rank slice when the correlation sequence contains the same frequencies as the data, i.e., it can be modeled as a sum of M decaying exponentials. Either matrix of the above types constructed with this slice will then have a rank of M as stated in the KT method [4]. Papadopoulos and Nikias [7] establish sufficient conditions on the \(a_j's\) and \(s_j's\) based on the zeros of complex polynomials to ensure that expressions of type (A.1), (A.2), or (A.3) in Appendix A hold, i.e., A(i), B(i), and C(i) are not equal to zero \(\forall i = 1, \ldots, M\).

Recently, the problem of identifiability of the AR parameter of a causal ARMA model using one slice was addressed by Swami and Mendel in [20]. Since an exponentially damped sinusoidal signal can be seen as the unit-sample response of a proper rational model [21], and taking into account that the stationary output cumulants of a linear system are given by the Brillinger–Rosenblatt formula [18] that allows the calculation of cumulants as the higher order correlations of the unit-sample response, both problems become equivalent. That is, if the correlation slice of a deterministic signal or an output cumulant slice of an ARMA process can be modeled as a sum of M decaying exponentials, then it is a full-rank slice. This result for the ARMA cumulant sequence was also obtained in [20] (Theorem 1). The novelty of this section resides in relating the deterministic nonstationary problem with the stationary random one and applying the results already obtained for the cumulant sequence of a random process to the deterministic correlation sequence to thus recover signal parameters from it.

Keeping in mind the above discussion, if [20] establishes the sufficiency of \(M + 1\) slices for ensuring the identifiability of the AR part of a causal ARMA process, equivalently, it can be seen as establishing the sufficiency of a finite set of slices to recover signal parameters. In the third-order case, we have the following proposition adapted from [20].

**Proposition:** The third-order moment slices \(r_x(\tau,b), b = 0,\ldots,M - 1\) suffice to identify signal parameters.

The above proposition establishes a sufficient number of slices to identify signal parameters, but fewer slices may be necessary, e.g., if a full-rank slice exists, only one slice is necessary. In practice, the procedure will be as follows:

- Construct the Hankel matrices defined in (20) \(R_b(b = 0,1,\ldots,M - 1)\) formed by the 1-D slices \(r_x(\tau,b)\). If there is a full-rank slice, then any of this matrix \(R_b\) will be of rank \(M\). In this case, both of these matrices will be valid.
- If there is no full-rank slice, then form the block Hankel matrix \(R = [R_0^T R_1^T \cdots R_{M-1}^T]^T\). The above proposition ensures identifiability from this specific set of slices.

A similar procedure was used in [22] for the identification of nonminimum phase systems and is applied here for the deterministic signal case. Extensions to fourth-order correlations are straightforward along the same lines. Note that the identifiability problem is not present in [16]. In that work, they are concerned about parameter estimation of a harmonic signal with random phase using multiple realizations rather than a unique one as it is here. In [16], the diagonal line is shown to behave like the autocorrelation sequence; therefore, identifiability using this one slice is ensured.

B. Algorithm Description

The procedure of parameter estimation of exponentially damped sinusoidal signals using higher order correlations can be summarized as follows:

1) From the sequence \(\{y(0)y(1)\cdots y(N - 1)\}\) of noisy measured data defined in (14), the sequence
\{y'(0) y'(1) \cdots y'(N - 1)\} is formed by subtracting the sample mean to the measured data.

2) Estimate the third/fourth-order correlations of signal \(x(n)\) along a prescribed slice by calculating the cumulant sequence of \(y'(n)\) defined using the cov-type estimator (13) with the data set \(\{y'(0) y'(1) \cdots y'(N - 1)\}\). If multiple realizations of the signal are available, we can estimate the correlations for each realization and then average to obtain the resulting moments sequence.

3) With the estimated correlation sequence, construct a Hankel matrix described above. Then, use a rank-indicator [23] to establish the rank of the matrix. If the chosen slice is full-rank, the constructed matrix will have an effective rank of \(M\), and if not, use the results of previous section to obtain a rank-\(M\) matrix. If the number of exponentials is not known, test with different matrices constructed using moments along the specific set of slices described above, or use the block-Hankel \(R\) matrix.

4) With either matrix used in 3), which is known as \(C_i\), form the linear system of equations

\[ C^t \cdot b = -c_x \]  

(22)

where \(C^t\) is matrix \(C\) without the first column, \(c_x\) is the first column, and \(b\) is the backward prediction filter. Then, compute the \(M\)-truncated least-squares solution of (22) according to the KT method [4], which in terms of a singular value decomposition is

\[ b = -\sum_{i=1}^{M} \frac{u_i^H C_x v_i}{\lambda_i} \]  

(23)

where \(\lambda_i\), \(u_i\), and \(v_i\) are the singular values and the left and right singular vectors, respectively. Since the correlation sequence behaves as an exponential sequence, the same results regarding the choice of the prediction-error filter order remain valid for this case.

5) Once the complex frequencies have been estimated, complex amplitudes can be evaluated easily by solving a Vandermonde system with the measured data in the least-squares sense [4].

Likewise, Sections II and III can be extended to higher order correlations, and all the results remain valid. The extension to fourth-order correlations or cumulants is particularly useful and is used in the numerical results.

VI. COMPUTER SIMULATION RESULTS

This section demonstrates the effectiveness of using the cov-type estimator to estimate fourth-order correlations or cumulants in the parameter-retrieval problem. We focus particularly on cases where the number of data is small and the SNR is low. For brevity’s sake, the results reported here refer to the fourth-order case since, in our simulations, this one shows a clear superiority compared with third-order correlations. These results agree with those presented in [7]. In the following examples, the effects of noise, data length, and colored-noise-spectrum sensibility are studied using the cov-type estimator and the standard-biased estimator along the diagonal line, assuming a single realization to be available. For purposes of comparison, the KT method is also employed using matrices of the same dimensions as in higher order-based methods. If more than a single realization of the signal is available, the additional averaging in the estimated cumulants provides much better results, but the global performance remains unchanged. For the sake of brevity, results considering multiple realizations of the signal are not reported here. In the following computer simulations, a VAX 6000 was employed, and the IMSL library was extensively used.

The simulated data are given by the formula

\[ y(n) = a_1 e^{s_1 n T} + a_2 e^{s_2 n T} + w(n T) \quad n = 0 \cdots N - 1 \]  

(24)

where \(s_1 = -0.2 + j2\pi 0.42\), and \(s_2 = -0.1 + j2\pi 0.52\), \(a_1 = 1\), and \(a_2 = 1\). This example was also used in [4] and [7]. The initial phases are zero; therefore, every slice is a full-rank slice by the sufficient condition given in [7]. The noise is an i.i.d. complex Gaussian process, with independent real and imaginary parts with variance \(\sigma^2\). SNR is defined as log 10(1/2\(\sigma^2\)). Note that this is peak SNR; therefore, the last samples of the measurement are strongly contaminated by noise. The colored noise was generated by passing white Gaussian noise through an FIR filter with an impulse response given by \(h = [0.5, 0.6, 0.7, 0.8, 0.7, 0.6, 0.5, 0.0, 0.0, 0.0, 0.5, 0.6, 0.7, 0.8, 0.7, 0.6, 0.5, 0.0, 0.0]\). This FIR filter was also used in [7] to generate a colored Gaussian process.

A statistical measure of the performance of the methods used in our experiments was obtained through Monte Carlo simulations, making 500 independent runs with the transient signal kept the same and generating independent noise realizations using different seeds. The mean square error (MSE) was computed as well as the bias of the estimated damping factors and frequencies. A threshold point occurs in the simulations when the bias is too large (100% of the true value) or more roots than the signal order of the prediction-error filter polynomial appear outside the unit circle, which result in large estimation errors or a break in MSE values. The threshold points usually happen at low SNR and constitute the practical limitation in SNR for these methods and are the break points in the MSE plots shown in this section. For this example, \(N\) sample data from (24) was used to estimate the \(n_{th}\) point fourth-order cumulant sequence along the diagonal line in the third quadrant using the cov-type estimator (COV-FOC method) and the standard biased estimator (BIA-FOC method).

First, a study is carried out of the working parameters involved in this problem, such as sampling period \(T\), prediction-error filter order \(K\), and number of data \(N\). Among them, the choice of the prediction-error filter order is very important at low SNR. From the simulations performed with \(N = 30\) data for this example, the existence of an optimum choice of filter order appears to be clear, namely, 17 or 18 (2\(N/3\)). When the SNR is higher, the filter order \(K\) is much less critical with a wide interval of possible filter orders appearing. This behavior agrees with the experimental results presented in [4].
### TABLE I

**Bias of Damping Factors and Frequencies for KT, COV-FOC, and BIA-FOC Methods Using 30 Data in White Noise**

<table>
<thead>
<tr>
<th>SNR (dB)</th>
<th>Bias of $\alpha_0 = 0.2$ for 500 runs</th>
<th>Bias of $f_0 = 0.42$ for 500 runs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>KT COV-FOC BIA-FOC</td>
<td>KT COV-FOC BIA-FOC</td>
</tr>
<tr>
<td>30</td>
<td>-1.15E-3 -2.40E-3 -1.82E-2</td>
<td>5.51E-5 1.05E-4 9.64E-4</td>
</tr>
<tr>
<td>25</td>
<td>-7.8E-3 -3.14E-3 -1.97E-2</td>
<td>1.44E-4 9.49E-4 1.18E-3</td>
</tr>
<tr>
<td>20</td>
<td>7.8E-2 -1.06E-2 3.68E-2</td>
<td>2.56E-2 1.03E-3 1.77E-3</td>
</tr>
<tr>
<td>15</td>
<td>-7.8E-2 -1.06E-2 3.68E-2</td>
<td>1.08E-3 1.17E-3 3.47E-3</td>
</tr>
<tr>
<td>10</td>
<td>* -3.79E-2 *</td>
<td>1.70E-3 *</td>
</tr>
<tr>
<td>5</td>
<td>* * *</td>
<td>* * *</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SNR (dB)</th>
<th>Bias of $\alpha_0 = 0.1$ for 500 runs</th>
<th>Bias of $f_0 = 0.52$ for 500 runs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>KT COV-FOC BIA-FOC</td>
<td>KT COV-FOC BIA-FOC</td>
</tr>
<tr>
<td>30</td>
<td>5.68E-4 1.05E-3 -5.53E-2</td>
<td>5.44E-2 -7.78E-3</td>
</tr>
<tr>
<td>25</td>
<td>1.38E-3 1.44E-3 -5.32E-3</td>
<td>* *</td>
</tr>
<tr>
<td>20</td>
<td>3.61E-3 3.53E-3 *</td>
<td>* *</td>
</tr>
<tr>
<td>15</td>
<td>9.84E-3 1.44E-3 *</td>
<td>* *</td>
</tr>
<tr>
<td>10</td>
<td>3.14E-2 1.39E-2 *</td>
<td>* *</td>
</tr>
<tr>
<td>5</td>
<td>* * *</td>
<td>* * *</td>
</tr>
</tbody>
</table>

### TABLE II

**Bias of Damping Factors and Frequencies for KT, COV-FOC, and BIA-FOC Methods Using 30 Data in Colored Noise**

<table>
<thead>
<tr>
<th>SNR (dB)</th>
<th>Bias of $\alpha_0 = 0.2$ for 500 runs</th>
<th>Bias of $f_0 = 0.42$ for 500 runs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>KT COV-FOC BIA-FOC</td>
<td>KT COV-FOC BIA-FOC</td>
</tr>
<tr>
<td>30</td>
<td>-3.60E-4 -1.85E-3 -1.75E-2</td>
<td>3.61E-5 6.15E-5 8.93E-4</td>
</tr>
<tr>
<td>25</td>
<td>-7.9E-4 -2.12E-3 -1.82E-2</td>
<td>1.25E-4 7.67E-5 1.04E-3</td>
</tr>
<tr>
<td>20</td>
<td>-3.96E-3 -3.13E-3 -2.08E-2</td>
<td>2.78E-4 3.10E-4 4.53E-3</td>
</tr>
<tr>
<td>15</td>
<td>-1.49E-2 -7.09E-3 -2.95E-2</td>
<td>1.58E-3 6.38E-4 3.15E-3</td>
</tr>
<tr>
<td>10</td>
<td>* -2.79E-2 -6.33E-2</td>
<td>* -4.39E-3 8.89E-3</td>
</tr>
<tr>
<td>5</td>
<td>* * *</td>
<td>* * *</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SNR (dB)</th>
<th>Bias of $\alpha_0 = 0.1$ for 500 runs</th>
<th>Bias of $f_0 = 0.52$ for 500 runs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>KT COV-FOC BIA-FOC</td>
<td>KT COV-FOC BIA-FOC</td>
</tr>
<tr>
<td>30</td>
<td>-2.41E-3 7.31E-4 -5.60E-2</td>
<td>5.7E-5 -4.89E-5 1.09E-4</td>
</tr>
<tr>
<td>25</td>
<td>3.06E-4 8.46E-4 -5.63E-2</td>
<td>1.38E-4 -6.10E-5 1.21E-4</td>
</tr>
<tr>
<td>20</td>
<td>1.78E-3 1.41E-3 -5.84E-2</td>
<td>1.00E-4 -1.03E-4 4.89E-3</td>
</tr>
<tr>
<td>15</td>
<td>8.90E-3 3.80E-3 -6.70E-2</td>
<td>2.64E-4 -1.93E-4 7.21E-3</td>
</tr>
<tr>
<td>10</td>
<td>6.18E-2 1.58E-2 *</td>
<td>3.19E-3 -3.34E-4</td>
</tr>
<tr>
<td>5</td>
<td>* -6.92E-2 *</td>
<td>1.12E-3 -5.26E-3</td>
</tr>
</tbody>
</table>

Other important parameters involved in this problem are the sampling interval and the number of available data to estimate the cumulant sequence. The biased estimator is quite sensitive to these parameters. From (12), it can be seen that when the amount of data or the sampling period decreases, the oscillating components vibrating as the sum of frequencies of the original signal become more important, thus making it difficult to extract the correct frequencies from the cumulant sequence, especially at high noise levels. This situation is shown in Fig. 1(a) and (b), where the inability of the BIA-FOC method to extract the correct value of frequency 0.52 when the number of data is less than 70 or the sampling period less than 0.9 is made evident.

Once the influence of the parameters related to this problem have been established, we concentrate on studying the parameter estimation achieved in a noisy environment by the BIA-FOC, COV-FOC, and KT methods when the data-record length is short ($N = 30$). These 30 data were used to estimate 25 cumulants. With 25 cumulants (or 25 data for the KT method), a Hankel matrix is constructed with $K = 17$, and the prediction-error filter polynomial is rooted to estimate frequencies and damping factors. $T$ was chosen as 0.5. The MSE and bias of estimated frequencies and damping factors in additive white and colored noise is computed from 500 independent runs for each method. The results for the bias are given in Tables I and II, and the MSE is plotted in Figs. 2(a)–(d) and 3(a)–(d) for white and colored noise, respectively. From Fig. 2(a)–(d), it is clear that the COV-FOC method outperforms the BIA-FOC method, which shows large estimation errors, above all in the damping factor $-0.1$ and frequency 0.52. The KT method shows good performance in the white noise case, but in addition, the COV-FOC method shows an advantage, specially with the frequency associated with the larger damping factor. The advantage of the COV-FOC method over the KT method is even greater in the colored noise case [Fig. 3(a)–(d)], where the KT method is more affected by noise and exhibits spurious peaks. In this case, the BIA-FOC method performs better than in the white noise case, but the modeling errors introduced by the estimation of cumulants results, once again, in a poor estimation, especially at the pole $(-0.1 + j2\pi 0.52)$.

Tables I and II show the bias for the above experiment from 500 independent noise runs at different white and colored noise levels using the COV-FOC, BIA-FOC, and KT methods. These tables make it clear that the BIA-FOC method is affected by a persistent bias in the second pole due to the imperfect
modeling of the cumulant sequence, which contains several deterministic components contributing to the singular values of the constructed Hankel cumulant matrix.

In both cases, the COV-FOC method shows the advantage of using HOS even when the data-record length is short. When the amount of data increases, the BIA-FOC method becomes progressively equivalent to the COV-FOC method since the extra vibrating terms become negligible when $N$ increases. This situation is confirmed in our simulations. In the case of a long data record, the advantage of cumulant-based methods over the KT method is greater, especially in the colored noise case.

VII. CONCLUSION

A new higher order correlation estimator (the covariance-type estimator) has been developed for the purpose of parameter estimation of exponentially damped sinusoids using higher order correlations. This estimator reduces deterministic errors associated with imperfect modeling of the correlation sequence with the standard biased estimator and has the useful property of retaining the damped exponential structure in certain correlation sequences. These sequences are the same as in the infinite-data-length case; therefore, the cov-type
estimator maintains the same property with a finite number of data. When only noisy observations are available, the cumulant sequence can be estimated to reduce the effect of Gaussian noise and, after the summation, can be taken over the nonstationary signal to obtain a 1-D correlation sequence. Since the estimated higher order correlation or cumulant sequence maintain the same data structure, linear-prediction-based methods can be applied to this sequence instead of data, with the added advantage that cumulant sequences of an order greater than two are insensitive to Gaussian noise.

We have also discussed the problem of identifiability from a 1-D slice using recent results developed for causal ARMA processes. This allows us to obtain a consistent algorithm to extract signal parameters. This algorithm was applied to recover complex frequencies of a standard simulation example. The results obtained here show the improvement achieved by the cov-type estimator, which reduces deterministic errors associated with imperfect modeling of the 1-D slice, as compared with the standard biased estimator of moments. Compared with the data-based KT method, numerical results show an increase of about 5-10 dB in threshold values in colored noise when short-data length and a single realization of the signal is available. When multiple realizations are available, results not reported here show an additional improvement over the KT method.

Future work will be directed toward new applications of this estimator and a perturbation analysis of matrices proposed here [24] in order to quantify the accuracy of the estimates.

**APPENDIX A**

**PROOF OF PROPOSITION 1**

Four cases can be distinguished, depending on the quadrant the 1-D slice lies in on the $r_1-r_2$ plane

**Case 1—The 1-D Slice Lies on the First Quadrant:** In this case, $\tau$ ranges from 0 to $\infty$, slope $a$ is greater than 0 ($a < 0$ would correspond to a line in the fourth quadrant with a change in the origin), and the intercept $b$ can be taken as being $> 0$ since a value of $b < 0$ would mean a change in the origin of $\tau$. Taking these values into account, $n_0$ defined in (8) is equal to 0; therefore, the third-order correlation sequence becomes

$$r(\tau) = \sum_{i,j,k=1}^{M} \frac{a_i a_j^* a_k^*}{1 - e^{(a_i + a_j^* + a_k^*) \tau}} e^{(s_i^* + s_j^* + s_k^*) \tau} e^{-a_i \tau}. \quad (A.1)$$

To model $r(\tau)$ as a damped exponential sinusoid with the same frequencies as the original signal, $a$ must be chosen as equal to 0, (i.e., horizontal slices); therefore, the moments sequence would be

$$r(\tau) = \sum_{j=1}^{M} A(j) e^{s_j^* \tau} \quad A(j) = \sum_{i,k=1}^{M} \frac{a_i a_j^* a_k^*}{1 - e^{(s_i + s_j^* + s_k^*) \tau}} e^{-a_i \tau}. \quad (A.2)$$

Provided that $A(j) \neq 0$ for all $j$, i.e., the 1-D slice is a full-rank slice, the moments sequence $r(\tau)$ behaves like the original energy signal.

**Case 2—Second Quadrant:** In this case, $\tau = 0, -1, \ldots, -\infty$, $a \leq 0$, and $b \geq 0$. Now, $n_0 = -\tau$, the moments sequence being

$$r(\tau) = \sum_{i,j,k=1}^{M} \frac{a_i a_j^* a_k^*}{1 - e^{(a_i + a_j^* + a_k^*) \tau}} e^{(s_i^* + s_j^* + s_k^*) \tau} e^{-(s_i + s_j + s_k) \tau}. \quad (A.3)$$

In this case, due to the last exponential term in (A.3), there is no slice that can be modeled as an exponential sum in this quadrant.

**Case 3—Third Quadrant:** $\tau = 0, -1, \ldots, -\infty$, $a \geq 0$, and $b \leq 0$. In this case, $n_0$, which is defined as $\max\{0, -\tau, -a\tau - b\}$, can take different values, depending on the values of the slopes:

- If $a > 1$, $-a\tau - b > -\tau$ and then $n_0 = -a\tau - b$
- If $a = 1$, $-\tau - b > -\tau$ and then $n_0 = -\tau - b$
- If $a < 1$, it depends on the value of $b$. For $\tau < b/(1-a)$, $n_0 = -\tau$.

Analyzing the different cases, the following results can easily be obtained:
- If $a > 1$ or $a < 1$, there is no slice that can be modeled with the same exponential model as the data.
- If $a = 1$ (diagonal slice), the third-order-correlation sequence behaves as follows:

$$r(\tau) = \sum_{i=1}^{M} A(i) e^{s_i^* \tau} \quad A(i) = \sum_{j,k=1}^{M} \frac{a_i a_j^* a_k^*}{1 - e^{(s_i^* + s_j^* + s_k^*) \tau}} e^{-b(s_i + s_j + s_k)}. \quad (A.4)$$

As can be seen, the diagonal slice allows signal parameters to be obtained. This diagonal slice was used by Papadopoulos and Nikias [7] with $b = 0$ to recover signal parameters, showing via matrix decomposition that the rank of square matrices constructed with these correlations is equal to the number of exponentials. The slice was assumed to be a full-rank slice. In this case, we reached the same conclusion for any rectangular matrix of correlations since these correlations can be considered to be new "data" to which the KT method can be applied.

**Case 4—Fourth Quadrant:** $\tau = 0, 1, \ldots, -\infty$, $a \leq 0$, and $b \leq 0$ (as before, note that $a \geq 0$ can be considered to be a line in the first quadrant with a change of origin, whereas $b \geq 0$ would correspond to a change in the origin of $\tau$). In this case, $n_0 = -a\tau - b$, and the third-order-correlation sequence is

$$r(\tau) = \sum_{i,j,k=1}^{M} \frac{a_i a_j^* a_k^* e^{-(s_i + s_j + s_k) \tau}}{1 - e^{(s_i + s_j^* + s_k^*) \tau}} e^{-a \tau}. \quad (A.5)$$

Since $a \leq 0$, $a$ must be chosen as equal to 0 (horizontal slice) to ensure an exponential model with the same frequencies. This case is consistent with the symmetry properties of moments since it is equivalent to slices with $a = 1$ (diagonal slices), which belong to the third quadrant.
APPENDIX B

PROOF OF PROPOSITION 2

If the three possible cases with infinite data length are analyzed when $N$ is finite, the following results are obtained:

a) **Horizontal slices in quadrant 1**: In this case, $S_1 = 0$ and $S_2 = \min(N-1, N-1 - \tau, N-1 - b)$. Substituting these values in (12), we find that

$$\hat{r}_x(\tau) = \frac{1}{N} \sum_{i,j,k=1}^{M} \frac{a_i a_j^* a_k^*}{1 - e^{s_i + s_j + s_k^*}} e^{s_j^* \tau} e^{s_k^* b} \times \left(1 - e^{(s_i + s_j^* + s_k^*)(S_2 + 1)}\right) \quad \tau = 0, 1, \ldots nm. \quad (B.1)$$

As can be seen in (B.1), unless $S_2 \neq S_2(\tau)$, vibrating terms as the sum of three frequencies would appear; therefore, the moments sequence would not preserve the data structure. If we choose $b \geq nm$ ($nm$ is the maximum computed correlation lag), then $S_2 = N-1 - b$, and $S_2$ turns out to be independent of $\tau$. Therefore, horizontal slices with $b \geq nm$ retain the exponential model for the estimation of $r_x(\tau)$. This case is studied and analyzed in [9]; it has the drawback of involving higher data lags to estimate the moments, and therefore, it is more sensitive to noise when the amount of available data is small.

b) **Diagonal slices in quadrant 3**: Now, $S_1 = -\tau - b$ and $S_2 = N - 1$, and the estimated moments sequence becomes

$$\hat{r}_x(\tau) = \frac{1}{N} \sum_{i,j,k=1}^{M} \frac{a_i a_j^* a_k^*}{1 - e^{s_i + s_j^* + s_k^*}} \times e^{-s_i \tau} e^{-b_s - b_{s_i}} \left(1 - e^{(s_i + s_j^* + s_k^*)(N + \tau + b)}\right) \quad \tau = 0, -1, \ldots - nm. \quad (B.2)$$

Here, the presence of the term in brackets in (B.2) indicates that the estimation of the third-moments sequence contains terms oscillating as the sum of the frequencies of the original signal, where it is impossible to extract the frequencies and damping factors of the signal. When the number of data becomes larger, the terms in brackets become negligible, and the signal parameters can be obtained.

c) **Horizontal slices in quadrant 4**: In this circumstance, $S_1 = -b$ and $S_2 = N - 1 - \tau$. Substituting these values in (12), it can be seen that the estimated-moments sequence again contains terms vibrating as the sum of frequencies, and therefore, the rank of any matrix formed with these correlations will be of $M(M + 1)/2$, which is the number of exponential terms in this estimated 1-D slice. As in the case of diagonal slices in the third quadrant, the estimation of the moment sequence does not retain the original structure of the data.

APPENDIX C

PROOF OF PROPOSITION 3

Proposition 3 can be demonstrated by studying the different cases with the covariance estimator defined in (13):

a) **Horizontal slices in the first quadrant ($a = 0$)**: In this case, we have $T_1 = 0$ and $T_2 = N - 1 - \tau_{max}$ or $T_2 = N - 1 - b$, depending on $b \leq \tau_{max}$ or $b \geq \tau_{max}$, respectively, where $\tau_{max} = nm$ is the maximum lag of moments to be estimated. Taking these limits into account, we obtain

$$\hat{r}_x(\tau) = \frac{1}{N} \sum_{n=0}^{T_2} x(n)x^*(n + \tau)x^*(n + b) = \sum_{j=1}^{M} A(j)e^{s_j \tau} \quad \tau = 0, 1, \ldots nm$$

$$A(j) = \frac{1}{N} \sum_{i,k=1}^{M} \frac{a_i a_j^* a_k^*}{1 - e^{s_i + s_j^* + s_k^*}} e^{s_j^* b} \times \left(1 - e^{(s_i + s_j^* + s_k^*)(T_2 + 1)}\right). \quad (C.1)$$

Since $A(j)$ is now independent of $\tau$ due to $T_2$ being fixed and provided that the slice is full-rank (i.e., $A(j) \neq 0, j = 1 \cdots M$), the estimated horizontal slice sequence retains the exponential structure of the original signal.

b) **Diagonal slices in the third quadrant ($a = 1$)**: In this case, $T_1 = -\tau - b$, and $T_2 = N + \tau_{max} - \tau - 1$, resulting in

$$\hat{r}_x(\tau) = \frac{1}{N} \sum_{n=-\tau-b}^{N+\tau_{max}-\tau-1} x(n)x^*(n + \tau)x^*(n + \tau + b) = \sum_{i=1}^{M} B(i)e^{-s_i \tau} \quad \tau = 0, -1, \ldots nm$$

$$B(i) = \frac{1}{N} \sum_{j,k=1}^{M} \frac{a_i a_j^* a_k^*}{1 - e^{s_i + s_j^* + s_k^*}} e^{-b_{s_j} - b_{s_k}^*} \times \left(1 - e^{(s_i + s_j^* + s_k^*)(N + \tau_{max} + b)}\right). \quad (C.2)$$

As above, $B(i)$ is independent of $\tau$, and therefore, the estimated sequence does not contain terms vibrating at other frequencies that are different from the data ones.

c) **Horizontal slices in the fourth quadrant**: Here, $T_1 = -b$, and $T_2 = N - 1 - \tau_{max}$, resulting in

$$\hat{r}_x(\tau) = \frac{1}{N} \sum_{n=-b}^{N-1-\tau_{max}} x(n)x^*(n + \tau)x^*(n + b) = \sum_{j=1}^{M} C(j)e^{s_j \tau} \quad \tau = 0, 1, \ldots nm$$

$$C(j) = \frac{1}{N} \sum_{i,k=1}^{M} \frac{a_i a_j^* a_k^*}{1 - e^{s_i + s_j^* + s_k^*}} e^{s_j^* b} \times \left(1 - e^{(s_i + s_j^* + s_k^*)(N - \tau_{max})}\right). \quad (C.3)$$
Since $C(j)$ does not depend on $\tau$ (as can be seen in (C.3)), the horizontal slices in the fourth quadrant remain valid when the cov-type estimator is used.

APPENDIX D
EXTENSION TO FOURTH-ORDER CORRELATIONS

A. Infinite-Data Length

The fourth-order correlations of an energy signal are defined as [18]

$$R_x(\tau_1, \tau_2, \tau_3) = \sum_{n=-\infty}^{\infty} x(n)x^*(n+\tau_1)x(n+\tau_2)x^*(n+\tau_3)$$

$$\tau_1, \tau_2, \tau_3 = 0, \pm 1, \pm 2 \cdots.$$  \hspace{1cm} (D.1)

Likewise, a 1-D slice in the space $(\tau_1, \tau_2, \tau_3)$ is defined as

$$r_x(\tau) = R_x(\tau, a\tau + b, c\tau + d) \hspace{1cm} \tau = 0, \pm 1, \pm 2 \cdots.$$  \hspace{1cm} (D.2)

where $a$ and $c$ are the slopes, and $b$ and $d$ the intercepts of the lines along the moments space $(\tau_1, \tau_2, \tau_3)$. We are interested in maintaining the same damped exponential structure in the fourth-order correlation sequence as in the original signal. In the case of infinite-data length, the following proposition is valid.

Proposition D-1: Provided that $r_x(\tau)$ is a full-rank slice, only the following slices retain the exponentially damped structure of the original signal:

1) **Horizontal slices in the first and fourth quadrant**

$$r_x(\tau) = R_x(\tau, b, d).$$  \hspace{1cm} (D.3)

2) **Diagonal slices in the third quadrant**

$$r_x(\tau) = R_x(\tau, b + \tau, d - \tau).$$  \hspace{1cm} (D.4)

Proof: The proof follows the same lines as the third-order case (see Appendix A).

B. Finite-Data Length

In the case where we have $N$ signal samples and are interested in estimating $nm$ correlations from the data, the following proposition, which is the fourth-order counterpart of the third-order case, still holds:

Proposition D-2: With the biased estimator defined as

$$\hat{R}_x(\tau_1, \tau_2, \tau_3) = \frac{1}{N} \sum_{n=S_2}^{S_1} x(n)x^*(n+\tau_1)$$

$$x(n+\tau_2)x^*(n+\tau_3)$$

$$S_1 = \max(0, -\tau_1, -\tau_2, -\tau_3)$$

$$S_2 = \min(N - 1, N - 1 - \tau_1,$$

$$N - 1 - \tau_2, N - 1 - \tau_3)$$  \hspace{1cm} (D.5)

the estimated 1-D slice sequence maintains the signal structure only for the horizontal slices in the first quadrant with $b$ or $d$ greater than $nm$. When the SNR is low, this estimation becomes poor due to the high lag used in the correlation estimator (keep in mind that it is a peak SNR).

The proof follows easily from Appendix B, taking into account the following equation obtained from (D.5) and (1) for the fourth-order correlation estimate

$$\hat{r}_x(\tau) = \frac{1}{N} \sum_{i,j,k,l=1}^{M} \frac{a_i a_j^* a_k a_l^*}{1 - e^{i\xi_i + i\xi_j + i\xi_k + i\xi_l}}$$

$$e^{i\tau_i} e^{i(\sigma + b)} e^{i(\sigma + d)}$$

$$\times e^{i(x_i + x_j + x_k + x_l)} e_i (1 - e^{i(x_i + x_j + x_k + x_l)}) (S_i - S_j + 1)$$

$$\tau = 0, \pm 1, \pm 2 \cdots.$$  \hspace{1cm} (D.6)

**Proposition D-3:** The covariance-type estimator defined by

$$\hat{R}_x(\tau_1, \tau_2) = \frac{1}{N} \sum_{n=T_1}^{T_2} x(n)x^*(n+\tau_1)x(n+\tau_2)x^*(n+\tau_3)$$

$$\tau_1 = 0, \pm 1, \ldots \tau_1_{\text{max}}, \tau_2 = 0, \pm 1, \ldots \tau_2_{\text{max}}$$

$$\tau_3 = 0, \pm 3 \tau_{\text{max}}$$

$$T_1 = \max(0, -\tau_1, -\tau_2, -\tau_3)$$

$$T_2 = \min(N - 1 - \tau_1_{\text{max}}, N - 1 - \tau_2_{\text{max}},$$

$$N - 1 - \tau_3_{\text{max}}, N + \tau_1_{\text{max}} - \tau_1 - 1,$$

$$N + \tau_2_{\text{max}} - \tau_2 - 1, N + \tau_3_{\text{max}} - \tau_3 - 1)$$  \hspace{1cm} (D.7)

where $\tau_1_{\text{max}}, \tau_2_{\text{max}},$ and $\tau_3_{\text{max}}$ are the maximum lags for $\tau_1, \tau_2,$ and $\tau_3$ respectively, allows the damped exponential structure for the estimated correlation sequence to be maintained in the same cases as in infinite-data length (Proposition D-1). This means that the signal parameters can be recovered from fourth-order correlations estimated from a finite number of data.

C. Finite-Length and Noisy Data

Let us suppose that the measured signal consists of $M$ damped sinusoids in noise of the type given in (14), where $u(n)$ is a zero-mean, complex Gaussian process with real and imaginary parts independent and identically distributed. Assuming a large number of realizations available for this process, and defining $y(n)$ as in (15), the fourth-order cumulant sequence of $y(n)$ is defined as [17]

$$C_y(n, \tau_1, \tau_2, \tau_3)$$

$$= E\{y(n)y^*(n+\tau_1)y^*(n+\tau_2)y^*(n+\tau_3)\}$$

$$- E\{y(n)y^*(n+\tau_1)\} E\{y(n+\tau_2)y^*(n+\tau_3)\}$$

$$- E\{y(n)y^*(n+\tau_2)\} E\{y(n+\tau_1)y^*(n+\tau_3)\}$$

$$- E\{y(n)y^*(n+\tau_3)\} E\{y(n+\tau_1)y^*(n+\tau_2)\}.$$  \hspace{1cm} (D.8)

Taking into account that the signal and the noise are independent and that cumulants higher than two for a Gaussian process are all zero, we get
\[ C_{\nu}(n, \tau_1, \tau_2, \tau_3) = 2(\nu)|x^n(\tau_1)x^n(\tau_2)x^n(\tau_3) (n + \tau_3). \] (D.9)

If the summation is taken over the index \( n \), (D.9) becomes

\[ C_{\nu}(\tau_1, \tau_2, \tau_3) = \frac{1}{N} \sum_{n=1}^{T_1} C_{\nu}(n, \tau_1, \tau_2, \tau_3) \]

\[ = \frac{1}{N} \sum_{n=1}^{T_1} x^n(\nu)x^{n+\tau_1}x^{n+\tau_2}x^{n+\tau_3} \]

\[ = -2\tilde{R}_{\nu}(\tau_1, \tau_2, \tau_3) \] (D.10)

where \( T_1 \) and \( T_2 \) are defined as in (D.7).

From (D.10), it can be seen that the fourth-order cumulant sequence \( C_{\nu}(\tau_1, \tau_2, \tau_3) \) is equivalent to the estimated fourth-order correlations of the signal and since these estimated correlations behave like the original signal, the signal parameters can be extracted from them. In the case in point here, where only a single realization is available, the second- and fourth-order moments in (D.8) are estimated as correlations using the cov-type estimator. In the expression of the fourth-order cumulant sequence, there would then appear crosscorrelations between the signal and the noise and the correlations of these terms behave like a "noise" added to the estimated fourth-order correlations of the signal. Even so, when the estimated cumulant matrix is combined with robust SVD techniques, better results are obtained than when working only with noisy data, especially in the colored noise case (Section VI).

REFERENCES


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