Quasi-co-Frobenius Coalgebras

José Gómez Torrecillas

Departamento de Algebra, Facultad de Ciencias, Universidad de Granada, E18071
Granada, Spain

AND

Constantin Năstăsescu

Facultatea de Matematică, Str. Academiei 14, 70109 Bucharest 1, Romania

Communicated by Kent R. Fuller

Received October 4, 1993

Throughout this paper \( C \) will be a coalgebra over a field \( k \) and \( \mathcal{M}^C \) (resp. \( \mathcal{C}^* \)) will denote the Grothendieck category of all right (resp. left) \( C \)-comodules. The dual space \( C^* = \text{Hom}_k(C, k) \) is endowed naturally of a structure of \( k \)-algebra such that every right \( C \)-comodule can be viewed as a left \( C^* \)-module (see [10, 3]). These left \( C^* \)-modules are called rational left \( C^* \)-modules and the full subcategory \( \text{Rat}(C^*-\text{Mod}) \) of \( C^*-\text{Mod} \) consisting of all rational left \( C^* \)-modules is isomorphic to \( \mathcal{M}^C \). Analogous statements can be made on \( \mathcal{C}^* \). The coalgebra \( C \) is said to be left semiperfect [7] when \( \mathcal{C}^* \) has enough projectives. A sufficient condition on \( C \) to be left semiperfect is that \( C \) is projective as a left \( C^* \)-module [7, Theorem 23]. This is the case for the so-called left co-Frobenius coalgebras [7, Theorem 23]. A coalgebra \( C \) is said to be left co-Frobenius if there is a left \( C^* \)-monomorphism from \( C \) to \( C^* \). The finite-dimensional left and right co-Frobenius coalgebras are precisely the dual coalgebras of the Frobenius algebras. In Section 1 we will extend the notion of co-Frobenius coalgebra in a very natural sense by the concept of left Quasi-co-Frobenius coalgebra (left \( QcF \)). However, these coalgebras will be recognized as the coalgebras for which \( C^*C \) is projective. The notion of \( QcF \) coalgebra is more general than the notion of co-Frobenius coalgebra (Remarks 1.5). The finite-dimen-
sional $QcF$ coalgebras are precisely the dual of finite-
dimensional $QF$ algebras (see Remarks 1.5). In fact, the behaviour of $QcF$
coalgebras is in many aspects similar to that of $QF$ rings.

In Section 1 we introduce the notion of (left) $QcF$ coalgebras and we
give some characterizations of left $QcF$ coalgebras in Theorem 1.3.

Section 2 investigates the exactness of the Rational functor. This study
allows us to provide a clean characterization of left and right $QcF$
coalgebras in terms of the category $\mathcal{C}_Q$ (Theorem 2.6). Moreover, the
results proved in this section are used in Section 3.

In Section 3 we characterize semiperfect coalgebras by the existence of a
duality in the sense of Colby and Fuller [2] between the categories of left
and right comodules (Theorem 3.5). The notion of Colby–Fuller duality is
a suitable extension of the concept of Morita duality to the framework of
Grothendieck categories. Finally, Theorem 3.12 provides a duality for $QcF$
coalgebras that fits the classical duality for $QF$ rings.

We will follow the coalgebraic notation of [3, 10]. For module-theoretic
notions we will follow [1].

1. QUASI-CO-FROBENIUS COALGEBRAS

DEFINITION. A coalgebra $C$ is said to be left $QcF$ (Quasi-co-Frobenius)
if there exists a monomorphism of left $C^*$-modules from $C_C$ to a free left
$C^*$-module.

Remark 1.1. It is evident that every left co-Frobenius coalgebra is left
$QcF$.

In Theorem 1.3, we will give some characterizations of $QcF$ coalgebras.
The following lemma, which is of independent interest, is used in the proof
of that theorem.

LEMMA 1.2. Every indecomposable projective right comodule has finite
dimension as $k$-vector space.

Proof. By the locally finite property for comodules [10, Theorem 2.1.3.b]
there is an epimorphism

$$f: \bigoplus \{M_i \mid i \in I\} \to E \tag{1}$$

of left $C^*$-modules or, equivalently, right $C$-comodules, where $\{M_i \mid i \in I\}$
is a family of finite-dimensional rational left $C^*$-modules. Since $E$ is a
projective right $C$-comodule, (1) is a splitting epimorphism. Thus,

$$\bigoplus \{M_i \mid i \in I\} \cong E \oplus E' \tag{2}$$
for some right $C$-comodule $E'$. Since $M_i$ is a left $C^*$-module of finite length, it has a decomposition as a direct sum of finitely many indecomposable left $C^*$-modules of finite length. Thus, $E \oplus E'$ has a decomposition as a direct sum of indecomposable left $C^*$-modules of finite length. Now we can apply Azumaya's Theorem (see [1, Theorem 12.6 and Lemma 12.2]) to obtain that $E$ is isomorphic to a direct summand of some $M_i$. In particular, $E$ has finite dimension as $k$-vector space.

A left $C^*$-module $M$ is said to be torsionless if $M$ embeds in a direct product of copies of $C^*$. Following [3] we will say that a bilinear form $b_i:C \times C \rightarrow k$ is $C$-balanced if the map $\theta:C \rightarrow C^*$ defined by $\theta(d)(c) = b_i(c,d)$ is left $C^*$-linear.

**Theorem 1.3.** The following conditions are equivalent for a coalgebra $C$.

(i) $C$ is left QcF.

(ii) $C$ is a torsionless left $C^*$-module.

(iii) There exists a family of $C$-balanced bilinear forms $\{b_i:C \times C \rightarrow k \mid i \in I\}$ such that for every nonzero $x \in C$ there is $i \in I$ such that $b_i(C, x) \neq 0$.

(iv) Every injective right $C$-comodule is projective.

(v) $C$ is a projective right comodule.

(vi) $C$ is a projective left $C^*$-module.

**Proof.** (i) $\Rightarrow$ (ii). This is obvious.

(ii) $\Rightarrow$ (iii). Let $\theta:C \rightarrow (C^*)^I$ be a monomorphism of left $C^*$-modules, with $(C^*)^I$ a direct product of copies of $C^*$. Let $p_i:(C^*)^I \rightarrow C^*$ denote the $i$th canonical map for each $i \in I$, and define $b_i:C \times C \rightarrow k$ by $b_i(c, d) = p_i(\theta(d)(c))$. From [3, Lemma 1], $b_i$ is a $C$-balanced bilinear form. Now, if $x$ is a nonzero element of $C$, then $\theta(x) \neq 0$. Hence, there is $i \in I$ such that $p_i(\theta(x)) \neq 0$ and, thus, $0 \neq p_i(\theta(x)(c)) = b_i(c, x)$ for some $c \in C$.

(iii) $\Rightarrow$ (ii). Define $\theta:C \rightarrow (C^*)^I$ by $\theta(c) = (\theta_i(c))_{i \in I}$, where $\theta_i(c):C \rightarrow k$ is given by $\theta(c)(d) = b_i(d, c)$. Since $b_i$ is a $C$-balanced bilinear form, $\theta_i$ is an homomorphism of left $C^*$-modules for every $i \in I$. It is easy to prove that $\theta$ is a monomorphism of left $C^*$-modules. Thus, $C$ is a torsionless left $C^*$-module.

(ii) and (iii) $\Rightarrow$ (vi). Let $\text{Soc}(C_C) = \oplus \{S_j \mid j \in J\}$ be the socle of the right $C$-comodule $C$. Then

$$C_C \cong \oplus \{E(S_j) \mid j \in J\},$$

(3)

where $E(S_j)$ denotes the injective hull in $\mathcal{M}_C$ of the minimal right coideal $S_j$ (see [3]).

The argument of [3, Theorem 1] works to prove that the injective hull $E(S)$ in $\mathcal{M}_C$ of every minimal right coideal $S$ of $C$ is finite dimensional.
whenever we make the following modification: Write $S = C^*x$ for some nonzero $x$ in $C$. Then there exist $i \in I$ and $c \in C$ such that $b_i(c, x) \neq 0$. The left coideal $(cC^*)^\perp = \{y \in C \mid b_i(y, z) = 0 \text{ for all } z \in cC^*\}$ satisfies that $(cC^*)^\perp \cap S = 0$ and then we can conclude that $E(S)$ is finite dimensional like in [3, Theorem 1.1].

For $S$ a minimal right $C$-coideal, we have a monomorphism of left $C^*$-modules

$$f : E(S) \hookrightarrow C \hookrightarrow (C^*)^\perp.$$  \hspace{1cm} (4)

We have that $E(S)$ is an artinian torsionless left $C^*$-module, and this implies that $E(S)$ embeds in a finitely generated free left $C^*$-module. But $E(S)$ is injective as a left $C^*$-module [1, Proposition 4], whence it is isomorphic to a direct summand of a free left $C^*$-module. Therefore, $E(S)$ is a projective left $C^*$-module and it follows from (3) that $C$ is a projective left $C^*$-module.

(vi) $\Rightarrow$ (v). This is clear.

(v) $\Rightarrow$ (iv). Let $V$ be an injective right $C$-comodule. Then $V$ is isomorphic in $\mathcal{A}$ to a direct summand of a direct sum of copies of $C_C$. Since $C_C$ is a projective right $C$-comodule this implies that $V$ is a projective right $C$-module.

(iv) $\Rightarrow$ (i). We have that the injective right $C$-comodule $C$ is projective. On the other hand,

$$C_c \cong \bigoplus \{E(S_j) \mid j \in J\},$$  \hspace{1cm} (5)

where the $E(S_j)$'s are injective hulls in $\mathcal{A}$ of minimal right $C$-coideals. Therefore each $E(S_i)$ is an indecomposable projective right $C$-comodule. By Lemma 1.1, $E(S_i)$ is finite dimensional as $k$-vector space for each $j \in J$. By [3, Proposition 4], $E(S_i)$ is a projective left $C^*$-module for every $j \in J$ and, thus, $C$ is a projective left $C^*$-module. In particular, $C$ is a left $QcF$ coalgebra.

**Corollary 1.4.** Every left $QcF$ coalgebra is left semiperfect.

**Proof.** If $C$ is a left $QcF$ coalgebra, then $C\cdot C$ is projective. By [7, Theorem 23], $C$ is left semiperfect.  \hspace{1cm} \blacksquare

**Remarks 1.5.**

(a) It follows from Theorem 1.3 and the Faith–Walker characterization of $QF$ rings (i.e., every injective module is projective [1, Theorem 31.9]) that a finite dimensional coalgebra $C$ is left $QcF$ if and only if the ring $C^*$ is $QF$. As a consequence, a finite-dimensional coalgebra is left
$QeF$ if and only if it is right $QeF$. Example 1.6 shows that it is not true in the infinite-dimensional case.

(b) In [9] it is shown that there is a finite-dimensional $QF$ algebra which is not a Frobenius algebra. Taking the dual coalgebra, we have a $QeF$ coalgebra which is not co-Frobenius.

Example 1.6 (Sweedler). Let $C$ be a vector space with basis $(g_k, d_k; k = 1, 2, \ldots)$ and define

$$
\Delta(g_k) = g_k \otimes g_k,
$$

$$
\Delta(d_k) = g_k \otimes d_k + d_k \otimes g_{k+1},
$$

$$
e(d_k) = 1; \quad e(g_k) = 0;
$$

then $(C, \Delta, e)$ is a coalgebra. In $C^*$ we define the elements

$$
g^*_k(g_i) = \delta_{ki}, \quad g^*_k(d_i) = 0
$$

$$
d^*_k(d_i) = \delta_{ki}, \quad d^*_k(g_i) = 0.
$$

Clearly, $e = \sum_i g^*_i$ and $g^*_i g^*_j = \delta_{ij} g^*_i$. Hence $C = \oplus_i C g^*_i$ as a left $C^*$-module and $C = \oplus_i g^*_i C$ as a right $C^*$-module. We have also $C g^*_k = \langle g_k, d_k \rangle$ (the vector space spanned by $g_k, d_k$); and $g^*_k C = \langle g_k, d_{k-1} \rangle$ if $k > 1$ and $g^*_1 C = \langle g_1 \rangle$. Hence $C$ is a left and right semiperfect coalgebra. By [7, Example 1], $C$ is left co-Frobenius and, thus, $C$ is left $QeF$. The socle of $C$ (left or right) as a $C^*$-module is $\langle g_k; k = 1, \ldots, \infty \rangle$. The set \{\langle g_k; k = 1, \ldots, \infty \rangle\} consists of all, up to isomorphism, simple comodules.

Clearly, $C g^*_k$ is an essential extension as a left $C^*$-module of the simple $\langle g_k \rangle$. Therefore, the set \{\langle C g^*_k; k = 1, \ldots, \infty \rangle\} is made of all the nonisomorphic indecomposable injective right comodules. Since $C = \oplus_i g^*_i C$ then in particular $g^*_1 C$ is an indecomposable injective right $C^*$-module. Assume now that $C$ is also right $QeF$. Then $C_{g^*}$ is projective and so is $g^*_1 C$. Put $M = g^*_1 C$. The right $C$-comodule $M^* = \text{Hom}_C(M, k)$ is indecomposable and injective. Thus, $M^* \cong C g^*_j$ for some $j$. But $\text{dim}(M^*) = 1$ and $\text{dim}(g^*_j C) = 2$, which is a contradiction.

A coalgebra $C$ has a canonical structure of a $C^*$-$k$-bimodule that induces on $C^* = \text{Hom}_C(C, k)$ a $k - C^*$-bimodule structure. It is easy to see that this right $C^*$-module structure on $C^*$ is the usual one. Now, since $k$ is an injective cogenerator for $k$-Mod, we obtain the following result.

Proposition 1.7. $C^*$ is a right self-injective ring if and only if $C$ is flat as a left $C^*$-module.

Proof. This follows from a well-known characterization of flat modules (see [11, Proposition 10.4]).
COROLLARY 1.8. If $C$ is a left $QcF$ coalgebra, then $C^*$ is a right self-injective ring.

2. THE RATIONAL FUNCTOR

Let $C$ be a coalgebra with dual algebra $C^*$. Each left $C^*$-module $M$ contains a unique maximal rational submodule $\text{Rat}(M)$. In fact, we can define a functor $\text{Rat}:C^*\text{-Mod} \to C^*\text{-Mod}$ that associates to each $C^*$-morphism $f:M \to N$ the restriction $f:\text{Rat}(M) \to \text{Rat}(N)$. This functor is a left exact preradical (see [7; 8; 11, Chap. VI]). From [7, Theorem 23] we know that for a right semiperfect coalgebra, $\text{Rat}$ is a left exact radical. We will prove in Proposition 2.2 that $\text{Rat}$ is indeed an exact functor in this case. To do this, we use Lemma 2. We include its proof since we do not know a suitable reference.

LEMMA 2.1. Let $C$ be a right semiperfect coalgebra and consider $P$ a rational left $C^*$-module. If $P$ is projective in $\text{Rat}(C^*\text{-Mod})$ then $P$ is projective in $C^*\text{-Mod}$.

Proof. By the locally finite property, there is a family $\{M_i | i \in I\}$ of finite-dimensional rational left $C^*$-modules and an epimorphism

$$\oplus \{M_i | i \in I\} \to P \to 0.$$ 

By [7, Theorem 10], for each $i \in I$ there is a finite-dimensional projective object $P_i$ of $\text{Rat}(C^*\text{-Mod})$ such that $M_i$ is an epimorphic image of $P_i$. Therefore, we have an epimorphism

$$\oplus \{P_i | i \in I\} \to P \to 0.$$ 

Since $P$ is projective in $\text{Rat}(C^*\text{-Mod})$, $\oplus \{P_i | i \in I\} \cong P \oplus P'$, for some $P' \in \text{Rat}(C^*\text{-Mod})$. By [3, Proposition 4], $P_i$ is projective in $C^*\text{-Mod}$ for every $i \in I$ and, thus, $P$ is a projective left $C^*$-module. 

We recall some ideas from [4] on localization in Grothendieck categories. A full subcategory $\mathcal{E}$ of a Grothendieck category $\mathcal{A}$ is said to be localizing if $\mathcal{E}$ is closed under extensions, direct sums, subobjects, and quotient objects. The objects of $\mathcal{E}$ are called the $\mathcal{E}$-torsion objects. For the localizing subcategory $\mathcal{E}$ of $\mathcal{A}$ it can be defined the quotient category $\mathcal{A}/\mathcal{E}$. There are functors $T:\mathcal{A} \to \mathcal{A}/\mathcal{E}$ and $S:\mathcal{A}/\mathcal{E} \to \mathcal{A}$ such that $T$ is an exact functor and $S$ is a right adjoint functor to $T$. Therefore, $S$ is left exact. If $\psi:1_\mathcal{A}: \to S \ast T$ is the unit of the adjunction, then $\psi_A:A \to (S \ast T)(A)$ has $\mathcal{E}$-torsion kernel and cokernel for every object $A$ of $\mathcal{A}$. Moreover, the
counit $\phi : T \circ S \to 1_{\text{loc}/C}$ of the adjunction is a natural isomorphism. When
the localizing subcategory $\text{loc}$ is closed under direct products, we will say
that $\text{loc}$ is a TTF class (see [11, Chap. VI] for details).

**Proposition 2.2.** Assume that the coalgebra $C$ is right semiperfect. The
functor $\text{Rat}: C^*\text{-Mod} \to C^*\text{-Mod}$ is exact. Therefore, $\text{Rat}(C^*\text{-Mod})$ is a
localizing subcategory of $C^*\text{-Mod}$.

**Proof.** We already know that $\text{Rat}$ is left exact. Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence in $C^*\text{-Mod}$. We have the commutative diagram

$$
\begin{array}{ccc}
0 & \to & \text{Rat}(M') \xrightarrow{\iota'} \text{Rat}(M) \xrightarrow{\iota''} \text{Rat}(M'') \\
& \downarrow{\iota'} & \downarrow{\iota''} \downarrow{\iota''} \\
0 & \to & M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0 \\
\end{array}
$$

where $\iota'$, $\iota''$, and $\iota'''$ are the inclusion maps. Since $\text{Rat}(C^*\text{-Mod})$ has enough
projectives, there is an epimorphism $f: P \to \text{Rat}(M''') \to 0$, where $P \in \text{Rat}(C^*\text{-Mod})$ is projective. By Lemma 2.1, $P$ is also projective in $C^*\text{-Mod}$. Then
there exists $g: P \to M$ such that $v \circ g = \iota''' \circ f$. But $1_M g \subseteq \text{Rat}(M)$
and if we denote by $g': P \to \text{Rat}(M)$ the corestriction of $g$, then we have
$g = \iota' \circ g'$. Hence, $v \circ i \circ g' = \iota'' \circ f$. Since $v \circ i = \iota' \circ v'$, we have $\iota'' \circ v' \circ g' = \iota''' \circ f$ and, so, $v' \circ g' = f$. But $f$ is an epimorphism; whence $v'$ is an
epimorphism. This gives the exactness of $\text{Rat}$. It is clear that $\text{Rat}(C^*\text{-Mod})$
is a localizing subcategory of $C^*\text{-Mod}$. \[\square\]

We denote by $\mathcal{F}$ the class of the torsionfree left $C^*$-modules with
respect to the localizing subcategory $\text{Rat}(C^*\text{-Mod})$. In other words,

$$\mathcal{F} = \{ M \in C^*\text{-Mod} \mid \text{Rat}(M) = 0 \}.$$ 

**Theorem 2.3.** Assume that $C$ is a right semiperfect coalgebra. The class $\mathcal{F}$
enjoys the following properties.

(i) $\mathcal{F}$ is a localizing subcategory of $C^*\text{-Mod}$.
(ii) $\mathcal{F}$ is a TTF class stable under injective envelopes.
(iii) The quotient category $C^*\text{-Mod}/\mathcal{F}$ is equivalent to $\text{Rat}(C^*\text{-Mod})$.
(iv) If we denote by $a = \text{Rat}(C,C^*)$, then $a$ is a two-sided ideal of the
ring $C^*$. Moreover, $C^*/a$ is a flat right $C^*$-module.

**Proof.**

(i) It is immediate from the exactness of $\text{Rat}$ that $\mathcal{F}$ is stable under
homomorphic images.
(ii) This is clear.

(iii) Let us denote by $T:C^*\text{-Mod} \to C^*\text{-Mod}/\mathcal{F}$ and $S:C^*\text{-Mod}/\mathcal{F} \to C^*\text{-Mod}$ the canonical functors. Consider the functor $F: \text{Rat}(C^*\text{-Mod}) \to C^*\text{-Mod}/\mathcal{F}$ given by $F = T \circ i$, where $i: \text{Rat}(C^*\text{-Mod}) \to C^*\text{-Mod}$ is the inclusion functor; and the functor $G:C^*\text{-Mod}/\mathcal{F} \to \text{Rat}(C^*\text{-Mod})$, defined by $G = \text{Rat} \circ S$. If $M \in \text{Rat}(C^*\text{-Mod})$, then the natural transformation $\psi:I \to S \circ T$ provides an exact sequence

$$0 \to \text{Ker}(\psi_M) \to M \to \text{ST}(M) \to \text{coKer}(\psi_M) \to 0$$

with $\text{Ker}(\psi_M), \text{coKer}(\psi_M) \in \mathcal{F}$. Now apply the exact functor $\text{Rat}$ to this sequence to obtain an isomorphism $\text{Rat}(\psi_M): M \cong \text{Rat}(\text{ST}(M))$. This implies that $G \circ F$ is naturally isomorphic to $1_{\text{Rat}(C^*\text{-Mod})}$. On the other hand, if $X \in C^*\text{-Mod}/\mathcal{F}$, then $S(X)/\text{Rat}(S(X)) \in \mathcal{F}$. Hence, $T(S(X)/\text{Rat}(S(X))) = 0$. Since $T$ is an exact functor, we obtain that $F \circ G$ is naturally isomorphic to $1_{C^*\text{-Mod}/\mathcal{F}}$.

(iv) It is well known that $a$ is a two-sided ideal of $C^*$. Moreover, $C^*/a \in \mathcal{F}$ since $\text{Rat}(C^*/a) = 0$. If $M \in C^*\text{-Mod}$, then $aM \in \text{Rat}(C^*\text{-Mod})$. Therefore $aM \cong 0$ for $M \in \mathcal{F}$. Thus, $a \subseteq I$ for every left ideal $I$ of $C^*$ such that $C^*/I \in \mathcal{F}$. It follows that $a$ is the unique minimal (left) ideal in the Gabriel filter associated to the localizing subcategory $\mathcal{F}$ of $C^*\text{-Mod}$. Hence, $\mathcal{F}$ is isomorphic with the category $C^*/a\text{-Mod}$. We consider the canonical ring morphism $\pi:C^* \to C^*/a$. The functor $(C^*/a) \otimes_{C^* \text{-Mod}} -:C^*/a\text{-Mod} \to C^*/a\text{-Mod}$ is a left adjoint to the functor restriction of scalars $\pi_*: C^*/a\text{-Mod} \to C^*\text{-Mod}$. Up to the isomorphism $\mathcal{F} \cong C^*/a\text{-Mod}$, $\pi_*$ is nothing but the inclusion functor $j: \mathcal{F} \to C^*\text{-Mod}$. Since $\mathcal{F}$ is stable under injective envelopes, the functor $(C^*/a) \otimes_{C^* \text{-Mod}} -$ has to be exact, i.e., $C^*/a$ is a flat right $C^*$-module.

**Corollary 2.4.** A Hopf algebra $H$ has a left (or right) integral if and only if the functor $\text{Rat}$ is exact.

**Proof.** By [7, Theorems 3 and 10], if $H$ has a left integral, then $H$ is right semiperfect. Now, apply Proposition 2.2. Conversely, if $\text{Rat}$ is an exact functor, then it can be easily proved that the statements of Theorem 2.3 hold. If $a = 0$, then $\mathcal{F} = H^*\text{-Mod}$, which is a contradiction. Therefore, $a \neq 0$, i.e., $H$ has a left integral.

To finish this section, we will use the $\text{Rat}$ functor to show that the coalgebras $C$ for which $C$ is a projective generator of the category $\mathcal{M}^C$ are precisely the left and right $\text{QF}$ coalgebras.

**Proposition 2.5.** If $C$ is a left $\text{QF}$ coalgebra, then $C$ is a generator for the category $\mathcal{M}^C$.

**Proof.** Since the finite-dimensional left $C$-comodules generate $\mathcal{M}^C$, it suffices to prove that every finite-dimensional left $C$-comodule is gener-
ated by $C$. If $M \in \mathcal{C}_\mathcal{M}$ is finite-dimensional, then there is an epimorphism of right $C^*$-modules $(C^*)^\vee \to M$. From Corollary 1.4 and Proposition 2.2, \( \text{Rat} : \text{Mod-C}^* \to \text{Mod-C}^* \) is an exact functor. Therefore, we obtain an epimorphism \( \text{Rat}(C^*)^\vee \to M \). By Corollary 1.8, $C^*$ is a right self-injective ring. It is not hard to see that \( \text{Rat}(C^*) \subseteq C^* \) is an injective object of the category \( \text{Rat}(\text{Mod-C}^*) \), i.e., \( \text{Rat}(C^*) \) is an injective left $C$-comodule. Since every left comodule embeds in a direct sum of copies of $C$ [3], \( \text{Rat}(C^*) \) is a direct summand (as left $C$-comodule) of some direct sum $C^{(1)}$ of copies of $C$. In particular, there is an epimorphism \( C^{(1)} \to \text{Rat}(C^*) \). But this implies that $C$ generates $M$. }

\textbf{THEOREM 2.6.} The following conditions are equivalent for a coalgebra $C$.

(i) $C$ is left and right QcF.

(ii) $C$ is a projective generator of $\mathcal{C}_\mathcal{M}$.

(iii) $C$ is a projective generator of $\mathcal{M}_\mathcal{C}$.

\textbf{Proof.} We have only to prove that (i) is equivalent to (ii).

(i) $\Rightarrow$ (ii). By Theorem 1.3 and Proposition 2.5. $C$ is left QcF. Moreover, $C$ is right QcF if and only if $C^*_C$ is torsionless. From [3] we know that $C^*_C$ has essential socle. Since $C^*_C$, is injective (Corollary 1.8), $C^*_C$ is torsionless if and only if every simple right $C^*$-submodule of $C^*_C$ is torsionless. Therefore, if we prove that every finite-dimensional left $C$-comodule is torsionless, then $C$ is right QcF. Let $M \in \mathcal{M}$ be finite-dimensional. Then $M \cong N^*$, where $N = M^* \in \mathcal{C}_\mathcal{M}$ [10, Lemma 5.1.4]. Since $C$ is a generator of $\mathcal{C}_\mathcal{M}$, there is an epimorphism $C^{(1)} \to N$. Hence, $M \cong N^*$ embeds in $(C^*)^l$, i.e., $M$ is torsionless. 

\textbf{Remark 2.7.} From the proof of Theorem 2.6 it can be deduced that the following condition (iv) is equivalent to conditions (i)–(iii) in that theorem.

(iv) $C^*$ is a left and right self-injective ring and $C$ is a generator for $\mathcal{C}_\mathcal{M}$ and $\mathcal{M}_\mathcal{C}$.

\section{3. Duality}

The concept of Morita duality for module categories has been extended to the framework of abelian categories by Colby and Fuller [2] in the following sense. A pair of contravariant functors

\[ D : \mathcal{A} \cong \mathcal{A}' : D' \]

between abelian categories $\mathcal{A}$ and $\mathcal{A}'$ is called right adjoint if there are natural isomorphisms $\eta_{A, A'} : \text{Hom}_\mathcal{A}(A, D'A') \to \text{Hom}_\mathcal{A}(A', DA)$, for each
pair of objects $A \in \mathcal{A}$ and $A' \in \mathcal{A}'$. From $\eta$ we have the arrows of right adjunction $\tau: 1_{\mathcal{A}} \to D' \circ D$ and $\tau': 1_{\mathcal{A}'} \to D \circ D'$, defined by $\tau_A = \eta_{A,D,A}(1_A)$ and $\tau'_A = \eta_{D',A',A}(1_{D'A'})$. These natural transformations $\tau$ and $\tau'$ determine the right adjunction. An object $A$ of $\mathcal{A}$ (resp. $A'$ of $\mathcal{A}'$) is said to be reflexive with respect to $\tau$ (resp. $\tau'$) if $\tau_A$ (resp. $\tau'_A$) is an isomorphism. If we denote by $\mathcal{A}_0$ and $\mathcal{A}'_0$ to the full subcategories of the reflexive objects, then $D$ and $D'$ define a contravariant equivalence between them. Assume that $\mathcal{A}$ and $\mathcal{A}'$ are Grothendieck categories. Following the terminology of [5] we will say that $D$ and $D'$ define a Colby–Fuller duality if both functors are exact and $\mathcal{A}_0$ and $\mathcal{A}'_0$ are closed under subobjects, quotient objects, and finite direct sums (i.e., they are finitely closed) and contain generating sets for $\mathcal{A}$ and $\mathcal{A}'$, respectively (i.e., they are generating). The aim of this section is to show the existence of Colby–Fuller dualities between $\text{Rat}(C^*\text{-Mod})$ and $\text{Rat}(\text{Mod-C}^*)$ (or, equivalently, between $\mathcal{A}^C$ and $\mathcal{A}'^C$) in the case that $C$ is a semiperfect or $Q\text{C}F$ coalgebra.

Associated to a coalgebra $C$ we will consider some classes of left $C^*$-modules of interest. We list them:

$$\mathcal{F} = \{ M \in C^*\text{-Mod} \mid \text{Rat}(M) = 0 \}$$

$$\mathcal{D} = \{ M \in C^*\text{-Mod} \mid \text{Hom}_{C^*}(M, C^*) = 0 \}$$

$$\mathcal{F}' = \{ M \in C^*\text{-Mod} \mid C \otimes C^*. M = 0 \}.$$

Of course, we can also consider the analogous classes of right $C^*$-modules, denoted by $\mathcal{F}'$, $\mathcal{D}'$, and $\mathcal{F}'$.

**Lemma 3.1.** If $M$ is any left $C^*$-module then $C \otimes C^*. M$ is a rational left $C^*$-module.

**Proof.** Let $F \to M$ be a free presentation of the left $C^*$-module $M$. By tensorizing on the left by $C^*$, we obtain an epimorphism of left $C^*$-modules $C \otimes C^*. F \to C \otimes C^*. M$. But $C \otimes C^*. F$ is isomorphic to a direct sum of copies of $C \otimes C^*. C$, whence it is a rational left $C^*$-module. Therefore, $C \otimes C^*. M$ is rational.

**Lemma 3.2.** For any coalgebra $C$, we have the equality $\mathcal{D} = \mathcal{F}'$.

**Proof.** We have to prove that if $M \in C^*\text{-Mod}$, then $\text{Hom}_{C^*}(M, C^*) = 0$ if and only if $C \otimes C^*. M = 0$. But this is a consequence of the natural isomorphism $\text{Hom}_{C^*}(M, C^*) \cong (C \otimes C^*. M)^*$. 

In what follows, we will consider the Rat functors on $C^*\text{-Mod}$ and $\text{Mod-C}^*$. We will not distinguish notations for left or right $C^*$-modules. For every $M \in C^*\text{-Mod}$ consider $\alpha_M: M \to M^{**}$ the morphism given by the canonical natural transformation $\alpha: 1 \to (-)^{**}$. On the other hand, the
inclusion \( i: \text{Rat}(M^*) \hookrightarrow M^* \) gives an epimorphism \( i^* : M^{**} \twoheadrightarrow \text{Rat}(M^*)^* \).

We can obtain a natural transformation \( \sigma: 1 \rightarrow (-)^* \circ \text{Rat} \circ (-)^* \) defined as \( \sigma_M = i^* \circ \alpha_M \) for every \( M \in C^\ast\text{-Mod} \). For \( M \) rational, we can in fact consider \( \sigma_M : M \rightarrow \text{Rat}(\text{Rat}(M^*)) \) and, therefore, we have a natural transformation \( \sigma: 1 \rightarrow \text{Rat} \circ (-)^* \circ \text{Rat} \circ (-)^* \). Of course, we can make the same definitions for right \( C^\ast \)-modules. It can be easily checked that the natural transformations \( \sigma \) come from the right adjoint pair

\[
\text{Rat} \circ (-)^* : \text{Rat}(C^\ast\text{-Mod}) \rightleftarrows \text{Rat}(\text{Mod-C}^\ast) : \text{Rat} \circ (-)^*.
\]

**Theorem 3.3.** The following conditions are equivalent for a coalgebra \( C \).

(i) \( \sigma_M : M \rightarrow (\text{Rat}(M^*))^* \) is a monomorphism for every \( M \in \text{Rat}(\text{Mod-C}^\ast) \).

(ii) \( C \) is right semiperfect.

(iii) \( \text{Rat}(C^\ast\text{-Mod}) \rightarrow C^\ast\text{-Mod} \) is an exact functor.

(iv) \( \text{Rat} \circ (-)^* : \text{Rat}(\text{Mod-C}^\ast) \rightarrow (C^\ast\text{-Mod}) \) is an exact functor.

**Proof.** (i) \( \Rightarrow \) (ii). By [7, Theorem 10] we have to prove that \( \text{Rat}(M^*) \) is dense in \( M^* \) for every \( M \in \text{Rat}(C^\ast\text{-Mod}) \). Let \( m \in M \) be such that \( f(m) = 0 \) for every \( f \in \text{Rat}(M^*) \). Now, \( \sigma_M(m) = (i^* \circ \alpha_M)(m) = \alpha_M(m) \circ i \). Thus, for \( f \in \text{Rat}(M^*) \), we have

\[
[\sigma_M(m)](f) = [\alpha_M(m) \circ i](f) = [\alpha_M(m)](f) = f(m) = 0.
\]

Therefore \( \sigma_M(m) = 0 \); whence \( m = 0 \). This shows that \( \text{Rat}(M^*) \) is dense in \( M^* \).

(ii) \( \Rightarrow \) (iii). This is Proposition 2.2.

(iii) \( \Rightarrow \) (iv). This is evident.

(iv) \( \Rightarrow \) (i). Assume that \( M \in \text{Rat}(C^\ast\text{-Mod}) \). Take \( m \in M \) such that \( \sigma_M(m) = 0 \). By the locally finite property for comodules, there exists a finite dimensional \( C^\ast \)-submodule \( N \) of \( M \) containing \( m \). Let \( j: N \hookrightarrow M \) denote the inclusion. Now \( \sigma_M \circ j = (\text{Rat}(j^*))^* \circ \sigma_N \). Since \( \text{Rat} \circ (-)^* \) is an exact functor, \( \text{Rat}(j^*)^* \) is a monomorphism. On the other hand, since \( N \) is finite dimensional, \( N^* \) is a rational right \( C^\ast \)-module and, therefore, the inclusion \( \text{Rat}(N^*) \hookrightarrow N^* \) is the identity map. Therefore, \( \sigma_N = \alpha_N \) and this last morphism is bijective. We conclude that \( \sigma_M \circ j \) is a monomorphism. But \( 0 = \sigma_M \circ j(m) \); whence \( m = 0 \).

**Lemma 3.4.** Assume that \( C \) is left and right semiperfect and let \( M \in \text{Rat}(C^\ast\text{-Mod}) \). If \( \text{Rat}(\alpha_M) : M \rightarrow \text{Rat}(M^{**}) \) is an epimorphism then \( \sigma_M : M \rightarrow \text{Rat}(\text{Rat}(M^*)) \) is an isomorphism.

**Proof.** By Theorem 3.3 we have only to prove that \( \sigma_M \) is an epimorphism. Recall that \( \sigma_M = \text{Rat}(i^* \circ \alpha_M) \), where \( i : \text{Rat}(M^*) \hookrightarrow M^* \) is the
inclusion map. Thus, $\sigma_M = \text{Rat}(i^*) \circ \text{Rat}(\alpha_M)$. By Proposition 2.2, $\text{Rat}(i^*)$ is an epimorphism. This concludes the proof of the lemma.

**Theorem 3.5.** A coalgebra $C$ is left and right semiperfect if and only if the functors

$$\text{Rat} \circ (-)^*: \text{Rat}(C^*-\text{Mod}) \rightarrow \text{Rat}(\text{Mod}-C^*) : \text{Rat} \circ (-)^*$$

give a Colby–Fuller duality. Moreover, $\varepsilon_C$ and $C_C$ are reflexive under this duality.

**Proof.** The “if” part is a consequence of Theorem 3.3. Assume that $C$ is a left and right semiperfect coalgebra. By Theorem 3.3, the functors $\text{Rat} \circ (-)^*$ are exact. If $M$ is a finite-dimensional rational left $C^*$-module, then $M^*$ is a finite-dimensional rational right $C^*$-module [10, Lemma 5.1.4] and, therefore, $M^{**}$ is a finite-dimensional rational left $C^*$-module. This implies that $\text{Rat}(M^{**}) = M^{**}$ and $\text{Rat}(\alpha_M) = \alpha_M$, which is an isomorphism. By Lemma 3.4, $\sigma_M$ is an isomorphism. Thus, every finite-dimensional rational left $C^*$-module is reflexive. By the locally finite property for rational comodules, any set of representatives of the isomorphism classes of the finite-dimensional rational left $C^*$-modules is a set of reflexive generators for $\text{Rat}(C^*-\text{Mod})$. Since $\text{Rat} \circ (-)^*$ is an additive functor, every finite direct sum of reflexive rational left $C^*$-submodules is reflexive. Now, if $M \in \text{Rat}(C^*-\text{Mod})$ is reflexive and $N \subseteq M$ is a $C^*$-submodule, then we have the diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N \longrightarrow 0 \\
\text{\sigma}_N \downarrow & & \text{\sigma}_M \downarrow & & \text{\sigma}_{MN} \downarrow & \\
0 & \longrightarrow & \text{Rat}(\text{Rat}(N^*)) & \longrightarrow & \text{Rat}(\text{Rat}(M^*)) & \longrightarrow & \text{Rat}(\text{Rat}(M/N)^*)) \longrightarrow 0
\end{array}
$$

with exact rows. Since $\sigma_M$ is an isomorphism and $\sigma_{M/N}$ is a monomorphism (Theorem 3.3), it follows that $\sigma_{M/N}$ and $\sigma_N$ are isomorphisms. Thus, we have proved that the full subcategory of $\text{Rat}(C^*-\text{Mod})$ whose objects are all the reflexive rational left $C^*$-modules is finitely closed. Analogous arguments can be made on right $C^*$-modules to show that the functors $\text{Rat} \circ (-)^*$ define a Colby–Fuller duality. To conclude, we have only to prove that $\varepsilon_C$ is reflexive. By Lemma 3.4, it suffices to show that $\text{Rat}(\alpha_C): C \rightarrow \text{Rat}(C^{**})$ is an epimorphism. But this is already known (see [7, p. 371]).

**Remark 3.6.** The functors $\text{Rat} \circ (-)^*$ have a rather easy description. It is possible to show that for $M \in \text{Rat}(C^*-\text{Mod})$, $\text{Rat}(M^*)$ is the sum of all the finite-dimensional right $C^*$-submodules of $M^*$ (the case $M = C$ is proved in [10, p. 100]).
If $M \in C^*\text{-Mod}$ then $M^* = \text{Hom}_{C^*}(M, C^*)$ is canonically a right $C^*$-module. Therefore, we have a contravariant functor $(\_)^+: C^*\text{-Mod} \to \text{Mod}-C^*$. Analogously, we have a contravariant functor $(\_)^+: \text{Mod}-C^* \to C^*\text{-Mod}$. We will prove that for $\mathcal{Q}cF$ coalgebras the functors $\text{Rat} \circ (\_)^+$ give a Colby–Fuller duality between the categories $\text{Rat}(C^*\text{-Mod})$ and $\text{Rat}(\text{Mod}-C^*)$. Since every $\mathcal{Q}cF$ coalgebra is semiperfect and $C^*$ is a left and right self-injective ring, we already know that the functors $\text{Rat} \circ (\_)^+$ are exact. For $M \in C^*\text{-Mod}$ we denote by $\beta_M: M \to M^{++}$ the map given by the canonical natural transformation $\beta: 1 \to (\_)^{++}$. Like for the definition of $\sigma$, we consider the natural transformation $\rho: 1 \to (\_)^{++} \circ \text{Rat} \circ (\_)^+$ that acts on $M \in C^*\text{-Mod}$ as the composition $i^+ \circ \beta_M$, where $i: \text{Rat}(M^+) \to M^+$ is the inclusion map.

**Lemma 3.7.** Let $C$ be a left and right semiperfect coalgebra. If $M \in C^*\text{-Mod}$ with $\text{Rat}(M^*) = 0$ then $\text{Rat}(M) = 0$. In particular, $\mathcal{D} \subseteq \mathcal{I}$.

**Proof.** Let $R = \text{Rat}(M) \in \text{Rat}(C^*\text{-Mod})$. We have an epimorphism $M^* \twoheadrightarrow R^*$. Since $\text{Rat}$ is exact and $\text{Rat}(M^*) = 0$, we have that $\text{Rat}(R^*) = 0$. By Theorem 3.3, $\sigma_{R^*}: R \to \text{Rat}(\text{Rat}(R^*)) = 0$ is a monomorphism, and we have obtained $R = 0$.

**Lemma 3.8.** Let $C$ be a left and right semiperfect coalgebra and consider $M \in C^*\text{-Mod}$. Then $M^+ = 0$ if and only if $\text{Rat}(M^+) = 0$.

**Proof.** Assume that $\text{Rat}(M^+) = 0$. Observe that $M^+ \cong (C \otimes_C M)^*$. By Lemma 3.7, we have $\text{Rat}(C \otimes_C M) = 0$. Hence, by Lemma 3.1, $C \otimes_C M = 0$ and this implies that $M^+ = 0$ (Lemma 3.2).

**Proposition 3.9.** Let $C$ be a left and right $\mathcal{Q}cF$ coalgebra. If $M \in C^*\text{-Mod}$ satisfies that $\beta_M: M \to M^{++}$ is an isomorphism, then $\rho_M: M \to (\text{Rat}(M^+))^+$ is an isomorphism.

**Proof.** Consider the exact sequence

$$0 \to \text{Rat}(M^+) \to M^+ \to N \to 0.$$

We claim that $N^+ = 0$. If $N^+ \neq 0$ then there is some homomorphic image $T$ of $N$ that embeds in $C^*C_*$. Let $j: T \hookrightarrow C^*C_*$ be the embedding. Thus we have an exact sequence

$$0 \to R \to M^+ \xrightarrow{\pi} T \to 0$$

that gives us a monomorphism $\pi^*: T^+ \to M^{++}$. Put $S = \text{Im}((\beta_M)^{-1} \circ \pi^+) \subseteq M$. We want to show that $\text{Ker} \pi = \{f \in M^* | f_{1S} = 0\}$. Let $f \in \text{Ker} \pi$ and $m \in S$. There is $g \in T^*$ such that $m = [(\beta_M)^{-1} \circ \pi^+](g)$. Now, $f(m) = \beta_M(m)(f) = \beta_M([(\beta_M)^{-1} \circ \pi^+](g))(f) = \pi^+(g(f)) = g(\pi(f)) = g(0) = 0$. Therefore $\text{Ker} \pi \subseteq \{f \in M^* | f_{1S} = 0\}$. For the other inclusion, let $f \in M^+$ such that $f_{1S} = 0$. Since $j \in T^*$, there exists $m \in S$ such that
\[ \pi^*(j) = \alpha_M(m) \]. Thus, \( j(\pi(f)) = (\pi^*(j))(f) = \alpha_M(m)(f) = f(m) = 0 \). Hence, \( \pi(f) = 0 \) and \( f \in \text{Ker } \pi \), which proves the claim.

Now let \( \psi = i^*: M^+ \rightarrow S^+ \), where \( i: S \rightarrow M \) is the inclusion. It is clear from the claim that \( \psi \) annihilates \( R \); whence we can induce \( \psi': T \rightarrow S^+ \) such that \( \psi' \circ \pi = \psi \). From the equality \( \text{Ker } \pi = \{ f \in M^+ | f_i = 0 \} \) it follows that \( \psi' \) is a monomorphism. Since \( C^* \) is left self-injective, \( i^* = \psi \) is an epimorphism. Therefore \( T \) is isomorphic to \( S^+ \). Moreover, \( \text{Rat}(T) = 0 \), since it is an homomorphic image of a member of \( \mathcal{S} \). Therefore, \( \mathcal{R}(S^+) = 0 \). From Lemma 3.8, \( S^+ = 0 \), that is, \( T = 0 \). This gives a contradiction. So, we have proved that \( N^+ = 0 \). Therefore, the canonical map \( M^{++} \rightarrow (\text{Rat}(M^+))^* \) is an isomorphism and, since \( \beta_M \) is also bijective, we can conclude that \( \rho_M: M \rightarrow (\text{Rat}(M^+))^* \) is an isomorphism.

**Corollary 3.10.** Assume that \( C \) is a left and right \( QcF \) coalgebra. If \( M \) is any finitely generated rational module, then \( \rho_M: M \rightarrow \text{Rat}(\text{Rat}(M^+))^* \) is an isomorphism.

**Proof.** Let \( M \in \text{Rat}(C^*\text{-Mod}) \) be finite dimensional. We already know that \( M \) is torsionless. Now \( M \) is finitely generated as a left \( C^* \)-module and \( C^* \) is left and right self-injective. It follows easily that \( \beta_M: M \rightarrow M^{++} \) is an isomorphism.

**Proposition 3.11.** \( \rho_M: M \rightarrow \text{Rat}(\text{Rat}(M^+))^* \) is a monomorphism for any rational left \( C^* \)-module \( M \) on a left and right \( QcF \) coalgebra \( C \).

**Proof.** The argument of the proof of (iv) \( \Rightarrow \) (i) of Theorem 3.3 works in this case.

**Theorem 3.12.** Let \( C \) be a left and right \( QcF \) coalgebra. The functors

\[
\text{Rat}^*(-)^*: \text{Rat}(C^*\text{-Mod}) \cong \text{Rat}(\text{Mod}-C^*) : \text{Rat}^*(-)^*
\]

give a Colby–Fuller duality. Moreover, \( c_C \) and \( C_c \) are reflexive under this duality.

**Proof.** A similar argument to that of the proof of Theorem 3.5 gives the Colby–Fuller duality. To see that \( c_C \) is reflexive, recall from Theorem 3.5 that \( C_c \) is a reflexive object for the Colby–Fuller duality given by the functors \( \text{Rat}^*(-)^* \). By [6, Theorem 3.2] (see also [3, Lemma 2]) the reflexive objects for a Colby–Fuller duality are precisely the linearly compact objects. But this condition is independent of the concrete functors defining the duality. So, \( C_c \) is also reflexive for the Colby–Fuller duality defined by the functors \( \text{Rat}^*(-)^* \).
REFERENCES