Generalized Local Duality

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Duality theory for locally noetherian schemes has been studied by
several authors [4, 6, 7]. We aim to extend the classical results to the
more general situation of schemes containing some generically stable
subset and verifying relative finiteness conditions, such as Krull
schemes, for example [9, 10, 15]. Thus we extend to the setting of
schemes the results on duality established in [1, 2], obtaining that
every dualizing functor is determined by an injective \( \mathcal{E}_X \)-module \( \mathcal{E} \) and giving a structure theorem of \( \mathcal{E} \) as a
direct sum of indecomposable injectives.

We start with a short introduction on localization theory for sheaves of
modules, referring to [8, 12, 14] for more information. Let \((X, \mathcal{E}_X)\) be a
ringed space and \(\mathcal{S}(X, \mathcal{E}_X)\) the category of sheaves of modules over \(X\). Let
\(Y\) be a generically stable subset of \(X\) and let \(Z = X \setminus Y\) be the comple-
ment of \(Y\). An \(\mathcal{E}_X\)-module \(\mathcal{F}\) is said to be \(Y\)-torsion if \(\text{Supp}(\mathcal{F}) \subset Z\). For
each \(\mathcal{E}_X\)-module \(\mathcal{F}\), it is possible to define an \(\mathcal{E}_X\)-submodule \(\Gamma_Z\mathcal{F}\) satisfying
the property that it is the largest sub-\(\mathcal{E}_X\)-module of \(\mathcal{F}\) which is
\(Y\)-torsion. We will denote by \(\mathcal{S}_Z(X, \mathcal{E}_X)\) the subcategory of \(\mathcal{S}(X, \mathcal{E}_X)\)
consisting of all \(Y\)-torsion \(\mathcal{E}_X\)-modules. The functor \(\Gamma_Z : \mathcal{S}(X, \mathcal{E}_X) \rightarrow
\mathcal{S}_Z(X, \mathcal{E}_X)\) is the right adjoint of the embedding functor \(i_Z : \mathcal{S}_Z(X, \mathcal{E}_X) \rightarrow
\mathcal{S}(X, \mathcal{E}_X)\). Therefore \(\mathcal{S}_Z(X, \mathcal{E}_X)\) is a localizing subcategory of \(\mathcal{S}(X, \mathcal{E}_X)\)
and with respect to this we may define the usual notions of torsion,
torsionfree, relatively injective, ...

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Let us denote by $S_{X/Z}(X, \mathcal{O}_X)$ the quotient category, that is, the full subcategory of $S(X, \mathcal{O}_X)$ of all $Y$-closed sheaves of $\mathcal{O}_X$-modules. The localization of $\mathcal{F}$ relative to the subcategory $S_{F}(X, \mathcal{O}_X)$ will be denoted by $\mathcal{C}l_{X/Z}(\mathcal{F})$. Thus we obtain the following situation of categories and functors

$$S(X, \mathcal{O}_X) \xleftarrow{\text{a}} S_{X/Z}(X, \mathcal{O}_X),$$

where $a = \mathcal{C}l_{X/Z}(-)$ is the left exact localization functor and $i$ is the inclusion. Using sheaf extensions it is possible to get $Y$-closed $\mathcal{O}_X$-modules from sheaves of modules over the ringed space $(Y, \mathcal{O}_X|_Y)$. More concretely, if $j: Y \to X$ is the inclusion map, then we have a commutative diagram of categories and functors

$$\begin{array}{c}
S_{X/Z}(X, \mathcal{O}_X) \\
\xleftarrow{\text{a}} \\
S(X, \mathcal{O}_X) \\
\xleftarrow{\text{i}} \\
S(Y, \mathcal{O}_Y|_Y)
\end{array}$$

In other words, this yields a clear way of obtaining the localization of an $\mathcal{O}_X$-module, with respect to $S_{F}(X, \mathcal{O}_X)$, by first restricting to $Y$ and then extending to $X$. See [8, 13].

Throughout this paper, we always assume $Y$ to be generically closed, and hence (equivalently) that $Z = X \setminus Y$ is closed under specialization. We denote by $Q(X, \mathcal{O}_X)$ the category of quasicoherent sheaves of $\mathcal{O}_X$-modules, by $Q_{F}(X, \mathcal{O}_X)$ the full subcategory of $Q(X, \mathcal{O}_X)$ whose objects are $Y$-torsion, and, finally, by $Q_{F/Z}(X, \mathcal{O}_X)$ the full subcategory of quasicoherent sheaves of $Y$-closed $\mathcal{O}_X$-modules.

In our presentation, the fundamental objects are the quasicoherent sheaves over a scheme $(X, \mathcal{O}_X)$. We remark that, in general, one cannot restrict to quasicoherent sheaves in the previous diagram, so that the functor $j_*$ does not necessarily map quasicoherent sheaves to quasicoherent sheaves.

1. LOCALIZATION ON SCHEMES

In this section, we will present a brief survey of localization on schemes. The reader may consult [5, 8, 12, 14] for more details.
(1.1) Definition. A subset $Y$ of a topological space $X$ is said to be \textit{generically stable} or \textit{closed under generization}, if the following equivalent conditions hold

1. $Y = \{ x \in X; [x]_{X}^{X} \cap Y \neq \emptyset \}$;

2. the complement of $Y$ in $X$, $Z = X \setminus Y$, is \textit{closed by specialization}, that is, for every $x \in Z$, $[x]_{X}^{X} \subseteq Z$.

So, if $(X, \mathcal{O}_{X})$ is a scheme and if $Y$ is a subset of $X$, then $Y$ is generically stable in $X$ if and only if $Y \cap U$ is generically stable in $U$ for every affine open subset $U \subseteq X$. Every generically stable subset $Y$ of a scheme $(X, \mathcal{O}_{X})$ induces for every affine open subset $U = \text{Spec}(A)$ a semicentered idempotent kernel functor $\kappa_{A} = \kappa_{Y \cap U}$ in $A$-$\text{mod}$ (cf. [8, 14]). Actually, if $\mathfrak{p}$ is a prime ideal in $A$ and $\kappa_{A, \mathfrak{p}}$ is the idempotent kernel functor defined by the multiplicatively closed set $A \setminus \mathfrak{p}$ of $A$, then $\kappa_{A}$ is the meet of all the idempotent kernel functors $\kappa_{A, \mathfrak{p}}$, where $\mathfrak{p}$ runs through $Y \cap U$.

(1.2) Definition. If $(X, \mathcal{O}_{X})$ is a scheme and $Y$ is a generically stable subset of $X$, then $Y$ is \textit{locally of finite type} if for each affine open subset $U = \text{Spec}(A) \subseteq X$, the idempotent kernel functor $\kappa_{Y \cap U}$ is of finite type over $A$, i.e., the Gabriel filter $\mathcal{G}(\kappa_{Y \cap U})$ has a filter basis consisting of finitely generated ideals.

We will say that $Y$ is of \textit{finite type}, if it is locally of finite type and, with the induced topology, is quasicompact.

There are interesting characterizations (see [14, (6.16)]) of generically stable subsets $Y$ of a scheme $(X, \mathcal{O}_{X})$ locally of finite type. In fact, the following assertions are equivalent:

1. $Y$ is locally of finite type;

2. there exists an affine open covering $(U_{\alpha} = \text{Spec}(A_{\alpha}); \alpha \in \Lambda)$ of $X$, such that $\kappa_{Y \cap U_{\alpha}}$ is of finite type, for every $\alpha \in \Lambda$;

3. there exists an affine open covering $(U_{\alpha} = \text{Spec}(A_{\alpha}); \alpha \in \Lambda)$ of $X$, such that $Y \cap U_{\alpha}$ is of finite type in $U_{\alpha}$, for every $\alpha \in \Lambda$;

4. there exists an affine open covering $(U_{\alpha} = \text{Spec}(A_{\alpha}); \alpha \in \Lambda)$ of $X$, such that $Y \cap U_{\alpha}$ is a quasicompact space, for every $\alpha \in \Lambda$.

Let $\kappa$ be an idempotent kernel functor in $A$-$\text{mod}$. An $A$-module $M$ is $\kappa$-noetherian if the set $\{ N \subseteq M; M/N$ is $\kappa$-torsionfree $\}$ satisfies the ascending chain condition, or equivalently, if every submodule $N \subseteq M$ contains a finitely generated submodule $N' \subseteq N$ such that $N/N'$ is $\kappa$-torsion. The ring $A$ is $\kappa$-noetherian if it is $\kappa$-noetherian as an $A$-module.
For example, if \( A \) is a noetherian ring, then \( A \) is \( \kappa \)-noetherian for every idempotent kernel functor \( \kappa \) in \( A \)-mod. As another example, let \( A \) be a Krull domain and let \( X^{(1)} \) denote the set of prime ideals of \( A \) of height less than or equal to one. Since \( X^{(1)} \) is generically stable, it determines an idempotent kernel functor \( \kappa_{X^{(1)}} \) in \( A \)-mod and \( A \) is \( \kappa_{X^{(1)}} \)-noetherian; cf. [11, Corollary XIII.4.6]. Furthermore, in general, a Krull domain is not noetherian.

(1.3) DEFINITION. We say that \( (X, \mathcal{O}_X) \) is locally \( Y \)-noetherian if for every affine open subset \( U = \text{Spec}(A) \) of \( X \), the ring \( A \) is \( \kappa_Y \cap U \)-noetherian.

If \( X \) is locally \( Y \)-noetherian and \( Y \) is quasicompact, then we say that \( X \) is \( Y \)-noetherian.

Clearly, if \( X \) is (locally) \( Y \)-noetherian, then \( Y \) is (locally) of finite type.

Just as for the notion of being of finite type, in [14, (5.30)] there are characterizations of locally \( Y \)-noetherian schemes. In fact, the following assertions are equivalent:

1. \( X \) is locally \( Y \)-noetherian;
2. there exists an affine open covering \( \{U_\alpha = \text{Spec}(A_\alpha); \alpha \in \Lambda \} \) of \( X \), such that the ring \( A_\alpha \) is \( \kappa_Y \cap A_\alpha \)-noetherian, for every \( \alpha \in \Lambda \);
3. there exists an affine open covering \( \{U_\alpha = \text{Spec}(A_\alpha); \alpha \in \Lambda \} \) of \( X \), such that \( U_\alpha \) is \( U_\alpha \cap Y \)-noetherian, for every \( \alpha \in \Lambda \).

Let \( (X, \mathcal{O}_X) \) be a scheme and let \( Y \) be a generically stable subset which is locally of finite type. Then for every \( \mathcal{F} \in \mathcal{Q}(X, \mathcal{O}_X) \), we have \( \text{Cl}_{X/Y}(\mathcal{F}) \in \mathcal{Q}(X, \mathcal{O}_X) \) and \( \Gamma_Y \mathcal{F} \in \mathcal{Q}(X, \mathcal{O}_X) \); cf. [8]. Thus they define endofunctors in \( \mathcal{Q}(X, \mathcal{O}_X) \).

(1.4) DEFINITION. Let \( Y \) be a generically stable subset of a scheme \( (X, \mathcal{O}_X) \), let \( Z = X \setminus Y \), and let \( L_Y \) denote the family of quasicoherent ideals of \( \mathcal{O}_X \) defining closed subsets of \( X \) contained in \( Z \) [3, E.G.A. I Chap. 1, 4.1], i.e.,

\[
L_Y = \{ \mathcal{I} \subset \mathcal{O}_X; \mathcal{I} \in \mathcal{Q}(X, \mathcal{O}_X), \text{Supp}(\mathcal{O}_X/\mathcal{I}) \subset Z \}.
\]

(1.5) LEMMA. Let \( (X, \mathcal{O}_X) \) be a quasiseparated scheme, \( Z \subseteq X \) a subset of \( X \) closed under specialization, \( U = \text{Spec}(A) \subseteq X \) an affine open subset, and \( T \subseteq U \) a closed subset in \( U \). If \( T \subseteq Z \), then \( \overline{T^X} \subseteq Z \).

Proof. For each affine open subset \( V = \text{Spec}(B) \subseteq X \), we have \( \overline{T^X} \cap \text{Spec}(B) = \overline{T} \cap \overline{V} \); so it suffices to show that \( \overline{T^X} \cap V \subseteq Z \). Since \( X \) is quasiseparated, it follows that \( U \cap V \) is quasicompact. Hence \( U \cap V = \bigcup_{j=1}^n V_i \) with \( V_i = \text{Spec}(A_{g_i}) \) for \( g_1, \ldots, g_n \in B \). Let \( s_i: B \to B_{g_i} \) be the
restriction morphisms and \( r_i: A \to B_i \), the induced morphisms \( (V_i \subseteq U) \) is an affine open subset. Let \( a \) be an ideal of \( A \) such that \( V_\alpha(a) = T \). For each \( 1 \leq i \leq n \), let \( a_i \) be the extension of \( a \) through \( r_i \) and \( b_i \), the contraction of \( a_i \) through \( s_i \). Let \( Y = X \setminus Z \) be the generically stable complement of \( Z \). The idempotent kernel functor in \( B_i \), induced through \( r_i \), by \( \kappa_{Y \cup U} \) is \( \kappa_{Y \cap V_i} \), and it coincides with the one in \( B_i \), induced through \( s_i \), by \( \kappa_{Y \cap Y} \). By assumption \( T \subseteq Z \), i.e., \( a \in \mathcal{L}(\kappa_{Y \cap U}) \), so \( a_i \in \mathcal{L}(\kappa_{Y \cap V_i}) \) and \( b_i \in \mathcal{L}(\kappa_{Y \cap Y}) \), for all \( 1 \leq i \leq n \). Therefore the ideal \( b = \bigcap_{i=1}^{n} b_i \) belongs to \( \mathcal{L}(\kappa_{Y \cap Y}) \). Now, if \( v \in T \cap V \), then \( v \in V_b(a) \) and so \( a \subseteq v \); hence \( v \in V_b \). This follows that \( b \subseteq v \) and so we obtain \( b \subseteq v \), so \( v \in V_b(b) \) and hence \( T \cap V \subseteq V_b(b) \). As \( b \in \mathcal{L}(\kappa_{Y \cap Y}) \), it follows that \( V_b(b) \subseteq Z \); hence \( T \cap V \subseteq Z \). 

1.6 Lemma. Let \( Y \) be a generically stable subset of a scheme \((X, \mathcal{O}_X)\) and \( U \subseteq X \) an open subset. If \( \mathcal{F} \) is a \( Y \)-closed \( \mathcal{O}_X \)-module, then \( \mathcal{F}|_U \) is a \( Y \cap U \)-closed \( \mathcal{O}_X|_U \)-module.

Proof. Let us denote by \( i: U \hookrightarrow X \) the inclusion and by \( i_*: \mathcal{S}(U, \mathcal{O}_X|_U) \to \mathcal{S}(X, \mathcal{O}_X) \) the extension by zero, outside of \( U \). The functor \( i_* \), is left adjoint to the functor \( i^* : \mathcal{S}(X, \mathcal{O}_X) \to \mathcal{S}(U, \mathcal{O}_X|_U) \). Let \( u: \mathcal{M} \to \mathcal{N} \) be a \( Y \cap U \)-isomorphism. The \( \mathcal{O}_X \)-module homomorphism \( i_* (u): i_*(\mathcal{M}) \to i_*(\mathcal{N}) \) is a \( Y \)-isomorphism. For each \( \mathcal{F} \in \mathcal{S}_{X/Y}(X, \mathcal{O}_X) \) the sequence of bijections

\[
\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, i^*(\mathcal{F})) \equiv \text{Hom}_{\mathcal{O}_X}(i_*(\mathcal{N}), \mathcal{F}) \equiv \text{Hom}_{\mathcal{O}_X}(i_*(\mathcal{M}), \mathcal{F})
\]

shows that \( i^*(\mathcal{F}) \) is \((Y \cap U)\)-closed.

1.7 Lemma. Let \( Y \) be a generically stable subset of a scheme \((X, \mathcal{O}_X)\). If \( \mathcal{F}, \mathcal{G} \in \mathcal{S}(X, \mathcal{O}_X) \) and \( \mathcal{F} \) is \( Y \)-closed, then \( \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \) is \( Y \)-closed.

Proof. Let \( u: \mathcal{M} \to \mathcal{N} \) be a \( Y \)-isomorphism in \( \mathcal{S}(X, \mathcal{O}_X) \) and let us consider the morphism of \( \mathcal{O}_X \)-modules

\[
\text{Hom}_{\mathcal{O}_X}(\mathcal{N}, \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \to \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})).
\]

By the adjunction

\[
\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{N} \to \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, -): \mathcal{S}(X, \mathcal{O}_X) \to \mathcal{S}(X, \mathcal{O}_X);
\]

to prove that this morphism is bijective, it suffices to show the bijectivity of

\[
\text{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{N}, \mathcal{G}) \to \text{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}, \mathcal{G}).
\]
But for this, it suffices to verify that the morphism $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{N}$ is a $Y$-isomorphism. However, this is true, since the fact that $u$ is a $Y$-isomorphism implies $1 \otimes_{\mathcal{O}_X} u$ to be one as well. 

(1.8) Proposition. If $(X, \mathcal{O}_X)$ is a quasiseparated locally $Y$-noetherian scheme, then for each $\mathcal{F} \in \mathcal{Q}(X, \mathcal{O}_X)$ we have

$$\text{Cl}_{X/Y}(\mathcal{F}) \cong \lim_{\mathcal{F} \in \mathcal{L}_Y} \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}/\Gamma_2 \mathcal{F}).$$

Proof. For each $\mathcal{F} \in \mathcal{L}_Y$ the homomorphism

$$\text{Cl}_{X/Y}(\mathcal{F}) \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \text{Cl}_{X/Y}(\mathcal{F})) \to \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \text{Cl}_{X/Y}(\mathcal{F}))$$

induced by the inclusion $\mathcal{F} \subseteq \mathcal{O}_X$ is an isomorphism. The morphism $\mathcal{F}/\Gamma_2 \mathcal{F} \to \text{Cl}_{X/Y}(\mathcal{F})$ yields a monomorphism

$$\varphi: \lim_{\mathcal{F} \in \mathcal{L}_Y} \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}/\Gamma_2 \mathcal{F}) \to \text{Cl}_{X/Y}(\mathcal{F}).$$

Consider a section $s \in \text{Cl}_{X/Y}(\mathcal{F})(U)$ over an affine open subset $U \subseteq X$, i.e., a homomorphism of $\mathcal{O}_X|_U$-modules

$$s: \mathcal{O}_X|_U \to \text{Cl}_{X/Y}(\mathcal{F})|_U.$$

Let $\mathcal{X}$ be the $\mathcal{O}_X|_U$-module with the property that

$$\begin{array}{ccc}
\mathcal{O}_X & \xrightarrow{s} & \text{Cl}_{X/Y}(\mathcal{F})|_U \\
\downarrow & & \downarrow \\
\mathcal{O}_X|_U & \xrightarrow{\text{Cl}_{X/Y}(\mathcal{F})|_U} & \text{Cl}_{X/Y}(\mathcal{F})|_U
\end{array}$$

is a pullback diagram. Since the sheaves of $\mathcal{O}_X|_U$-modules occurring in the previous diagram are quasicoherent, $\mathcal{X}$ is a quasicoherent sheaf of ideals of $\mathcal{O}_X|_U$.

The inclusion $i: U \hookrightarrow X$ is a quasicompact morphism, so

$$\begin{array}{ccc}
\mathcal{O}_X \to i_* (\mathcal{X}) \\
\downarrow & & \downarrow \\
\mathcal{O}_X|_U \to i_* (\mathcal{O}_X|_U)
\end{array}$$

is a diagram in $\mathcal{Q}(X, \mathcal{O}_X)$; cf. [3, E.G.A. 1 (6.9.2)]. Now, the closed subset $\text{Supp}(\mathcal{O}_X|_U/\mathcal{X})$ of $U$ defined by $\mathcal{X}$ is contained in $Z$ and by Lemma 1.5

$$\text{Supp}(\mathcal{O}_X/\mathcal{X}) \subseteq \text{Supp}(i_* (\mathcal{O}_X|_U/i_* \mathcal{X}) \subseteq \text{Supp}(i_* (\mathcal{O}_X|_U/\mathcal{X})) \subseteq \text{Supp}(\mathcal{O}_X|_U/\mathcal{X}) \subseteq Z,$$
so we have $L \in \mathcal{L}_V$. Moreover, $L|_U = \mathcal{F}$ and $\tilde{s} \in \text{Hom}_{\mathcal{O}_X}(L|_U, \mathcal{F}/\Gamma_E \mathcal{F}|_U)$ is such that its image under the composition

$$\text{Hom}_{\mathcal{O}_X}(L, \mathcal{F}/\Gamma_E \mathcal{F})(U) \rightarrow \lim_{\mathcal{F} \in \mathcal{L}_V} \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}/\Gamma_E \mathcal{F})(U)$$

$$\xrightarrow{\tilde{s}(U)} \text{Cl}_{X/E}(\mathcal{F})(U)$$

is $s$. 

This result is the counterpart in $\mathcal{Q}(X, \mathcal{O}_X)$ of the well-known description of the module of quotients of a module with respect to an idempotent kernel functor in $\mathcal{A}$-mod; cf. [11, Chap. IX].

2. REPRESENTABILITY

(2.1) DEFINITION. We say that $\mathcal{F} \in \mathcal{S}(X, \mathcal{O}_X)$ is of $Y$-finite type, if for every $x \in X$, there exists an open neighbourhood $U$ and a sheaf of $\mathcal{O}_X|_U$-modules of finite type $\mathcal{F}'$, together with an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}|_U \rightarrow \mathcal{F}' \rightarrow 0$$

such that $\mathcal{F}'$ is $Y \cap U$-torsion. Equivalently, there exists a positive integer $n > 0$ and an exact sequence

$$(\mathcal{O}_X|_U)^n \rightarrow \mathcal{F}|_U \rightarrow \mathcal{F} \rightarrow 0,$$

where $\mathcal{F}$ is $Y \cap U$-torsion.

(2.2) LEMMA. If $(X, \mathcal{O}_X)$ is a locally Noetherian scheme and $\mathcal{M}$ and $\mathcal{N}$ are quasi-coherent sheaves with $\mathcal{M}$ of $Y$-finite type, then $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is quasi-coherent.

Proof. Let $U = \text{Spec}(A) \subseteq X$ be an affine open subset and let $M$ and $N$ be $A$-modules such that $\mathcal{M}|_U = \tilde{M}$ and $\mathcal{N}|_U = \tilde{N}$. Since $M$ is $\kappa_{Y \cap U}$-finitely generated and $A$ is a $\kappa_{Y \cap U}$-Noetherian ring, the module $M$ is $\kappa_{Y \cap U}$-finitely presented. So, there exists a coherent $\mathcal{O}_X|_U$-module $\mathcal{F}$ and a $Y \cap U$-isomorphism $f: \mathcal{F} \rightarrow \mathcal{M}|_U$. As $\mathcal{F} = \tilde{F}$, where $\tilde{F}$ is a finitely presented $A$-module,

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})|_U \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{M}|_U, \mathcal{N}|_U) \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{N}|_U)$$

$$\cong \text{Hom}_A(\tilde{F}, N).$$
The proof of the next lemma is essentially the same as that of Lemma 6.27 in [14].

(2.3) Lemma. Let \((X, \mathcal{O}_X)\) be a quasiseparated locally \(Y\)-noetherian scheme, \(Y\) a generically stable subset, and \(U \subseteq X\) an open subset. For each quasicoherent sheaf \(\mathcal{F}\) in \((X, \mathcal{O}_X)\) and each quasicoherent \(\mathcal{O}_Y|_U\)-submodule \(\mathcal{G} \subseteq \mathcal{F}|_U\) of \(Y \cap U\)-finite type, there exists a \(\mathcal{O}_X\)-submodule of \(Y\)-finite type \(\mathcal{G}' \subseteq \mathcal{F}\) such that \(\mathcal{G}'|_U \cong \mathcal{G}\).

(2.4) Definition. Let \((X, \mathcal{O}_X)\) be a scheme, let \(Z \subseteq Z'\) be subsets of \(X\) closed by specialization, and let \(Y' = X \setminus Z' \subseteq Y = X \setminus Z\).

Assume that \((X, \mathcal{O}_X)\) is quasiseparated and a locally \(Y\)-noetherian scheme. Then we denote by \(\mathbf{Q}^{Y, \text{fin}}_X(X, \mathcal{O}_X)\) the subcategory of \(\mathbf{Q}_X(X, \mathcal{O}_X)\) consisting of all sheaves of \(Y\)-finite type.

(2.5) Definition. Let \(\mathcal{C}\) and \(\mathcal{C}'\) be subcategories of \(\mathbf{S}(X, \mathcal{O}_X)\). A functor \(T: \mathcal{C} \to \mathcal{C}'\) is said to be local if whenever \(\mathcal{M}, \mathcal{N} \in \mathcal{C}\) satisfy \(\mathcal{M}|_U \cong \mathcal{N}|_U\), for some \(U\) open in \(X\), then \(T(\mathcal{M}|_U) \cong T(\mathcal{N}|_U)\).

Let \((X, \mathcal{O}_X)\) be a quasiseparated locally \(Y\)-noetherian scheme and

\[ T: \mathbf{Q}^{Y, \text{fin}}_X(X, \mathcal{O}_X) \to \mathbf{Q}_X(X, \mathcal{O}_X) \]

a contravariant local functor. For \(\mathcal{M} \in \mathbf{Q}^{Y, \text{fin}}_X(X, \mathcal{O}_X)\), we let the \(\mathcal{O}_X\)-module structure on \(T(\mathcal{M})\) be induced by \(\mathcal{M}\).

So, let us consider an affine open subset \(U \subseteq X\). As \(\mathcal{M}\) is \(Y'\)-torsion, every section \(s \in \mathcal{M}(U)\) defines and is defined by a morphism of \(\mathcal{O}_X|_U\)-modules

\[ s: \mathcal{O}_X/\mathcal{J}|_U \to \mathcal{M}|_U \]

for some \(\mathcal{J} \in \mathcal{L}_{Y'}\). Since

\[ \beta_s: \mathcal{O}_X/\mathcal{J} \to i_*(\mathcal{O}_X/\mathcal{J}|_U) \to i_*(\mathcal{M}|_U) \]

is a morphism of quasicoherent sheaves of \(\mathcal{O}_X\)-modules, there exists some \(\mathcal{K} \subseteq i_*(\mathcal{M}|_U)\) quasicoherent of finite type, such that \(\beta_s\) factorizes as

\[
\begin{array}{ccc}
\mathcal{O}_X/\mathcal{J} & \xrightarrow{\beta_s} & \mathcal{K} \\
\downarrow \beta_i & & \downarrow \\
i* (\mathcal{M}|_U) \\
\end{array}
\]

with \(\mathcal{K}\) of \(Y'\)-torsion and \(\mathcal{K} \in \mathbf{Q}^{Y, \text{fin}}_X(X, \mathcal{O}_X)\). Applying the functor \(T\), we obtain the morphism

\[ T(\mathcal{K}) \xrightarrow{T(\beta_i^T)} T(\mathcal{O}_X/\mathcal{J}) \],
which, when restricted to $U$, yields the morphism
\[ T(\mathcal{F})|_U \cong T(i_*\mathcal{H}|_U)|_U \to T(\mathcal{T})|_U \to T(\mathcal{O}_X/\mathcal{F})|_U, \]
whose construction does not depend on $\mathcal{T}$. Let us denote it by $e^U$. The compatibility of the $\mathcal{O}_X$-structure of $T(\mathcal{F})$ and $\mathcal{H}$ shows that $e^U$ is a morphism of $\mathcal{O}_X{|}_U$-modules and, for each section $r \in \mathcal{O}_X(U)$, it can be proved that $e^{U}_r = re^U_r$.

(2.6) Proposition. Let $(X, \mathcal{O}_X)$ be a quasiseparated locally $Y$-noetherian scheme. If $T : \mathcal{O}_X^{\mathbb{C}_Y} \to \mathcal{Q}_{X/\mathbb{C}}(X, \mathcal{O}_X)$ is a contravariant local functor, then there exists a natural transformation
\[ \Phi : T \to \text{Hom}_{\mathcal{O}_X}(-, \mathcal{E}), \]
where
\[ \mathcal{E} = \lim_{\mathcal{F} \in \mathcal{L}_Y} T(\mathcal{O}_X/\mathcal{F}). \]

Proof. We know that $\mathcal{E}$ is a $Y$-closed $\mathcal{O}_X$-module and, as a consequence of Lemma 2.2, it makes sense to consider the functor
\[ \text{Hom}_{\mathcal{O}_X}(-, \mathcal{E}) : \mathcal{O}_X^{\mathbb{C}_Y} \to \mathcal{Q}_{X/\mathbb{C}}(X, \mathcal{O}_X). \]

Let $\mathcal{A} \in \mathcal{O}_X^{\mathbb{C}_Y}(X, \mathcal{O}_X)$ and let $\mathcal{I}_{\mathcal{A}}$ be the family of all $\mathcal{O}_X$-submodules $\mathcal{A}'$ of $\mathcal{A}$, which are quasicoherent, of finite type, and such that $\mathcal{A}/\mathcal{A}'$ is $Y$-torsion. For each $\mathcal{N} \in \mathcal{I}_{\mathcal{A}}$, there exists an ideal $\mathcal{F} \in \mathcal{L}_Y$, such that $\mathcal{A} = 0$, i.e., such that $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{F}, \mathcal{N}) = \mathcal{N}$.

Let $\tilde{\varphi}_r : \mathcal{N} \to \text{Hom}_{\mathcal{O}_X}(T(\mathcal{A}), T(\mathcal{O}_X/\mathcal{F}))$ be the homomorphism of $\mathcal{O}_X$-modules defined on every open subset $U = \text{Spec}(\mathcal{A}) \subseteq X$ by
\[ \mathcal{N}(U) \to \text{Hom}_{\mathcal{O}_X}(T(\mathcal{A})|_U, T(\mathcal{O}_X/\mathcal{F})|_U) : s \mapsto e^{U}_r \]
and let
\[ \varphi_r : T(\mathcal{A}) \to \text{Hom}_{\mathcal{O}_X}(\mathcal{N}, T(\mathcal{O}_X/\mathcal{F})) \]
be the adjoint morphism of $\tilde{\varphi}_r$. Let us consider the composition
\[ \begin{array}{ccc}
\text{Hom}_{\mathcal{O}_X}(\mathcal{N}, T(\mathcal{O}_X/\mathcal{F})) & \xrightarrow{(\phi^{\mathcal{A}})^{-1}} & \text{Hom}_{\mathcal{O}_X}(\mathcal{A}, T(\mathcal{O}_X/\mathcal{F})) \\
\varphi_r \downarrow & & \downarrow \\
T(\mathcal{A}) & \xrightarrow{\text{lim}_{\mathcal{F} \in \mathcal{L}_Y}} & \text{Hom}_{\mathcal{O}_X}(\mathcal{A}, T(\mathcal{O}_X/\mathcal{F})) \\
\phi_{\mathcal{F}} \downarrow & & \\
T(\mathcal{A}) & \xrightarrow{\Phi_r} & \text{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{E})
\end{array} \]
where \( j: \mathcal{N} \rightarrow \mathcal{M} \) denotes the inclusion. It is easy to see that the homomorphism \( \Phi_\mathcal{M} \) thus constructed does not depend on the \( \mathcal{O}_X \)-module \( \mathcal{N} \) chosen in the family \( \mathcal{I}_x \) and that \( \varphi: T \rightarrow \Hom_{\mathcal{O}_X}(\mathcal{N}, \mathcal{E}) \) is a natural transformation.

(2.7) Lemma. Let \((X, \mathcal{O}_X)\) be a quasiregular locally \( Y \)-Noetherian scheme. If \( T: \mathcal{Q}_{X_1}^{\mathcal{O}_X}(X, \mathcal{O}_X) \rightarrow \mathcal{Q}_{X_2}(X, \mathcal{O}_X) \) is a local left exact contravariant functor that maps \( Y \)-isomorphisms to isomorphisms, then \( T(\mathcal{M}) \in \mathcal{Q}_{X_2}(X, \mathcal{O}_X) \), for every \( \mathcal{M} \in \mathcal{Q}_{X_1}(X, \mathcal{O}_X) \).

Proof. If \( \mathcal{M} \in \mathcal{Q}_{X_1}(X, \mathcal{O}_X) \), then there exists an \( \mathcal{O}_X \)-submodule of finite type \( \mathcal{N} \subseteq \mathcal{M} \), such that \( \mathcal{M}/\mathcal{N} \) is \( Y \)-torsion. Moreover, there exists \( \mathcal{F} \in \mathcal{I}_Y \) such that \( \mathcal{N} = 0 \). We thus have a morphism

\[
\varphi_\mathcal{M}: T(\mathcal{N}) \rightarrow \Hom_{\mathcal{O}_X}(\mathcal{M}, T(\mathcal{O}_X/\mathcal{F})).
\]

Let us verify that \( \varphi_\mathcal{M}_U \) is a monomorphism for every affine open subset \( i: U \hookrightarrow X \). Consider an epimorphism \( (\mathcal{O}_X/\mathcal{F})^n \rightarrow \mathcal{M}_U \) in \( \mathcal{O}_X |_U \)-mod. There exists an epimorphism \( g: \mathcal{G} \rightarrow \mathcal{N} \) in \( \mathcal{O}_X \)-mod with \( \mathcal{G} \in \mathcal{Q}_{X_1}(X, \mathcal{O}_X) \), such that \( \mathcal{G}_U \equiv (\mathcal{O}_X/\mathcal{F})^n \), fitting into a commutative diagram

\[
\begin{array}{ccc}
\mathcal{G} & \rightarrow & \mathcal{N} \\
\downarrow & & \downarrow \\
i_* (\mathcal{G}_U) & \rightarrow & i_* (\mathcal{N}_U)
\end{array}
\]

Applying \( T \) yields

\[
\begin{array}{ccc}
T(\mathcal{N}) & \rightarrow & T(\mathcal{G}) \\
\varphi_\mathcal{N} & \downarrow \varphi_\mathcal{G} \\
\Hom_{\mathcal{O}_X}(\mathcal{M}, T(\mathcal{O}_X/\mathcal{F})) & \rightarrow & \Hom_{\mathcal{O}_X}(\mathcal{G}, T(\mathcal{O}_X/\mathcal{F}))
\end{array}
\]

\( \varphi_\mathcal{M}_U \) is an isomorphism in \( \mathcal{O}_X |_U \)-modules, since it is the composition

\[
T(\mathcal{G}_U) \equiv T((\mathcal{O}_X/\mathcal{F})^n)_U \equiv \Hom_{\mathcal{O}_X}(\mathcal{G}_U, T(\mathcal{F}_U))
\]

so \( \varphi_\mathcal{M}_U \) is a monomorphism.

Therefore \( T(\mathcal{N}) \) is an \( \mathcal{O}_X \)-submodule of \( \Hom_{\mathcal{O}_X}(\mathcal{M}, T(\mathcal{O}_X/\mathcal{F})) \), so \( T(\mathcal{N}) \) is \( Y \)-torsion. As \( T \) maps \( Y \)-isomorphisms to isomorphisms, it follows that \( T(\mathcal{M}) \equiv T(\mathcal{N}) \) and that it is \( Y \)-torsion.

Let us assume that \( T \) is left exact and that \( T \) maps \( Y \)-isomorphisms to isomorphisms. Let \( i_A: U = \text{Spec}(A) \hookrightarrow X \) be an open affine subset of \( X \) and let \( \kappa_A = \kappa_U \cap Y \) and \( \kappa'_A = \kappa_U \cap Y' \) be the idempotent kernel functors.
defined by the generically stable subsets $Y \cap U$ resp. $Y' \cap U$. If $N$ is a finitely generated $\kappa'_A$-torsion $A$-module, then the $\mathcal{O}_X$-module $i_{A_*}(\tilde{N})$ is $Y'$-torsion. Let $\mathcal{F} \subseteq i_{A_*}(\tilde{N})$ be a quasicoherent $\mathcal{O}_X$-module of finite type, such that $\mathcal{F}|_U = \tilde{N}$. Put $T_\mathcal{F}(N) = T(\mathcal{F}|U)$. Then $T_\mathcal{F}(N)$ does not depend upon the chosen sheaf as $T$ is local.

Given a homomorphism $f: N \to M$ of $A$-modules, let us consider the homomorphism of $\mathcal{O}_X$-modules

$$i_{A_*}(\tilde{f}): i_{A_*}(\tilde{N}) \to i_{A_*}(\tilde{M}).$$

Since $\mathcal{F} \subseteq i_{A_*}(\tilde{N})$ is quasicoherent and of finite type, the image of the composition

$$\mathcal{F} \to i_{A_*}(\tilde{N}) \xrightarrow{i_{A_*}(\tilde{f})} i_{A_*}(\tilde{M})$$

is quasicoherent and of finite type, as well. Hence, there exists some quasicoherent $\mathcal{O}_X$-module of finite type $\mathcal{G}$ such that $\mathcal{G}|_U = \tilde{M}$ and so $i_{A_*}(\tilde{f})$ factorizes as

$$\xymatrix{ \mathcal{F} \ar[d]_{i_{A_*}(\tilde{f})} \ar[r]^f & \mathcal{G} \ar[d]_{i_{A_*}(\tilde{M})} \\
 i_{A_*}(\tilde{N}) \ar[r]_{i_{A_*}(\tilde{f})} & i_{A_*}(\tilde{M})}$$

Let us put $T_\mathcal{F}(f) = T(\tilde{f}|U)$. As $T$ maps $Y$-isomorphisms to isomorphisms, $T_\mathcal{F}$ may be defined on $\kappa_A$-finitely generated and $\kappa'_A$-torsion $A$-modules. As in [2], let us denote by $T^{\kappa_A-\text{f.t.}}_\mathcal{F}$ the full subcategory of $\mathcal{A}$-mod which consists of all $\kappa_A$-finitely generated and $\kappa'_A$-torsion $A$-modules. Let $M \in T^{\kappa_A-\text{f.t.}}_\mathcal{F}$ and let $N$ be a finitely generated submodule of $M$ such that $M/N$ is $\kappa'_A$-torsion, and let us define $T_\mathcal{F}(M) = T_\mathcal{F}(N)$. It is then obvious that

$$T_\mathcal{F}: T^{\kappa_A-\text{f.t.}}_\mathcal{F} \to (\mathcal{A}, \kappa_A)\text{-mod}$$

is a left exact contravariant functor that maps $\kappa_A$-isomorphisms to isomorphisms in $(\mathcal{A}, \kappa_A)\text{-mod}$.

(2.8) LEMMA. Let $(X, \mathcal{O}_X)$ be a quasiseparated locally $Y$-noetherian scheme. If $T: \mathcal{Q}^{\kappa_A-\text{f.t.}}_Y(X, \mathcal{O}_X) \to \mathcal{Q}^{\kappa_A}_Y(X, \mathcal{O}_X)$ is a local left exact contravariant functor that maps $Y$-isomorphisms to isomorphisms, then for every affine open subset $U = \text{Spec}(A) \subseteq X$

$$\mathcal{E}(U) = \lim_{\text{f.t.}} T_\mathcal{F}(A/I).$$
Proof. We have

$$\mathcal{E}|_U = \left( \lim_{\mathcal{F} \in \mathcal{U}(Y)} T(\mathcal{O}_X/\mathcal{F}) \right)|_U = \lim_{\mathcal{F} \in \mathcal{U}(Y)} (T(\mathcal{O}_X/\mathcal{F})|_U).$$

So,

$$\mathcal{E}(U) = \Gamma(U, \mathcal{E}|_U) = \Gamma\left(U, \lim_{\mathcal{F} \in \mathcal{U}(Y)} (T(\mathcal{O}_X/\mathcal{F})|_U)\right)$$

$$= \lim_{\mathcal{F} \in \mathcal{U}(Y)} \Gamma(U, T(\mathcal{O}_X/\mathcal{F})|_U)$$

$$= \lim_{\mathcal{F} \in \mathcal{U}(Y)} T(\mathcal{O}_X/\mathcal{F})(U) = \lim_{I \in \mathcal{I}(Y)} T_A(A/I)$$

because $X$ is a quasiseparated scheme and the sheaves that appear are quasicoherent.

When $U = \text{Spec}(A)$ is an affine open subset, we will just write $E_A$ for $\mathcal{E}(U)$.

(2.9) Theorem. Let $(X, \mathcal{O}_X)$ be a quasiseparated locally $Y$-noetherian scheme. If $T: \mathcal{Q}_{Y/1}(X, \mathcal{O}_X) \to \mathcal{Q}_{X/Z}(X, \mathcal{O}_X)$ is a local left exact contravariant functor that maps $Y$-isomorphisms to isomorphisms, then the natural transformation $\Phi: T \Rightarrow \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{E})$ is a natural isomorphism.

Proof. Let $\mathcal{M} \in \mathcal{Q}_{Y/1}(X, \mathcal{O}_X)$, and let us consider the morphism

$$\Phi_\mathcal{M}: T(\mathcal{M}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{E}).$$

For every affine open subset $U = \text{Spec}(A) \subseteq X$ we have

$$\Phi_\mathcal{M}(U): T(\mathcal{M})(U) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{E})(U).$$

Let $M$ be a $\mathcal{O}_X$-torsion $A$-module of $\kappa_A$-finite type such that $\mathcal{M}|_U = \mathcal{M}$. Then the previous morphism reduces to

$$\phi: T_A(M) \to \text{Hom}_A(M, E_A)$$

and this is an isomorphism by [2, (2.3), p. 163].

(2.10) Theorem. Let $(X, \mathcal{O}_X)$ be a quasiseparated locally $Y$-noetherian scheme. There exists an equivalence $\eta$, with inverse $\nu$, between the category

$$\mathcal{Q}_{Y/1}(X, \mathcal{O}_X), \mathcal{Q}_{X/Z}(X, \mathcal{O}_X)$$
of local left exact contravariant additive functors that map \( Y \)-isomorphisms to isomorphisms and the category \( \mathcal{Q}_{X/Z}(X, \mathcal{O}_X) \) of quasicoherent \( Y \)-closed \( Y' \)-torsion sheaves, given by

\[
\eta(T) = \lim_{\mathcal{F} \in \mathcal{U}(Y')} T(\mathcal{O}_X/\mathcal{F})
\]

and

\[
\nu(\mathcal{N}) = \mathcal{F} \text{om}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{N}).
\]

**Proof.** As we have already seen, \( T(\mathcal{O}_X/\mathcal{F}) \) is \( Y' \)-torsion, therefore \( \eta(T) \in \mathcal{Q}_{X/Z}(X, \mathcal{O}_X) \). It is also clear that \( \nu(\mathcal{N}) \) is a local left exact contravariant functor. We thus have that

\[
\nu\eta(T) = \nu\left( \lim_{\mathcal{F} \in \mathcal{U}(Y')} T(\mathcal{O}_X/\mathcal{F}) \right) = \mathcal{F} \text{om}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{N}) = \mathcal{F} \text{om}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{N}) = T
\]

and

\[
\eta\nu(\mathcal{N}) = \eta(\mathcal{F} \text{om}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{N})) = \lim_{\mathcal{F} \in \mathcal{U}(Y')} \mathcal{F} \text{om}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{F}, \mathcal{N}) = \Gamma'Y(\mathcal{N}) = \mathcal{N}.
\]

3. DUALITY

Let \( (X, \mathcal{O}_X) \) be a scheme and let \( Y' \subseteq Y \) be generically stable subsets such that \( Z' \setminus Z \) only contains maximal elements of \( Z' \).

Let \( x \in Z' \setminus Z \). If \( (X, \mathcal{O}_X) \) is a locally \( Y \)-noetherian scheme, then the generic closure of \( x \) is given by \( X_{(x)} = \{x\} \subseteq Y \), so \( (X, \mathcal{O}_X) \) is a locally \( X_{(x)} \)-noetherian scheme. Moreover, \( X_{(x)} \) is a locally noetherian scheme. Indeed, if \( Y = \text{Spec}(A) \) is an affine open neighbourhood of \( x \) with \( \mathcal{O}_X|_Y = \hat{A} \), then \( \mathcal{O}_X|_{X_{(x)} = A_p} \), identifying \( x \) with the prime ideal \( p \) of \( A \).

Since \( A \) is \( \kappa_{X_{(x)} \cap Y} \)-noetherian and \( \kappa_{X_{(x)} \cap Y} = \kappa_{A_p} \), it thus follows that \( A_p \) is a noetherian ring. Since \( \kappa_{A_p} \) is perfect, it follows that \( X_{(x)} \) is a locally noetherian scheme; actually \( X_{(x)} \cong \text{Spec}(\mathcal{O}_{X_{(x)}}) \).

As the scheme morphism \( i: X_{(x)} \to X \) is quasicompact, we may extend quasicoherent sheaves from \( X_{(x)} \) to \( X \).
Consider on $X_{(x)}$ the quasicoherent sheaf $\overline{k(x)}$, where $k(x)$ is the residue field of the local ring $\mathcal{O}_{X_{(x)}}$. Then $i_*(\overline{k(x)})$ is a quasicoherent sheaf on $X$.

As $\text{Supp}(i_*(\overline{k(x)})) = \{x\}^X$ (cf. [3, E.G.A. I (2.5.8)]), it clearly follows that for $U$ open in $X$, we have [6, p. 123]

$$\Gamma(U, i_*(\overline{k(x)})) = \begin{cases} k(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

(3.1) LEMMA. Let $(X, \mathcal{O}_X)$ be a scheme and let $Y' \subseteq Y$ be generically stable subsets such that $Z' \setminus Z$ only contains elements which are maximal in $Z'$ and that $X$ is locally $Y$-noetherian. If $x \in Z' \setminus Z$, then $i_*(\overline{k(x)})$ is $Y'$-torsion and of $Y$-finite type.

Proof. (1) Since $Z'$ is closed by specialization and $x \in Z'$, we have:

$$\text{Supp}(i_*(\overline{k(x)})) = \{x\}^X \subseteq Z'.$$

(2) Let $i: X_{(x)} \hookrightarrow X$ be the scheme morphism and the associated morphism $u: \mathcal{O}_X \rightarrow i_*(\overline{k(x)})$. If $\mathcal{I} = \text{Ker}(u)$, then we obtain the following exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X \xrightarrow{u} i_*(\overline{k(x)}) \xrightarrow{v} \mathcal{O}_{X/\mathcal{I}}.$$

To see that $i_*(\overline{k(x)})$ is of $Y$-finite type, let us consider the short exact sequence

$$0 \rightarrow \mathcal{O}_{X/\mathcal{I}} \xrightarrow{\phi} i_*(\overline{k(x)}) \rightarrow \text{Coker}(v) \rightarrow 0.$$

We have to verify that $\text{Supp}(\text{Coker}(v)) \subseteq Z$.

First, it is clear that $x \notin \text{Supp}(\text{Coker}(v))$, and that

$$\text{Supp}(\text{Coker}(v)) \subseteq \text{Supp}(i_*(\overline{k(x)})) = \{x\}^X.$$

So, to conclude we only have to take into account that $x \in Y$ is maximal with respect to specializations in $Y$; hence $\{x\}^X \subseteq Z \cup \{x\}$.

It thus follows that $\text{Supp}(\text{Coker}(v)) \subseteq \{x\}^X \setminus \{x\} \subseteq Z$. 

(3.2) THEOREM. Let $(X, \mathcal{O}_X)$ be a scheme and let $Y' \subseteq Y$ be generically stable subsets such that $Z' \setminus Z$ only contains maximal elements $Z'$ and that $X$
is locally $Y$-noetherian. Let $T \in \mathcal{O}_{Y^{-1}}(X, \mathfrak{O}_X), \mathcal{Q}_{X/Z}(X, \mathfrak{O}_X)$. Then the following assertions are equivalent:

1. For every $\mathcal{F} \in \mathcal{Q}_{Y^{-1}}(X, \mathfrak{O}_X)$ there is a natural isomorphism $\text{Cl}_{X/Z}(\mathcal{F}) \cong TT(\mathcal{F})$;

2. $T$ is exact and for every $x$ belonging to $Z' \setminus Z$ there exists an isomorphism $T(i_*(\overline{k(x)})) \cong i_*(\overline{k(x)})$.

Proof. (1) $\Rightarrow$ (2) Let us show that $T$ is right exact. Consider a monomorphism $i : \mathcal{F}' \to \mathcal{F}$ in $\mathcal{Q}_{Y^{-1}}(X, \mathfrak{O}_X)$. Then there exists an exact sequence of sheaves $T(\mathcal{F}) \to T(\mathcal{F}') \to \mathcal{E} \to 0$. Applying $T$ a second time yields an exact sequence

$$0 \to T(\mathcal{E}) \to TT(\mathcal{F}') \cong \text{Cl}_{X/Z}(\mathcal{F}') \xrightarrow{\text{Cl}_{X/Z}(i)} TT(\mathcal{F}) \cong \text{Cl}_{X/Z}(\mathcal{F}).$$

As $\text{Cl}_{X/Z}(i)$ is a monomorphism, we obtain that $T(\mathcal{E}) = 0$ and so $\text{Cl}_{X/Z}(\mathcal{E}) \equiv TT(\mathcal{E}) = 0$, from which it follows that $\mathcal{E}$ is $Y$-torsion. So,

$$T(\mathcal{F}) \to T(\mathcal{F}') \to 0$$

is an exact sequence in $\mathcal{Q}_{X/Z}(X, \mathfrak{O}_X)$.

To prove that $T(i_*(\overline{k(x)})) \cong i_*(\overline{k(x)})$ for every $x \in Z' \setminus Z$, let us consider an affine open subset $U \subseteq X$. If $x \not\in U$, then

$$T(i_*(\overline{k(x)}))(U) = 0 = i_*(\overline{k(x)})(U).$$

If $x \in U$, let us suppose that $U = \text{Spec}(A)$, and let us denote by $i_A : X(x) \to U$ the inclusion. Then it follows that

$$i_*(\overline{k(x)})(U) = i_*(\overline{k(x)})(U) = Q_{\kappa, \mathfrak{p}}(A/\mathfrak{p});$$

hence

$$T(i_*(\overline{k(x)}))(U) = T_A(Q_{\kappa, \mathfrak{p}}(A/\mathfrak{p})) \cong T_A(A/\mathfrak{p})$$

$$\cong Q_{\kappa, \mathfrak{p}}(A/\mathfrak{p}) = i_*(\overline{k(x)})(U) = i_*(\overline{k(x)})(U)$$

by [1, (1.4)]

(2) $\Rightarrow$ (1) For every $\mathcal{F} \in \mathcal{Q}_{Y^{-1}}(X, \mathfrak{O}_X)$ and every affine open subset $U = \text{Spec}(A)$, if $\mathcal{F} = M$, then it is clear that $TT(\mathcal{F}(U)) = T_A(T_A(M))$ with $M \in T_A^{\mathfrak{p}}$. Moreover $\text{Cl}_{X/Z}(\mathcal{F}(U)) = Q_{\kappa, \mathfrak{p}}(M)$. So, from the conditions in (2) one deduces that $T_A$ is exact and for each $p \in \mathcal{K}(\kappa'_{A}) \setminus \mathcal{K}(\kappa'_{A})$, one has $T_A(A/\mathfrak{p}) \cong Q_{\kappa, \mathfrak{p}}(A/\mathfrak{p})$. So we get $TT(\mathcal{F}) \equiv \text{Cl}_{X/Z}(\mathcal{F})$, indeed. $\blacksquare$
(3.3) Definition. A functor in \( \langle Q_{X^{Y},1}(X, \mathcal{O}_X), Q_{Y/X}(X, \mathcal{O}_X) \rangle \) is \((Y, Y')\)-dualizing if it satisfies the equivalent conditions of the previous theorem.

We say that a sheaf \( \mathcal{E} \in Q_{X/Z}(X, \mathcal{O}_X) \) is \((Y, Y')\)-dualizing if the associated functor \( \mathcal{F} \) is \((Y, Y')\)-dualizing.

To conclude, let us give a characterization of \((Y, Y')\)-dualizing sheaves in the particular case of a locally noetherian scheme.

(3.4) Theorem. Let \((X, \mathcal{O}_X)\) be a scheme, and \(Y' \subseteq Y\) generically stable subsets such that \(Z' \setminus Z\) contains only maximal elements of \(Z'\) and that \(X\) is locally noetherian. If \(D\) is a Y-closed quasicoherent sheaf \(Y'\)-torsion, then the following statements are equivalent:

1. \(D\) is \((Y, Y')\)-dualizing,
2. \(D \cong \bigoplus \{ \mathcal{E}(i_!(k(x))); x \in Z' \setminus Z \}\),

where \(\mathcal{E}(i_!(k(x)))\) represents the injective hull of \(i_!(k(x))\).

Proof. \((1) \Rightarrow (2)\). Let \(\mathcal{F}\) be the injective hull of \(D\). As \(D\) is torsionfree, so is \(\mathcal{F}\). The irreducible components of \(\mathcal{F}\) are \(i_!(E(x))\), where \(E(x)\) is the injective hull of \(k(x)\) in \(\mathcal{O}_{X,x}\), with \(x \in Y\). Since \(D\) is \(Y'\)-torsion, none of these elements \(x\) belongs to \(Y'\), i.e., \(x \in Z' \setminus Z\) and we have an isomorphism

\[ \mathcal{F} \cong \bigoplus \{ i_!(k(x)) \}^{n_x}, \]

where \(n_x \geq 0\).

If \(x \in Z' \setminus Z\), then \(\mathcal{F}_x = E(\mathcal{O}_x)\), the injective hull of \(D\) in \(\mathcal{O}_{X,x}\)-mod. Then \(\mathcal{D}_x\) is a dualizing \(\mathcal{O}_{X,x}\)-module and so

\[ \mathcal{F}_x = \mathcal{D}_x = E(k(x)), \]

where \(n_x = 1\) and

\[ \mathcal{F} = \bigoplus \{ \mathcal{E}(i_!(k(x))); x \in Z' \setminus Z \}. \]

Let us consider the canonical monomorphism \(j: D \to \mathcal{F}\). For every \(x \in Z' \setminus Z\) we have that \(\mathcal{F}_x = \mathcal{D}_x = E(k(x))\) is indecomposable injective; thus \(j_x: \mathcal{D}_x \to \mathcal{F}_x\) which is an isomorphism. To conclude, since \(D\) and \(\mathcal{F}\) are \(Y\)-closed, \(j\) is isomorphism.

\((2) \Rightarrow (1)\). Let \(T = \mathcal{F} \text{Hom}_{\mathcal{O}_X}(-, D)\). Then \(T\) is an exact contravariant local functor that maps \(Y\)-isomorphisms to isomorphisms. We will see that for every \(x \in Z' \setminus Z\) there exists an isomorphism

\[ T(i_!(k(x))) \cong i_!(k(x)). \]
Let $U = \text{Spec}(A)$ an affine open subset. Then
\[
T(i_* \overline{(k(x))})(U) = T_A(Q_{x, A}(A/\mathfrak{p})) = Q_{x, A}(A/\mathfrak{p})
\]
\[
= (i_A)_*(\overline{k(x)}) = i_*(\overline{k(x)})(U)
\]
when we identify $x$ with $\mathfrak{p}$.

(3.5) EXAMPLE. A quasicompact integral scheme $(X, S_X)$ is a Krull scheme if it is $X^{(1)}$-closed, $X^{(1)}$-noetherian, and $X^{(1)}$-regular, where $X^{(1)}$ is the set of points in $X$ such that the Krull dimension of $S_{x, X}$ is less or equal than one.

It is clear that a Krull scheme satisfies the conditions imposed in the foregoing theorems, when we take $Y = X^{(1)}$; cf. [9, 10].

REFERENCES