SUB- AND SUPER-SOLUTIONS IN BIFURCATION PROBLEMS

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1. INTRODUCTION

In this paper we point out the relationship between sub- and super-solutions and branches of solutions of certain elliptic problems of the type

\[ -\Delta u(x) = f(\lambda, x, u(x)), \quad \text{in } \Omega, \]
\[ u(x) = 0, \quad \text{on } \partial \Omega, \]

where \( \Omega \) is a regular, bounded domain in \( \mathbb{R}^N \), and \( f \) is locally Lipschitz.

It is well-known that there must always exist a solution of such a problem between a sub-solution \( \underline{u} \) and a super-solution \( \bar{u} \) such that \( \underline{u}(x) \leq u(x) \leq \bar{u}(x) \) for all \( x \in \Omega \) (see [1]). Here we give conditions for a branch of solutions of \( (P) \) to lie pointwise below a suitable branch of super-solutions, or above a branch of sub-solutions (see theorem 2.2). This simple result gives us a very useful tool to study the behaviour of certain branches of solutions of some bifurcation problems.

In particular, in Section 3 we use our result to study the asymptotical behaviour of a branch of positive solutions for a problem studied in [2], where function \( f \) is of the type

\[ f(\lambda, x, u) = \lambda u - g(x)u^\alpha, \quad \text{with } \alpha > 1, \text{ and } g(x) \geq 0, \text{ g vanishes at } \Omega_0 \subset \Omega. \]

In Section 4 we apply our result to obtain an a priori bound for the solutions of some approximating problems of one elliptic equation stated on all of \( \mathbb{R}^N \). This estimate allows us to infer, by an approximating procedure as in [3], the existence of a branch of positive solutions for the problem in \( \mathbb{R}^N \).

2. A PRIORI BOUNDS GIVEN BY SUB- AND SUPER-SOLUTIONS

Consider the problem

\[ -\Delta u(x) = f(\lambda, x, u(x)), \quad \text{in } \Omega, \]
\[ u(x) = 0, \quad \text{on } \partial \Omega, \]

where \( \Omega \) is a regular, bounded domain, and \( f \) is locally Lipschitz.

The unknown in problem \( (P) \) is the pair \( (\lambda, u) \in \mathbb{R} \times C_0^2(\bar{\Omega}) \), where \( C_0^2(\bar{\Omega}) \) is the space of twice differentiable functions, vanishing at the boundary of the domain \( \Omega \). \( (P)_\lambda \) will denote problem \( (P) \), where \( \lambda \) is fixed.

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From now on, we denote by $\mathcal{P}$ the closed cone of positive functions in $C^1_0(\bar{\Omega})$:
\[ \mathcal{P} = \{ u \in C^1_0(\bar{\Omega}) : u(x) \geq 0 \ \forall \ x \in \bar{\Omega} \}. \]

It is known that the interior of the cone $\mathcal{P}$ is given by
\[ \mathring{\mathcal{P}} = \left\{ u \in C^1_0(\bar{\Omega}) : u(x) > 0 \ \forall \ x \in \Omega, \ \frac{\partial u}{\partial n_e}(x) < 0 \ \forall \ x \in \partial \Omega \right\}, \]
where $n_e$ is the outer-normal vector at $x \in \partial \Omega$.

**Lemma 2.1.** Assume that $f$ is locally Lipschitz, and let $\bar{u} \in C^2_0(\bar{\Omega})$ be a super-solution of $(P)_\lambda$. Suppose that $u \in C^2_0(\bar{\Omega})$ is a solution of $(P)_\lambda$, $u \neq \bar{u}$. Then, $\bar{u} - u \notin \partial \mathcal{P}$.

**Proof.** Suppose, by contradiction, that $v := \bar{u} - u \in \partial \mathcal{P}$. In particular, $\bar{u}(x) \geq u(x), \ \forall \ x \in \bar{\Omega}$. Let $L$ be the Lipschitz constant for $f$ in $[|\lambda|] \times \bar{\Omega} \times [\min_\Omega u, \max_\Omega \bar{u}]$. Then, for all $x \in \bar{\Omega}$,
\[ -\Delta v(x) + Lv(x) \geq f(\lambda, x, \bar{u}(x)) - f(\lambda, x, u(x)) + L(\bar{u}(x) - u(x)) \geq 0. \]
By the strong maximum principle, as we know that $v \neq 0$, we obtain that $v \in \mathring{\mathcal{P}}$, a contradiction. ■

**Theorem 2.2.** Assume that $f$ is locally Lipschitz. Suppose that $I \subset \mathbb{R}$ is an interval, and let $\Sigma \subset I \times C^2_0(\bar{\Omega})$ be a connected set of solutions of $(P)$. Consider a continuous map $\bar{U} : I \rightarrow C^2_0(\bar{\Omega})$ such that $\bar{U}(\lambda)$ is a super-solution of $(P)_\lambda$ for every $\lambda \in I$, but not a solution. If $u_0 \leq (\neq) \bar{U}(\lambda_0)$ in $\bar{\Omega}$, for some $(\lambda_0, u_0) \in \Sigma$, then $u < \bar{U}(\lambda)$ in $\Omega$, for all $(\lambda, u) \in \Sigma$.

**Proof.** Consider the continuous map $T : I \times C^1_0(\bar{\Omega}) \rightarrow C^1_0(\bar{\Omega})$ given by $T(\lambda, u) = \bar{U}(\lambda) - u$. Since $\bar{U}$ is continuous, $T$ is a continuous operator. Then it follows that $T(\Sigma)$ is a connected set. By lemma 2.1, we conclude $T(\Sigma) \cap \partial \mathcal{P} = \emptyset$. In particular, $T(\Sigma)$ lies either completely in $\mathcal{P}$ or completely out of $\mathcal{P}$. Since $T(\lambda_0, u_0) \in \mathcal{P}$, we conclude that $T(\Sigma) \subset \mathcal{P}$, i.e. $u < \bar{U}(\lambda)$ in $\Omega$, for all $(\lambda, u) \in \Sigma$. ■

**Remarks.**
- In the same way, if $U : I \rightarrow C^2_0(\bar{\Omega})$ is a continuous map of sub-solutions, but not solutions, such that $u_0 \geq (\neq) U(\lambda_0)$ in $\Omega$ for some $(\lambda_0, u_0) \in \Sigma$, then $u > U(\lambda)$ in $\Omega$, for all $(\lambda, u) \in \Sigma$.
- Observe that here we have only considered sub- and super-solutions lying in $C^2_0(\bar{\Omega})$. This is not very restrictive, if we take into account that, for a given sub- or super-solution in $C^2(\bar{\Omega})$, the first step of the classical construction of sequences for the method of sub- and supersolutions (see [11]) provides a new sub- (or super-) solution, this time in $C^2_0(\bar{\Omega})$. Moreover, in many cases, the sub- or super-solutions will be given by solutions of a suitable different problem in $C^2_0(\bar{\Omega})$.

3. FIRST APPLICATION

Consider the problem
\[ -\Delta u(x) = \lambda u(x) - g(x)h(u(x)), \quad \text{in} \ \Omega, \]
\[ u(x) = 0, \quad \text{on} \ \partial \Omega, \quad (3.1) \]
where $\Omega$ is a bounded, regular domain, and $g: \bar{\Omega} \to \mathbb{R}$ is a continuous function such that
\[
g(x) > 0 \quad \text{for all } x \in \Omega \setminus \bar{\Omega}_0, \quad (G)
\]
where $\Omega_0 = \text{int}\{x \in \Omega: g(x) = 0\}$. We will assume that $\Omega_0$ is a regular domain, $\Omega_0 \neq \Omega$. Also, $h: \mathbb{R} \to \mathbb{R}$ is a regular function satisfying
\[
h(s) > 0 \quad \text{for all } s > 0, \quad \lim_{s \to 0} \frac{h(s)}{s} = 0, \quad \lim_{s \to +\infty} \frac{h(s)}{s} = +\infty. \quad (H)
\]

Existence and nonexistence results for problem (3.1) have been stated previously in [2] by using variational methods. The bifurcation curve of positive solutions of problem (3.1) has been described in [4], for the particular case $h(u) = u^p$.

For every $q \in C(\bar{\Omega})$, we define $\eta_1(\Omega, q)$ as the principal eigenvalue of the eigenvalue problem
\[
-\Delta u(x) + q(x)u(x) = \eta u(x), \quad \text{in } \Omega, \\
u(x) = 0, \quad \text{on } \partial \Omega,
\]
whose associated eigenfunction $\theta_1(\Omega, q) \in C^1_0(\bar{\Omega})$ can be chosen strictly positive in $\Omega$, with $\|\theta_1(\Omega, q)\|_{L^\infty} = 1$.

It is known that
\[
\eta_1(\Omega, q) = \inf_{\phi \in C^1_0(\bar{\Omega}) \setminus \{0\}} \frac{\int_\Omega |\nabla \phi|^2 + \int_\Omega q \phi^2}{\int_\Omega \phi^2}.
\]

In the particular case of $q \equiv 0$, we recall the classical notation $\eta_1(\Omega, 0) = \lambda_1(\Omega)$, and $\theta_1(\Omega, 0) = \phi_1(\Omega)$.

One can easily obtain the following a priori estimate.

**Lemma 3.1.** Assume (G) and (H). If $(\lambda, u)$ is a solution of (3.1) with $u$ positive, then
\[
\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0).
\]

**Proof.** Consider $(\lambda, u)$, a solution of problem (3.1), and observe that
\[
\lambda_1(\Omega) = \inf_{\phi \in C^1_0(\bar{\Omega}) \setminus \{0\}} \frac{\int_\Omega |\nabla \phi|^2}{\int_\Omega \phi^2} \leq \lambda \frac{\int_\Omega u^2 - \int_\Omega g h(u) u}{\int_\Omega u^2} < \lambda.
\]

Also, calling $w$ to the extension by 0 of $\phi_1(\Omega_0)$,
\[
\lambda = \eta_1(\Omega_0, g \frac{h(u)}{u}) = \inf_{\phi \in C^1_0(\bar{\Omega}) \setminus \{0\}} \frac{\int_\Omega |\nabla \phi|^2 + \int_\Omega g(h(u)/u) \phi^2}{\int_\Omega \phi^2} \leq \frac{\int_\Omega |\nabla w|^2}{\int_\Omega w^2} = \lambda_1(\Omega_0).
\]

This completes the proof of lemma 3.1. ■
THEOREM 3.2. Assume (G) and (H). There exists an unbounded branch \( \Sigma \subset (\lambda_1(\Omega), \lambda_1(\Omega_0)) \times C_0^2(\Omega) \) of positive solutions of (3.1) bifurcating from \((\lambda_1(\Omega), 0)\), such that if \((\lambda_n, u_n) \in \Sigma\), with \(\|u_n\| \to +\infty\), then \(\lambda_n \to \lambda_1(\Omega_0)\). In particular, \(\text{Proj}_\Sigma = (\lambda_1(\Omega), \lambda_1(\Omega_0))\).

**Proof.** The existence of the unbounded branch \( \Sigma \) of positive solutions of (3.1) bifurcating from \((\lambda_1(\Omega), 0)\) follows immediately from the global bifurcation theorem of Rabinowitz [5]. It remains to prove the qualitative behaviour of the obtained branch. For this, we divide the proof in two steps.

**Step 1.** Assume that function \( h \) is strictly increasing. In this case, by the results in [6], we know that, for every \( \lambda \) fixed, problem (3.1) admits at most one positive solution. We now prove the existence of a solution of (3.1) for every \( \lambda \in (\lambda_1(\Omega), \lambda_1(\Omega_0)) \).

Fix \( \lambda_* \in (\lambda_1(\Omega), \lambda_1(\Omega_0)) \), and observe that \( u = \varepsilon \phi_1(\Omega) \) is a sub-solution of problem (3.1)\( \lambda_* \), whenever \( h(s)/s \leq (\lambda_* - \lambda_1(\Omega))/\sup_{\Omega} g \) holds for all \( s \in (0, \varepsilon] \). This can be obtained for \( \varepsilon \) small enough.

On the other hand, taking into account the continuous dependence of \( \lambda_1(\Omega) \) with respect to the domain \( \Omega \), we deduce that there exist regular domains \( \Omega_1, \Omega_2, \) such that

\[
\Omega_0 \subset \subset \Omega_1 \subset \subset \Omega_2 \subset \subset \Omega \quad \text{and} \quad \lambda_* < \lambda_1(\Omega_2).
\]

Let \( \varphi \in C^2(\Omega) \) be a strictly positive function such that

\[
\varphi(x) = \begin{cases} 
1 & \text{if } x \in \Omega \setminus \Omega_2, \\
\phi_1(\Omega_2) & \text{if } x \in \Omega_1.
\end{cases}
\]

Then the function \( \tilde{u}_K := K \varphi \) (where \( K \in \mathbb{R} \)) is a super-solution of problem (3.1)\( \lambda_* \), for all \( \lambda \leq \lambda_* \), provided that \( K \) is sufficiently big.

So, due to the uniqueness, we can conclude that the unbounded branch \( \Sigma \) of positive solutions must satisfy the hypothesis of theorem 3.2. Moreover, this branch can also be given as the graph of a continuous function \( K: (\lambda_1(\Omega), \lambda_1(\Omega_0)) \to C_0^2(\Omega) \), \( K(\lambda) = u_\lambda \), the unique solution of problem (3.1)\( \lambda \), when \( h \) is strictly increasing.

**Step 2.** If \( h \) is not strictly increasing, consider

\[
\tilde{h}(s) := \frac{s}{s + 1} \inf_{t \in [s, +\infty)} h(t).
\]

Observe that \( \tilde{h} \) is strictly increasing, and also \( \tilde{h}(s) < h(s) \), for all \( s > 0 \).

Applying the result of the previous step to problem (3.1) (changing \( h \) to \( \tilde{h} \)), we obtain a continuous map \( \tilde{K}: (\lambda_1(\Omega), \lambda_1(\Omega_0)) \to C_0^2(\Omega) \), of solutions of problem (3.1). We now construct a continuous map of super-solutions for problem (3.1), \( \tilde{U}: (\lambda_1(\Omega), \lambda_1(\Omega_0)) \to C_0^2(\Omega) \), as:

\[
\tilde{U}(\lambda) := \begin{cases} 
\tilde{K}\left(\frac{\lambda_1(\Omega) + \lambda_1(\Omega_0)}{2}\right) & \text{if } \lambda < \frac{\lambda_1(\Omega) + \lambda_1(\Omega_0)}{2}, \\
\tilde{K}(\lambda) & \text{if } \lambda \geq \frac{\lambda_1(\Omega) + \lambda_1(\Omega_0)}{2}.
\end{cases}
\]

Since we know by lemma 3.1 that our unbounded branch \( \Sigma \) of positive solutions of problem (3.1) lies in \((\lambda_1(\Omega), \lambda_1(\Omega_0)) \times C_0^2(\Omega)\), we can now apply theorem 2.2, with \( \lambda_0 = \lambda_1(\Omega) \), and
$u_0 = 0$. This proves that for every $(\lambda, u) \in \Sigma$, we have $u < U(\lambda)$ in $\Omega$. Hence, we deduce the existence of a continuous function $M: [\lambda_1(\Omega), \lambda_1(\Omega_0)] \to \mathbb{R}$, with $\lim_{\lambda \to \lambda_1(\Omega_0)} M(\lambda) = +\infty$, such that, for all $(\lambda, u) \in \Sigma$, $\|u\|_{C^1_{0,0}} \leq M(\lambda)$.

Then the branch tends asymptotically to $(\lambda_1(\Omega_0), \infty)$. This completes the proof of theorem 3.2. ■

**Remark.** This result improves the results in [2] in that we prove the existence of the bifurcating branch, and its qualitative behaviour.

### 4. Second Application

Consider the problem

$$
-\Delta u(x) = \lambda g(x)f(u(x)), \quad \text{in } \mathbb{R}^N, \\
u(x) \to 0, \quad \text{as } |x| \to \infty,
$$

where $N \geq 3$, $\lambda$ is a real positive parameter, and $f: [0, +\infty) \to \mathbb{R}$ satisfies

$$
f \text{ is a Lipschitz function, differentiable at } 0, \quad \{ \begin{align*}
f(0) &= f(1) = 0, f(s) > 0 \text{ for } s \in (0, 1), \quad f(s) = 0 \text{ for } s \geq 1. \\
F1
\end{align*}
$$

Also, $g: \mathbb{R}^N \to \mathbb{R}$ is a continuous function such that

$$
g^+ \neq 0, \quad g^+ \in L^{N/2}(\mathbb{R}^N). \quad \text{(G1)}
$$

Problem (4.1) has been previously studied in [3, 7]. The idea in [3] is to approximate the problem in $\mathbb{R}^N$ by problems of type

$$
-\Delta u(x) = \lambda g(x)f(u(x)), \quad \text{in } \Omega, \\
u(x) = 0, \quad \text{on } \partial \Omega, \quad \text{(4.1)$_\Omega$}
$$

with $\Omega = B_R$, the Euclidean ball of radius $R$, and make $R \to +\infty$. This is the reason we will look for a priori bounds of the solutions of problem (4.1)$_\Omega$, independent of $\Omega$.

From now on, we denote by $\mathcal{D}^{1,2}(\mathbb{R}^N)$ the completion of $C^\infty_0(\mathbb{R}^N)$ in the norm $\|u\|_{\mathcal{D}^{1,2}}^2 = \int_{\mathbb{R}^N} |\nabla u|^2$. It can also be characterized as

$$
\mathcal{D}^{1,2}(\mathbb{R}^N) = \{ u \in L^{2N/(N-2)}(\mathbb{R}^N); \nabla u \in L^2(\mathbb{R}^N) \},
$$

and it is continuously embedded in $L^{2N/(N-2)}(\mathbb{R}^N)$.

For every function $p \in C(\bar{\Omega})$, such that $p(x_0) > 0$ at some $x_0 \in \Omega$, we define $\lambda_1(\Omega, p)$ as the principal positive eigenvalue of the weighted eigenvalue problem

$$
-\Delta u(x) = \lambda p(x)u(x), \quad \text{in } \Omega, \\
u(x) = 0, \quad \text{on } \partial \Omega,
$$

whose associated eigenfunction $\phi_1(\Omega, p) \in C^1_0(\bar{\Omega})$ can be chosen strictly positive in $\Omega$, with $\|\phi_1(\Omega, p)\|_{L^\infty} = 1$. It is known that

$$
0 < \lambda_1(\Omega, p) = \left( \sup_{\phi \in C^1_0(\bar{\Omega}) \setminus \{0\}} \frac{\int_{\Omega} p \phi^2}{\int_{\Omega} |\nabla \phi|^2} \right)^{-1} < +\infty.
$$

Observe that $\lambda_1(\Omega) = \lambda_1(\Omega, 1)$, and $\phi_1(\Omega) = \phi_1(\Omega, 1)$. 

In the same way, if $p \in L^{N/2}(\mathbb{R}^N)$, by the result in [8], it is known that there exists a first eigenvalue $\lambda_1(p)$ of

$$-\Delta w(x) = \lambda p(x)w(x), \quad \text{in } \mathbb{R}^N,$$

$$w(x) \to 0, \quad \text{as } |x| \to \infty$$

and one has

$$\lambda_1(p) = \left( \sup_{\phi \in \mathcal{D}^{1,2}} \frac{\int_{\mathbb{R}^N} p\phi^2}{\int_{\mathbb{R}^N} |\nabla \phi|^2} \right)^{-1}.$$

It is also known that the supremum is attained at the regular eigenfunction $\phi_1(p) \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, which can be chosen strictly positive in $\mathbb{R}^N$, with $\|\phi_1(p)\|_{L^\infty} = 1$.

By the result in [9], for every bounded regular domain $\Omega \subset \mathbb{R}^N$ such that $g^+$ is nontrivial in $\Omega$, an unbounded branch $\Sigma_0$ of positive solutions of problem (4.1)\_0 bifurcates from zero at $\lambda = \lambda_1(\Omega, g)$.

For $\Lambda \geq \lambda_1(\Omega, g)$ fixed, we define $\Sigma_0^\Lambda$ as the connected component of the set

$$\Sigma_0 \cap ([0, \Lambda] \times C_0^2(\bar{\Omega}))$$

which contains the bifurcation point $(\lambda_1(\Omega, g), 0)$.

**Theorem 4.1.** Assume (F1) and (G1), and fix $\Lambda \in (0, +\infty)$. Then there exists a function $w_\Lambda \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that for every bounded domain $\Omega \subset \mathbb{R}^N$ with $g^+$ nontrivial in $\Omega$, and for every $(\lambda, u) \in \Sigma_0^\Lambda$, we have $u \leq w_\Lambda$.

**Proof.** For every $r > 0$, define $g_r: \mathbb{R}^N \to \mathbb{R}$ by

$$g_r(x) = \begin{cases} 0 & \text{if } x \in B_r, \\ g^+ & \text{if } x \in \mathbb{R}^N \setminus B_r. \end{cases}$$

Observe that function $r \to \|g_r\|_{L^{N/2}}$ is continuous with $\lim_{r \to +\infty} \|g_r\|_{L^{N/2}} = 0$. Hence, it is possible to find $R > 0$ such that

$$0 < \|g_R\|_{L^{N/2}} \leq \frac{1}{L\Lambda \alpha^2},$$

where $L$ is the Lipschitz constant for $f$, and $\alpha$ is the norm of the continuous immersion of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ into $L^{N/(N-2)}(\mathbb{R}^N)$. Consider $\lambda_1(g_R)$, and observe that

$$\frac{1}{\lambda_1(g_R)} = \frac{\int_{\mathbb{R}^N} g_R \phi_1^2(g_R)}{\int_{\mathbb{R}^N} |\nabla \phi_1(g_R)|^2} \leq \frac{\|g_R\|_{L^{N/2}} \|\phi_1(g_R)\|_{L^{2N/(N-2)}}^2}{\|\phi_1(g_R)\|_{C_0^2}^2} \leq \alpha^2 \|g_R\|_{L^{N/2}} \leq \frac{1}{L\Lambda}.$$

This is, $\lambda_1(g_R) \geq L\Lambda$.

Consider $K > 0$ sufficiently big to obtain $K\phi_1(g_R)(x) \geq 1$, for all $x \in B_R$. We now prove that $w_\Lambda := K\phi_1(g_R)$ satisfies the hypothesis of theorem 4.1.

In fact, choose a bounded regular domain $\Omega \subset \mathbb{R}^N$, such that $g^+$ is nontrivial in $\Omega$. Then, $G := \max_{x \in \bar{\Omega}} g(x) > 0$. Observe that we can assume that $\lambda_1(\Omega, g) \leq \Lambda$. Otherwise, $\Sigma_0^\Lambda$ is empty, and the result is trivial. We define $\bar{u}_\Omega$ as the unique solution of the boundary value problem

$$-\Delta \bar{u}_\Omega(x) + \Lambda GL\bar{u}_\Omega(x) = \Lambda g^+(x)f(w_\Lambda(x)) + \Lambda GLw_\Lambda(x), \quad \text{in } \Omega,$$

$$\bar{u}_\Omega(x) = 0, \quad \text{on } \partial\Omega.$$
We claim that \( \tilde{u}_\lambda \) satisfies \( 0 < \tilde{u}_\lambda(x) < w_A(x) \), for all \( x \in \Omega \), and moreover, \( \tilde{u}_\lambda \) is a supersolution of problem (4.1)\( _0 \) in \( \Omega \), for all \( \lambda \leq \Lambda \).

In fact, \(-\Delta \tilde{u}_\lambda + \Lambda GL \tilde{u}_\lambda > 0 \) in \( \Omega \), and

\[
-\Delta (w_A - \tilde{u}_\lambda) + \Lambda GL (w_A - \tilde{u}_\lambda) = \lambda_1(g_R)g_R w_A - \Lambda g^+ f(w_A).
\]

This expression vanishes in \( \Omega \cap B_R \), and out of that ball we have

\[
-\Delta (w_A - \tilde{u}_\lambda) + \Lambda GL (w_A - \tilde{u}_\lambda) \geq \Lambda g^+ w_A - \Lambda g^+ f(w_A) \geq 0.
\]

Then, \( 0 < \tilde{u}_\lambda(x) < w_A(x) \), for all \( x \in \Omega \).

Moreover,

\[
-\Delta \tilde{u}_\lambda - \Lambda g^+ f(\tilde{u}_\lambda) = \Lambda g^+ f(w_A) + \Lambda LG w_A - \Lambda LG \tilde{u}_\lambda - \Lambda g^+ f(\tilde{u}_\lambda) \\
\geq \Lambda g^+ [f(w_A) + Lw_A - f(\tilde{u}_\lambda) - L\tilde{u}_\lambda] \geq 0.
\]

That is,

\[
-\Delta \tilde{u}_\lambda(x) \geq \Lambda g^+ f(\tilde{u}_\lambda(x)) \geq \lambda g(x) f(\tilde{u}_\lambda(x)) \quad \text{in} \quad \Omega,
\]

\[
\tilde{u}_\lambda(\equiv 0) \geq 0 \quad \text{on} \quad \partial \Omega.
\]

We now define the continuous (constant) map \( \bar{U}: [0, \Lambda] \to C^2(\overline{\Omega}) \) of supersolutions (but not solutions) of problem (4.1)\( _0 \) as \( \bar{U}(\lambda) : = \tilde{u}_\lambda \), for all \( \lambda \in [0, \Lambda] \).

Our branch \( \Sigma^A_\lambda \) lies in \([0, \Lambda] \times C^0(\overline{\Omega})\). Then, we can conclude, by theorem 2.2, with \( \lambda_0 = \lambda_1(\Omega, g) \), and \( u_0 = 0 \), that, for all \( (\lambda, u) \in \Sigma^A_0 \),

\[
u \leq \bar{U}(\lambda) = \tilde{u}_\lambda \leq w_A \quad \text{in} \quad \Omega.
\]

And the proof is completed. \( \blacksquare \)

Now, with this a priori bound, we obtain the following result.

**Theorem 4.2.** Assume (F1), (G1), \( g \in L^{N/2}(\mathbb{R}^N) \) and, moreover,

\[
f \in C^{1,\alpha}(0, \varepsilon), \quad \text{for some} \quad \varepsilon > 0, \quad g \text{ is bounded.} \quad (B)
\]

Then there exists an unbounded branch \( \Sigma \subseteq \mathbb{R} \times \mathbb{D}^{1,2}(\mathbb{R}^N) \) of positive solutions of problem (4.1), bifurcating from \((\lambda_1(g)/f'(0), 0)\). Moreover, setting

\[
\text{Proj}_\Sigma : = \{ \lambda \in \mathbb{R} : \exists \, u \in \mathbb{D}^{1,2}(\mathbb{R}^N), \, (\lambda, u) \in \Sigma \},
\]

one has

\[
\left[ \frac{\lambda_1(g^+)}{L}, +\infty \right] \supset \text{Proj}_\Sigma \supset \left[ \frac{\lambda_1(g)}{f'(0)}, +\infty \right].
\]

**Proof.** The proof follows the same approximating procedure of that in [3], using the a priori bound given in theorem 4.1. \( \blacksquare \)

**Remark.** The previous theorem slightly improves the result in [3, theorem 3.1], because we do not require here that \( g^+ \in L^{2N/(N+2)}(\mathbb{R}^N) \).

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