Radial Fučík Spectrum of the Laplace Operator*

M. Arias and J. Campos

Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain
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1. Introduction

The existence of solutions of the equation

\[ Lu = g(u) + f, \]

where \( L : D(L) \subseteq H \to H \) is a linear operator defined on a function space \( H, f \in H \) and \( g : H \to H \) is the Nemisky operator associated to a continuous function \( g : \mathbb{R} \to \mathbb{R} \) with \( y^{-1}g(y) \to a(b) \) as \( y \to +\infty(-\infty) \), is often closely related to the existence of solutions of

\[ Lu = au^+ - bu^- + f, \]

where \( u^+ = \max\{u, 0\}, u^- = \max\{-u, 0\} \).

One only expects a similar behaviour of the equation for all \( f \in H \) if the homogeneous equation

\[ Lu = au^+ - bu^-, \]

has only trivial solutions (see [D1, D2]). Then, it is very interesting to study the set

\[ A_0 = \{(a, b) \in \mathbb{R}^2 \mid (3) \text{ has a nontrivial solution}\}. \]

The above set is called the Fučík spectrum of \( L \). Unfortunately one has little information on \( A_0 \) even in concrete cases. In [D3] Dancer obtains

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some properties of $A_0$ when $L$ is an unbounded self-adjoint operator on $H = L^2(\Omega)$ with compact resolvent. He proves that every connected component of $A_0$ with an element $(\lambda, \lambda)$, where $\lambda$ is an eigenvalue of $L$ is unbounded. However, he does not know if $A_0$ contains an open set or if every component of $A_0$ contains an element $(\lambda, \lambda)$.

These questions are interesting even in concrete case. If we take $L = -\Delta$, the Laplace operator on a bounded open set $\Omega$ of $\mathbb{R}^N$ and we denote by $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ the eigenvalues of $-\Delta$ on $H^1_0(\Omega)$, then $A_0$ clearly contains each $(\lambda_k, \lambda_k)$ and the two lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$. It is easy to prove that $A_0 \subset (\lambda_1, +\infty) \times (\lambda_1, +\infty) \cup \{\mathbb{R} \times \lambda_1\} \cup \{\lambda_1 \times \mathbb{R}\}$.

Recently, De Figueiredo and Gossez [F-G] have obtained a variational characterization of the first nontrivial curve in the Fučík spectrum of the Laplacian, but their proofs strongly use the positivity of the first eigenfunction and do not seem to extend easily to the other components of $A_0$.

Fučík, in [F], obtains a complete description of $A_0$ for the one-dimensional case.

In this paper, we study the radial Fučík spectrum of the Laplacian when $\Omega$ is a ball of the N-dimensional Euclidean space $\mathbb{R}^N$. Concretely we study the existence of nontrivial radial solutions, i.e., $u = u(|x|)$, where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^N$, of the problem

$$\begin{align*}
(P) \quad & \begin{cases}
\Delta u + au^- - bu^- = 0 & \text{on } B = \{x \in \mathbb{R}^N : |x| < 1\} \\
u = 0 & \text{in } \partial B = \{x \in \mathbb{R}^N : |x| = 1\}
\end{cases}
\end{align*}$$

with $N \geq 2$, and we characterize the set

$$A^R_0 = \{(a, b) \in \mathbb{R}^2 : (P) \text{ has a nontrivial radial solution}\}.$$

It is well known that $u(x) = u(|x|)$ is a radial solution of $(P)$ if and only if, the function $v(r)$ is a solution of the singular problem:

$$\begin{align*}
(SP) \quad & \begin{cases}
v'' + \frac{N-1}{r}v' + av^+ - bv^- = 0 & \text{on } (0, 1) \\
\limsup_{r \to 0^+} |v(r)| < +\infty, \\
v(1) = 0.
\end{cases}
\end{align*}$$

Thus,

$$A^R_0 = \{(a, b) \in \mathbb{R}^2 : (SP) \text{ has a nontrivial solution}\}.$$

Note that $(P)$ has nonradial eigenvalues, hence $A^R_0 \subset A_0$ with strict inclusion. In Section 2, we study the linear equation associated to $(SP)$,
(L_\alpha) v'' + \frac{N - 1}{r} v' + \alpha v = 0,

obtaining some of its properties that we use in Section 3 to describe A_0^R. We obtain A_0^R as the union of an infinite number of monotone analytic curves all of them containing a radial eigenvalue of (P) and we study their asymptotic behaviour. Concretely, the shape of A_0^R is like the curves shown in Fig. 1, where \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots denote the radial eigenvalue of (P). In Section 4 we include some numerical examples of curves of A_0^R for N = 2, N = 3, N = 4, and N = 5.

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2. The Linear Equation

Let \alpha be a positive number and consider the equation

(L_\alpha) v'' + \frac{N - 1}{r} v' + \alpha v = 0, \quad r \in (0, +\infty).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}
We state some known properties of \((L_\alpha)\) that we use in the following:

(P1) The functions \(v^\alpha(r) = r^{1-N/2} J_\eta(\sqrt{\alpha r})\), where \(J_\eta(r)\) is the Bessel functions of the first kind and index \(\eta = (N-2)/2\), is a nontrivial solution of \((L_\alpha)\) bounded at zero. Every solution bounded at zero of \((L_\alpha)\) depends linearly on \(v^\alpha\).

(P2) The set of positive zeros of each nontrivial solution of \((L_\alpha)\) is infinite and discrete.

(P3) The function \((\alpha, r) \in (0, +\infty) \times (0, +\infty) \rightarrow v^\alpha(r) \in \mathbb{R}\) is an analytic function in both variables.

The properties (P1) and (P3) are an immediate consequence of the fact that \(v\) is a solution of \((L_\alpha)\) if and only if \(v(r) = r^{1-N/2} w(r)\), where \(w\) is a solution of the Bessel equation of index \(\eta\) (see, for instance, [B-N]). (P2) follows from (P1) and the Sturm comparison theorems, S.C.T. (see, for instance, [C-L]).

For a fixed \(s > 0\), let \(v(\cdot, \alpha, s)\) denote the solution of \((L_\alpha)\) satisfying

\[(I_s) \quad v(s; \alpha, s) = 0; \quad v'(s; \alpha, s) = 1.\]

We define

\[\phi_\alpha(s) = \min\{t > s : v(t; \alpha, s) = 0\};\]

i.e., \(\phi_\alpha(s)\) is the next zero of a solution of \((L_\alpha)\) that vanishes at \(s\).

**Remarks.** 1. \((L_\alpha)\) is an homogeneous equation, then every solution of \((L_\alpha)\) that is zero at \(s\) depends linearly on \(v(\cdot, \alpha, s)\).

2. If \(v(r)\) is a solution of \((L_\alpha)\), \(w(r) = v(r/\sqrt{\alpha})\) is a solution of \((L_1)\); thus

\[\phi_\alpha(s) = \frac{1}{\sqrt{\alpha}} \phi(\sqrt{\alpha} s),\]

where \(\phi = \phi_1\).

**Proposition 1.** The function

\[(\alpha, s) \in (0, +\infty) \times (0, +\infty) \rightarrow \phi_\alpha(s) \in (0, +\infty)\]

is well defined, analytic in both variables and

\[\frac{\partial \phi_\alpha}{\partial \alpha} (s) < 0, \quad \frac{\partial \phi_\alpha}{\partial s} (s) > 0, \quad \text{for all } s > 0, \alpha > 0.\]
Proof. From (P2), $\phi_\alpha(s)$ is well defined and the Theorem of Differentiability, with respect to initial conditions (see, for example, [H]), implies that $\phi_\alpha(s)$ is analytic.

Moreover, using the Sturm comparison theorems (S.C.T.) one has that $\phi_\alpha(s)$ is strictly increasing on $s$ and strictly decreasing on $\alpha$. (Note that $(L_\alpha)$ can be written as $(r^{N-1}v')' + ar^{N-1}v = 0$). Then, $(\partial\phi_\alpha/\partial\alpha)(s) \leq 0$ and $(\partial^2\phi_\alpha/\partial\alpha^2)(s) \geq 0$, $\alpha > 0$, $s > 0$.

We are going to prove that the above inequalities are strict. The function $v(r)$ is a solution of $(L_\alpha)$ if and only if $u(r) = r^{(N-1)/2} v(r)$ is a solution of

$$(C_\alpha) \ u'' + \left( \alpha + \frac{(N-1)(3-N)}{4r^2} \right) u = 0, \quad r \in (0, +\infty)$$

and, since $v$ and $u$ have the same zeros, $\phi_\alpha(s)$ can be expressed as

$$\phi_\alpha(s) = \min\{t > s : u(t; \alpha, s) = 0\},$$

where $u(\cdot; \alpha, s)$ is the solution of $(C_\alpha)$ that verifies $u(s, \alpha, s) = 0$, $u'(s; \alpha, s) = 1$.

By integrating $(C_\alpha)$, we have

$$u(r; \alpha, s) = (r - s) - \int_s^r \int_s^\tau \left( \alpha + \frac{(N-1)(3-N)}{4\mu^2} \right) u(\mu; \alpha, s) \, d\mu \, d\tau, \quad (*)$$

and by differentiating with respect to $\alpha$,

$$\frac{\partial u}{\partial \alpha}(r; \alpha, s) = -\int_s^r \int_s^\tau u(\mu; \alpha, s) \, d\mu \, d\tau$$

$$- \int_s^r \int_s^\tau \left( \alpha + \frac{(N-1)(3-N)}{4\mu^2} \right) \frac{\partial u}{\partial \alpha}(\mu; \alpha, s) \, d\mu \, d\tau.$$ 

I.e., if we write $w(r; \alpha, s) = (\partial u/\partial \alpha)(r; \alpha, s)$, $w(\cdot; \alpha, s)$ is a solution of

$$w'' + \left( \alpha + \frac{(N-1)(3-N)}{4r^2} \right) w = -u(r; \alpha, s),$$

$$w(s) = w'(s) = 0.$$

Let $\varphi_1(r)$ be the solution of $(C_\alpha)$ with $\varphi_1(s) = 1$, $\varphi_1'(s) = 0$ and let $\varphi_2(r) = u(r; \alpha, s)$. Using the formula of variation of the constants,
\[ w(r; \alpha, s) = \varphi_1(r) \int_s^r u^2(\mu; \alpha, s) \, d\mu - u(r; \alpha, s) \int_s^r \varphi_1(\mu) u(\mu; \alpha, s) \, d\mu \]

and

\[ w(\phi_\alpha(s); \alpha, s) = \varphi_1(\phi_\alpha(s)) \int_s^{\phi_\alpha(s)} u^2(\mu; \alpha, s) \, d\mu. \]

But \( \varphi_1(\phi_\alpha(s)) < 0 \) because, as a consequence of S.C.T., \( \varphi_1 \) has only one zero in \((s, \phi_\alpha(s))\). Thus \( w(\phi_\alpha(s); \alpha, s) < 0 \).

By differentiating with respect to \( \alpha \) the identity \( u(\phi_\alpha(s); \alpha, s) = 0 \), we have

\[ u'(\phi_\alpha(s); \alpha, s) \frac{\partial \phi_\alpha}{\partial \alpha} (s) + \frac{\partial u}{\partial \alpha} (\phi_\alpha(s); \alpha, s) = 0, \]

but \((\partial u/\partial \alpha)(\phi_\alpha(s); \alpha, s) < 0\) and \( u'(\phi_\alpha(s); \alpha, s) < 0\); thus \((\partial \phi_\alpha/\partial \alpha)(s) < 0\).

Let us now denote \( x(r; \alpha, s) = (\partial u/\partial s)(r; \alpha, s) \). From (\#), \( x(r; \alpha, s) \) is the solution of \((C_\alpha)\) satisfying \( x(s; \alpha, s) = -1, x'(s; \alpha, s) = 0 \).

From the S.C.T., \( x(\cdot; \alpha, s) \) has only one zero at \((s, \phi_\alpha(s))\) and \( x(\phi_\alpha(s); \alpha, s) > 0 \).

By differentiating with respect to \( s \),

\[ u'(\phi_\alpha(s); \alpha, s) \frac{\partial \phi_\alpha}{\partial s} (s) + \frac{\partial u}{\partial s} (\phi_\alpha(s); \alpha, s) = 0 \]

and \((\partial \phi_\alpha/\partial s)(s) > 0\).

The following lemma explains the behaviour of the limits of \( \phi_\alpha(s) \).

**Lemma 2.** (a) For every \( \alpha > 0 \), \( \lim_{\alpha \to 0^+} \phi_\alpha(s) = \xi_1/\sqrt{\alpha}, \) where \( \xi_1 \) denote the first positive zero of \( J_\eta(t), \) \( \eta = (N - 2)/2 \).

(b) For every \( s > 0 \), \( \lim_{\alpha \to +\infty} \phi_\alpha(s) = s \).

**Proof.** (a) Let \( \alpha > 0 \) fixed, since \( \phi_\alpha(s) : (0, +\infty) \to \mathbb{R}^+ \) is a monotone increasing continuous function, there exists \( \lim_{s \to 0^+} \phi_\alpha(s) = L \). \( L \geq \xi_1/\sqrt{\alpha} \) is a consequence of the assertion:

Every solutions of \((L_\alpha)\) has at most one zero in \((0, \xi_1/\sqrt{\alpha})\).

In fact, let \( v \) be a solution of \((L_\alpha)\) and suppose that \( v(r_1) = v(r_2) = 0 \) in \( r_1, r_2 \in (0, \xi/\sqrt{\alpha}) \). From S.C.T., \( v^\alpha(r) \) has to vanish in \((r_1, r_2)\), but \( v^\alpha(r) \) has its first positive zero at \( \xi_1/\sqrt{\alpha} \).
We are going to prove that \( L = \xi_1/\sqrt{\alpha} \). For this, we will prove that

if \( u \) is a solution of \((L_0)\) and \( u(\xi_1/\sqrt{\alpha}) \neq 0 \),

there exists \( r_0 \in (0, \xi_1/\sqrt{\alpha}) \) such that \( u(r_0) = 0 \).

Let us suppose that there exists \( u \) solution of \((L_0)\) with \( u(r) > 0 \) on \((0, \xi_1/\sqrt{\alpha})\). Let \( r_1 \in (0, \xi_1/\sqrt{\alpha}) \) and \( \lambda > 0 \) be such that \( \lambda u(r_1) < u^a(r_1) \). Then, \( w(r) = \lambda u(r) - u^a(r) \) is a solution of \((L_0)\) with \( w(r_1) < 0 \), \( w(\xi_1/\sqrt{\alpha}) > 0 \), and there exists \( r_2 \in (r_1, \xi_1/\sqrt{\alpha}) \) such that \( w(r_2) = 0 \). But using the above assertion, \( w(r) < 0 \) on \((0, r_2)\) and

\[
0 < \lambda u(r) < u^a(r), \quad r \in (0, r_2),
\]

but this is not possible because of \((P1)\).

To prove \((b)\) it is sufficient to consider the expression

\[
\phi_a(s) = \frac{1}{\sqrt{\alpha}} \phi(\sqrt{\alpha}s)
\]

and note that, by the properties of the functions \( J_n(t) \), \( \phi(s) - s \to \pi \) as \( s \to +\infty \) (see, for instance, [A-S]).

Remark. In the above proof we have shown the following property

(P4) Every nontrivial solution of \((L_0)\) has only one zero in \((0, \xi_1/\sqrt{\alpha})\).

3. Main Results

In this section we are going to obtain a complete description of \( A^R_0 \) using the function \( \phi_a(s) \). For this description it is necessary to note that every nontrivial solution of \((SP)\) has a finite number of zeros in \((0, 1)\). This result follows from \((P2)\) and

Lemma 3. Let \( a, b \) be positive numbers and let \( u \) be a nontrivial solution of

\[
(NL) \quad u'' + \frac{N - 1}{r} u' + au^+ - bu^- = 0 \quad (0, +\infty).
\]

Then, \( u \) has at most one zero in \((0, t_0)\), where \( t_0 = \min(\xi_1/\sqrt{\alpha}, \xi_1/\sqrt{b}) \).

The proof of this lemma is trivial from \((P4)\).
Let \( a, b \in \mathbb{R}^+ \) fixed and let \( v \) be a nontrivial solution, bounded at zero of (NL). Suppose, for example, \( \nu(0^+) > 0 \). Then, \( v \) is a bounded solution of (L\(_a\)) as long as it is positive and \( \nu(r) = Cr^\alpha, r \in (0, \sqrt[\alpha]{\xi_1}), C > 0 \).

If \( \xi_1/\sqrt{a} = 1 \) then \( v \) is a solution of (SP) and \( (a, b) \in A_0^R \). If \( \xi_1/\sqrt{a} > 1 \) then \( v \) cannot be solution of (SP) and if \( \xi_1/\sqrt{a} < 1 \), since \( v'(\xi_1/\sqrt{a}) < 0, v \) is solution of (L\(_b\)) as long as it is negative, i.e., on \( (\xi_1/\sqrt{a}, \phi_b(\xi_1/\sqrt{a})) \).

If \( \phi_b(\xi_1/\sqrt{a}) = 1, (a, b) \in A_0^R \). If \( \phi_b(\xi_1/\sqrt{a}) > 1 \), \( v \) cannot be solution of (SP) and if \( \phi_b(\xi_1/\sqrt{a}) < 1, v \) is again solution of (L\(_a\)) on \( (\phi_b(\xi_1/\sqrt{a}), \phi_a(\phi_b(\xi_1/\sqrt{a})) \)).

Denoting by \( R(a, b) = \phi_a \circ \phi_b \), and repeating this process we obtain

\[
v \text{ is a nontrivial solution of (SP) if and only if there exists } n \in \mathbb{N} \cup \{0\} \text{ such that either } R^n(a, b)(\xi_1/\sqrt{a}) = 1 \text{ or } \phi_b \circ R^n(a, b)(\xi_1/\sqrt{a}) = 1, \text{ where } R^0(a, b) \text{ denotes, as usual, the identity function on } \mathbb{R}.
\]

If \( \nu(0^+) < 0 \), we obtain by a similar reasoning

\[
v \text{ is a nontrivial solution of (SP) if and only if there exists } n \in \mathbb{N} \cup \{0\} \text{ such that either } R^n(b, a)(\xi_1/\sqrt{b}) = 1 \text{ or } \phi_a \circ R^n(b, a)(\xi_1/\sqrt{b}) = 1.
\]

One has thus the following result.

**Theorem 4.** The point \( (a, b) \in A_0^R \) if, and only if, there exists \( n \in \mathbb{N} \cup \{0\} \) such that one of the following identities is satisfied:

(i) \( R^n(a, b)(\xi_1/\sqrt{a}) = 1 \),
(ii) \( \phi_b \circ R^n(a, b)(\xi_1/\sqrt{a}) = 1 \),
(iii) \( R^n(b, a)(\xi_1/\sqrt{b}) = 1 \),
(iv) \( \phi_a \circ R^n(b, a)(\xi_1/\sqrt{b}) = 1 \).

**Remark.** Note that, if \( (a, b) \in A_0^R \) and either \( a \neq \xi_1^2 \) or \( b \neq \xi_2^2 \), then \( a > \xi_1^2 \) and \( b > \xi_2^2 \). (Observe that \( \phi_a(s) > s \), for each \( s > 0, \alpha > 0 \).) In this case, \( \xi_1^2 \) is the first eigenvalue of (P).

For convenience, we will denote in the following

\[
R_i^n(a, b) = \begin{cases} 
R^n(a, b)(\xi_1/\sqrt{a}) & i = 2n, \quad j = 1, \\
R^n(b, a)(\xi_1/\sqrt{b}) & i = 2n, \quad j = 2, \\
\phi_b \circ R^n(a, b)(\xi_1/\sqrt{a}) & i = 2n + 1, j = 1, \\
\phi_a \circ R^n(b, a)(\xi_1/\sqrt{b}) & i = 2n + 1, j = 2,
\end{cases}
\]
and we will call $C_i^j = \{(a, b) \in \mathbb{R}^+ \times \mathbb{R}: R_i^j(a, b) = 1\}$, $i = 1, 2, 3, \ldots$, $j = 1$ or 2. One has the following.

**Theorem 5.** For each $i = 1, 2, 3, \ldots, j = 1$ or 2, there exists a strictly decreasing analytic homeomorphism $c_i^j : (\alpha_i^j, +\infty) \rightarrow (\beta_i^j, +\infty)$ such that $C_i^j = \{(a, c_i^j(a)) : a \in (\alpha_i^j, +\infty)\}$. Moreover, $(\lambda, \lambda) \in C_i^j$ if, and only if, $\lambda = \xi_{j+1}^i$, where $\xi_1 < \xi_2 < \cdots < \xi_k < \cdots$ denote the sequence of positive zeros of $J_n$.

**Proof.** From the implicit expression of $C_i^j$ and the properties of $\phi_a(s)$ (see Proposition 1), it is easy to see that $C_i^j$ is composed by segments of curves that are graphs of analytic functions.

Let $a > 0$ fixed. The function $b \rightarrow R_i^j(a, b)$ is strictly decreasing; thus there exists at most one $b \in \mathbb{R}^+$ such that $R_i^j(a, b) = 1$.

So, we have defined an analytic function $c_i^j$ on an open set of $\mathbb{R}$, $I_i^j$, such that $(a, b) \in C_i^j$ if, and only if, $c_i^j(a) = b$. Moreover, $(c_i^j)'(a) < 0$ for all $i = 1, 2, 3, \ldots, j = 1$ or 2 because $(\partial \phi_a / \partial \alpha)(s) < 0$, $(\partial \phi_a / \partial s)(s) > 0$.

To see that $I_i^j = (\alpha_i^j, +\infty)$, $\alpha_i^j > 0$, it is necessary to note that $C_i^j$ is a closed set. In fact, if $(a_n, b_n) \in C_i^j$, $a_n \rightarrow a_0$, $b_n \rightarrow b_0$ as $n \rightarrow +\infty$, since $a_n > \xi_1^i$ and $b_n > \xi_2^i$, $a_0 \geq \xi_2^i > 0$, $a_0 \geq \xi_1^i > 0$, and $R_i^j(a_0, b_0)$ is well defined and $R_i^j(a_0, b_0) = 1$; i.e., $(a_0, b_0) \in C_i^j$.

Suppose that there exists $a_0 \in \mathbb{R} - I_i^j$ and $\varepsilon > 0$ such that $(a_0 - \varepsilon, a_0) \in I_i^j$.

Let $\{a_n\} \rightarrow a_0$, $a_n \in (a_0 - \varepsilon/2, a_0)$, $n \in \mathbb{N}$. Since $c_i^j$ is decreasing and $c_i^j(a) > \xi_1^i$, $a \in I_i^j$, one has

$$\xi_1^i < c_i^j(a_n) < c_i^j(a_0 - \varepsilon/2)$$

and there exists a subsequence, relabeled $\{a_n\}$, such that $c_i^j(a_n) \rightarrow b$. Now, $(a_n, c_i^j(a_n)) \in C_i^j$, $n \in \mathbb{N}$ and $(a_n, c_i^j(a_n)) \rightarrow (a_0, b)$, then $(a_0, b) \in C_i^j$ and, in particular, $a_0 \in I_i^j$ which is not possible. One has then $I_i^j = (\alpha_i^j, +\infty)$.

To prove that $c_i^j(I_i^j) = (\beta_i^j, +\infty)$, it is sufficient to observe that $c_i^j$ is strictly decreasing. If we suppose that $\lim_{a \rightarrow \alpha_i^j} c_i^j(a) = b < +\infty$, following the above argument, $(\alpha_i^j, b) \in C_i^j$ and $\alpha_i^j \in I_i^j$.

We will prove now that $C_i^j \cap \{(a, a) : a \in \mathbb{R}\} = \{(\xi_{j+1}^i, \xi_{j+1}^i)\}$, $i = 1, 2, 3, \ldots, j = 1$ or 2.

Since $c_i^j$ is monotone, such a set has at most one point. On the other hand, if $a = b$, $R_i^j(a, a) = \phi_a(\xi_i^j / \sqrt{a}) = \phi_a^{j-1}(\phi(\xi_i^j / \sqrt{a}) = \cdots = (1/\sqrt{a}) \phi(\xi_i^j) = (1/\sqrt{a})\xi_{j+1}^i$, $i = 1, 2, 3, \ldots, j = 1$ or 2, as, by definition, $\phi(\xi_k^j) = \xi_{k+1}^j$, $k \in \mathbb{N}$. Then $R_i^j(a, a) = 1$ if, and only if, $a = \xi_{j+1}^i$ and $(\xi_{j+1}^i, \xi_{j+1}^i) \in C_i^j$.

On the asymptotic behaviour of the curves $C_i^j$, we obtain the following.
**Proposition 6.**

\[
\beta_i = \begin{cases} 
\xi^2_{n+1}, & i = 2n + 1, j = 1 \text{ or } 2, n = 0, 1, 2, \ldots, \\
\xi^2_n, & i = 2n, j = 1, n = 1, 2, \ldots, \\
\xi^2_{n+1}, & i = 2n, j = 2, n = 1, 2, \ldots. 
\end{cases}
\]

Moreover, \(\alpha^1_i = \beta^1_i, \alpha^2_i = \beta^1_i, \) \(i = 1, 2, 3, \ldots.\)

**Proof.** Since \((a, b) \in A^0_0\) then \(a, b \geq \xi^2_1, \alpha^i_i, \beta^i_i \geq \xi^2_1, \ i = 1, 2, 3, \ldots, j = 1 \text{ or } 2.\) The expression of \(\beta^i_i\) is obtained taking limit as \(a \to +\infty\) on \(R^i_i(a, c^i_i(a)) = 1\) and considering

(a) for each \(\{\alpha_n\} \to 0, \{\alpha_n\} \to \alpha, \phi_{\alpha_n}(s_n) \to \xi^2_1/\sqrt{a},\)

(b) for each \(\{\alpha_n\} \to s > 0, \{\alpha_n\} \to +\infty, \phi_{\alpha_n}(s_n) \to s,\)

that are consequences of Lemma 2 and the monotonicity of \(\phi_{\alpha}(s)\)

The relationship between \(\alpha^1_i\) and \(\beta^1_i\) is trivial because \((a, b) \in C^1_i\) if and only if \((b, a) \in C^1_i, \ i = 1, 2, 3, \ldots.\)

We will end this section by studying the possible crossings between the curves of \(A^0_0.\)

**Proposition 7.** Let \(a, b > 0\) and \(j_1, j_2 \in \{1, 2\}.\) If \(k < i,\) one has

\[
R^1_k(a, b) < R^2_i(a, b).
\]

**Proof.** It is enough to note that \(R^1_k(a, b) < R^2_{i+1}(a, b),\) for each \(i = 1, 2, 3, \ldots, j_1, j_2 \in \{1, 2\}\) which is a consequence from \(\phi_{\alpha}(s) > s, s > 0, \alpha > 0\) and the monotonicity of \(\phi.\)

From the above proposition one can obtain

\[
C^1_k \cap C^2_i = \emptyset, \quad k \neq i, j_1, j_2 \in \{1, 2\}.
\]

**Remarks.**

1. We are not able to say anything about \(C^1_i \cap C^2_i.\) We know that \((\xi^2_{i-1}, \xi^2_{i+1}) \in C^1_i \cap C^2_i,\) but as we will see in the following section, it is possible that if \(i\) is even that \(C^1_i = C^2_i.\)

2. Observe that in the above results we only have referred to the nontrivial curves of \(A^0_0;\) i.e., we have considered \(i \neq 0.\) The curves \(C^1_0\) and \(C^2_0\) are the known lines \(\xi^2_1 \times \mathbb{R}\) and \(\mathbb{R} \times \xi^2_1,\) respectively.
Figure 2
N=4

N=5

Fig. 2—Continued
4. Examples

In this section we present the set $A_0^R$ when $N = 2$, $N = 3$, $N = 4$, and $N = 5$ (Fig. 2). The figures have been obtained by numerical methods. However, the special form of the Bessel function of index $\frac{1}{2}$ gives in the case $N = 3$, that $\phi(s) = s + \pi$ and the curves of $A_0^R$ can be obtained explicitly.

Note that the set $A_0^R$ for $N = 3$ coincide with the Fučík spectrum of $(P)$ in the one-dimensional case and $C_{2n+1}^1 \equiv C_{2n+1}^2$, $n \in \mathbb{N}$. Observe that in $N = 5$ this is not true.

References