A topological method for geodesic connectedness of space–times: Outer Kerr space–time

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The geodesic connectedness of outer Kerr space–time is proven by using a topological method. The proof is based on arguments about Brouwer’s degree for the solutions of functional equations. The applicability of these topological arguments for dealing with geodesics in space–times is stressed. © 2002 American Institute of Physics. [DOI: 10.1063/1.1506403]

I. INTRODUCTION

For a space–time, the question whether two causally related points can be joined by means of a causal geodesic has a clear physical meaning. More geometrically, geodesic connectedness (the possibility of joining any two points by a geodesic of any causal type) is a basic property. Some techniques introduced to study geodesic connectedness are applicable to related problems of physical interest, such as the gravitational lensing effect, or the connectedness of two submanifolds by a causal geodesic (see, e.g., Refs. 1–5).

Different tools have been introduced to study geodesic connectedness of Lorentzian manifolds. The first nontrivial examples were spaceforms, which become specially relevant from the geometrical viewpoint.6 A complete positive Lorentzian spaceform is geodesically connected if and only if it is not time-orientable. In particular, de Sitter space–time $S^n$ is not geodesically connected; this happens in spite of the fact that it is globally hyperbolic and, thus, each two causally related points in $S^n$ can be joined by a causal geodesic.7,8 The results about geodesic connectedness of manifolds endowed with an affine connection9 (see also Refs. 10–12) are potentially applicable to any manifold endowed with a nondegenerate metric. Geodesics of Lorentzian tori provide interesting examples related to connectedness.13–15

But the systematic study of the geodesic connectedness of physically relevant space–times was carried out after the introduction of some variational methods in Lorentzian geometry.16 These methods permit one to prove the geodesic connectedness of stationary and splitting type manifolds under reasonable conditions (see the book—Ref. 17 or the survey—Ref. 18). Moreover, with different improvements, they ensure the connectedness of outer Schwarzschild space–time,19 intermediate Reissner–Nordström,20 Gödel type,21 and other space–times. Recently, the authors have obtained the necessary and sufficient condition for the connectedness of generalized Robertson–Walker space–times.22 Moreover, topological arguments have been developed which prove the connectedness of multiwarped space–times, under sufficient conditions which are close to necessary conditions23 (see also Refs. 24 and 25). In particular, not only space–times as Schwarzschild black hole are shown to be geodesically connected,23 but also new proofs of the geodesic connectedness of space–times such as intermediate Reissner–Nördstrom23 and outer Schwarzschild26 are obtained.

Significantly, geodesic connectedness of outer Kerr space–time has not been studied yet.27 It is not difficult to prove that the stationary part of Kerr space–time is not geodesically connected; moreover, fast rotating Kerr space–time (i.e., Kerr space–time with parameters $a^2 > m^2$) is not geodesically connected.26 On the other hand, for values of the parameter $a$ close to 0, the hyper-
surface between the stationary part and the ergosphere has some good properties from a variational viewpoint (it is time and light-convex, essentially). This ensures some properties of causal geodesics, such as the multiplicity of connecting timelike or lightlike geodesics for big time-separation (Ref. 17, Theorem 7.2.4).

Nevertheless, the inexistence of a privileged time function or timelike Killing vector field on the ergosphere introduces difficulties for its study. In fact, the following result holds.\(^{26}\) Let \(r_+\) be the outer radius which determines the first event horizon of Kerr space–time. For any \(v > 0\) the region \(r > r_+ + v\) is not geodesically connected. In this article, we show that the topological arguments introduced by the authors in Ref. 23 are applicable to study geodesic connectedness of Kerr space–time, and prove that the whole region \(r > r_+\) is geodesically connected. Even though our study is restricted to Kerr space–time for simplicity, the method would work for some different generalizations of this space–time.

This article is organized as follows.

In Sec. II Kerr space–time is briefly recalled. In Sec. III we give a general overview of our proof, which may serve as a guide for the following sections. The importance of the topological arguments first introduced in Ref. 23 (for the rather different class of multiwarped space–times) is stressed. In Sec. IV general properties of the geodesics of Kerr space–time are recalled, and those geodesics relevant under our approach are selected. Then, we reduce the problem of geodesic connectedness to an analytic problem (Lemma 4.3), among other technical results.

In Secs. V and VI geodesic connectedness is proven for the slow rotating Kerr \(a^2 < m^2\). In Sec. V the case when one of the two points to be connected lies in the symmetry axis \(z\), is solved. Otherwise, the problem is technically more complicated, and solved in Sec. VI. Some concluding remarks are given in Sec. VII.\(^{28}\)

II. KERR SPACE–TIME

Kerr space–time is the standard relativistic model of the gravitational field of a rotating massive object. The simplest description of the Kerr metric tensor is in terms of the time coordinate \(t\) on \(\mathbb{R}\) and spherical coordinates \(r, \theta, \phi\) on \(\mathbb{R}^3\) (\(\theta\) denotes colatitude and \(\phi\) longitude), which are called Boyer–Lindquist coordinates. Following Ref. 31, Sec. 1.1 we treat the coordinate \(\phi\) as circular, so the coordinate system \(\theta, \phi\) covers all of the sphere except its north and south poles \((0,0, \pm 1)\). The coordinate function \(\phi\) is undefined at the poles, but its coordinate vector field \(\partial_\phi\) is well-defined and smooth on the entire sphere and is zero at the poles. Defining \(\theta(0,0,+)\! =\! 0\) and \(\theta(0,0,1)\! =\! \pi\) extends the function \(\theta\) to the entire sphere with \(0 \leq \theta \leq \pi\). At the poles, \(\theta\) is only continuous, but \(\cos \theta\) and \(\sin \theta\) are smooth (indeed, analytic) everywhere.

Let \(m > 0\) and \(a\) be two constants, such that \(m\) represents the mass of the object and \(ma\) the angular momentum as measured from infinity. In previous coordinates, Kerr metric takes the form

\[
ds^2 = g_{t,t} dt^2 + g_{r,r} dr^2 + g_{\theta,\theta} d\theta^2 + g_{\phi,\phi} d\phi^2 + 2 g_{\phi,t} d\phi dt
\]

(2.1)

with

\[
g_{r,r} = \frac{\lambda(r, \theta)}{\Delta(r)}, \quad g_{\phi,\phi} = \left[ r^2 + a^2 + \frac{2mra^2 \sin^2 \theta}{\lambda(r, \theta)} \right] \sin^2 \theta, \quad g_{t,t} = -1 + \frac{2mr}{\lambda(r, \theta)},
\]

\[
g_{\theta,\theta} = \lambda(r, \theta), \quad g_{\phi,t} = -\frac{2mra^2 \sin^2 \theta}{\lambda(r, \theta)}
\]

and using

\[
\lambda(r, \theta) = r^2 + a^2 \cos^2 \theta, \quad \Delta(r) = r^2 - 2mr + a^2.
\]

Halting the rotation by setting \(a = 0\), Kerr space–time becomes Schwarzschild space–time; if, then, the mass is removed \((m = 0)\) only (empty) Minkowski space–time remains.
In this article we only consider the case $0 < a^2 < m^2$, we mean, slow rotating Kerr space–time. Fast Kerr space–time is simpler; its (non)geodesic connectedness is studied in Ref. 26. Note that the above-given formulas show that Kerr metric fails when either $\lambda(r, \theta) = 0$ or $\Delta(r) = 0$. The function $\Delta(r)$ has the zeroes $r_+ = m + \sqrt{m^2 - a^2}$ and $r_- = m - \sqrt{m^2 - a^2}$ and

$$\Delta(r) = (r-r_-)(r-r_+).$$

So the hypersurfaces $\mathbb{R} \times \{x \in \mathbb{R}^3 : r = r_+\}$ and $\mathbb{R} \times \{x \in \mathbb{R}^3 : r = r_-\}$ are singular for (2.1) and are event horizons. From now on, we will consider the exterior Kerr space–time $K$, we mean the region without singularities in the metric tensor defined by imposing $r > r_+$. Recall that the region between the two event horizons has a strange physical behavior: matter might disappear in finite proper time, or suddenly appear from nowhere. Beyond the second event horizon, the ring singularity appears with its associated time machine.

### III. OVERVIEW

The relation between geodesic connectedness and our topological arguments can be seen intuitively as follows. Consider two points $p_0 \neq p_1$ of a Lorentzian manifold $M$, and fix a topological sphere of the tangent space to $p_0$, $S \subset T_{p_0}M$, such that the vector 0 is included in the interior of $S$. Consider now the subset $\exp_{p_0}sS, s \in \mathbb{R}$, yielded by the geodesics emanating at $p_0$ ($\exp_{p_0}$ is the exponential map at $p_0$). Initially, for small $s$, $p_1$ is outside $\exp_{p_0}sS$, but for some bigger $s$, $p_1$ may lie inside $\exp_{p_0}sS$. This topological change (from being outside to being inside the exponential of a sphere) reflects that $p_0$ and $p_1$ can be connected by a geodesic.

In order to be more precise mathematically, let us see a variation of this idea. Assume that $M$ is an open subset of $\mathbb{R}^5$, and consider the function

$$F: D \subset T_{p_0}M = \mathbb{R}^5 \rightarrow \mathbb{R}^5, \quad F(v) = \gamma_v(1) - p_1,$$

where $\gamma_v$ is the unique geodesic starting at $p_0$ satisfying $\gamma_v'(0) = v$, for any $v \in T_{p_0}M$, and $\gamma_v$ is defined at 1 for all $v$ in the domain $D$. Now, the zeroes of the function $F$ correspond with geodesics connecting $p_0$ and $p_1$. If $F$ satisfies certain conditions at the boundary of $D$ then topological arguments may imply the existence of a zero. In dimension $k = 1$ these conditions will be quite trivial: if $[a, b] \subset D$ and $F(a) \cdot F(b) < 0$ then $F$ will have a zero. For $k = 2$, and, say, $[a, b] \times [a', b'] \subset D$, $F = (F^1(x, y), F^2(x, y))$, if $F^1(a, y) \cdot F^1(b, y) < 0, \forall y \in [a', b']$, $F^2(x, a') \cdot F^2(x, b') < 0, \forall x \in [a, b]$, then the degree of $F$ will be $\neq 0$, and $F$ will have a zero; natural extensions of these conditions will be needed for $k \geq 3$. More exactly, we will need a variation of this argument. We will consider a sequence of increasing intervals $[a_n, b_n], [a'_n, b'_n]$. Under the condition for $F^1, F^1(a_n, y) \cdot F^1(b_n, y) < 0, \forall y \in [a'_n, b'_n]$, a connected set $C_n$ of zeroes of $F^1$ which joins the horizontal lines $y = a'_n, y = b'_n$ can be found. Then, we will look for a zero of $F^2$ in $C_n$ for $n$ big enough.

Clearly, a crucial step in this procedure is to ensure the boundary conditions on $F$. Thus, it is important to have a partial integration of the geodesic equations, as in most classical space–times.

For Kerr space–time $K$, the geodesic equations admit four independent first integrals [see (4.1)]. But the problem is still complicated, and previous arguments will be used by including some subtleties and technical computations. Our approach can be summarized in the following steps.

1. In order to connect two given points $p_0 = (t_0, r_0, \theta_0, \phi_0), p_1 = (t_1, r_1, \theta_1, \phi_1)$, we will consider all the geodesics emanating from $p_0$. Each geodesic $\gamma(s)$ is determined by its initial velocity $\gamma'(s_0)$. In Kerr space–time, this initial velocity is essentially characterized by the constants of motion $q, K, L, E$ (see Sec. IV). More precisely, $\gamma'(s_0)$ yields naturally $q, K, L, E$, and fixing the values of $q, K, L, E$, one obtains $t'(s_0), r'(s_0)^2, \theta'(s_0)^2, \phi'(s_0)$ [see Eq. (4.1)]. The sign
of \( r'(s_0), \theta'(s_0) \) will be canonically fixed \((r'(s_0) \leq 0, \theta'(s_0) \geq 0)\) for the geodesics which are relevant under our approach. Recall that there are restrictions for the possible values of \( q, K, L, E \) because \( r'(s_0)^2, \theta'(s_0)^2 \) calculated from (4.1) cannot be negative.

Geodesics with \( L=0, E=0 \) will be used to connect points in the simple case \( t_0=t_1 \) and either \( p_0 \) or \( p_1 \) lying in the \( z \) axis. In what follows we will consider geodesics with \( E^2+L^2>0 \). Moreover, we will also assume for simplicity \( r_0=r_1 \), \( t_0=t_1 \) (see Remark 4.5). Among geodesics with \( L \neq 0 \) those normalized with \( L=1 \) will always be chosen; if \( L=0 \) the chosen normalization will be \( E=1 \). These last geodesics will be useful when at least one of the points \( p_0, p_1 \) lies in the \( z \) axis; otherwise, geodesics with \( L=1 \) will be enough.

Summing up, we have three degrees of freedom for geodesics starting at \( p_0 \), corresponding to the set of directions in the tangent space to \( p_0 \), which will be described either by parameters \((q, K, L, 1, E)\) or by \((q, K, L, 0, E=1)\).

(2) The dimension of the manifold is four and, thus, there are four geodesic equations; nevertheless, the reparametrization of the geodesics is not relevant for geodesic connectedness and one of the equations will be dropped.

Concretely, we will prove that \( p_0, p_1 \) can be geodesically connected by using geodesics with \( r'(s) \neq 0 \) for all \( s \) in the domain of \( \gamma \) except at a point \( s^* \) such that \( r^* = r(s^*) \) satisfies \( r_0 < r^* < r_0 \). Thus, taking into account this singular point \( r^* \), any such geodesic \( \gamma(s) = (t(s), r(s), \theta(s), \varphi(s)) \) [characterized by its initial conditions as explained in step (1)] will be reparametrized with \( r \), that is, \( \gamma(r) = \gamma(s(r)) = (t(r), r, \theta(r), \varphi(r)) \) for \( r > r^* \).

The other two steps depend on if at least one of the points \( p_0, p_1 \) lies in the \( z \) axis or not. The first case is simpler; assume that, say, \( p_1 \) lies in the \( z \) axis.

\( (3^A) \) Because of the importance of the returning point \( r^* \) the initial parameters \( q, K \) (\( L = 0, E = 1 \)) will be changed by other parameters, \( (r^*, S) \) in a domain \( \hat{D} = \hat{D}_0 = (r_+, r_0) \times (0, \infty) \) [parameter \( S \) will also be related with properties of \( r^* \), see formula (5.2)].

Given a value \( (r^*, S) \in \hat{D} \) we can recover the values of \( q, K \) [Eq. (5.3)]. Nevertheless, the domain \( \hat{D} \) will be restricted to a subset \( \tilde{D} = (r_+, r_0) \times (0, \infty) \subseteq \hat{D} \). The value of \( r^*_L \) is \( (r_+, r_0) \), which characterizes \( \tilde{D} \) is calculated to ensure the following properties:

\( (3^A I) \) As commented in step (1), the values of \( q, K \) have some restrictions to characterize \( \gamma'(s_0) \). When \( (r^*, S) \) belongs to \( D \), these restrictions are satisfied for the corresponding \( q, K \). [In fact, \( h(r_0) \) in Eq. (5.1) will be positive, and Eq. (5.6) will hold; thus, from (4.1) we will be able to choose \( r'(s_0) < 0, \theta'(s_0) > 0 \).]

\( (3^A II) \) As commented in step (2), the behavior of \( r(s) \) for the relevant geodesics will be: first \( r'(s) = r_0 \) then \( r(s) \) decreases until \( r = r^* \) and finally \( r(s) \) increases until \( r_1 \). This behavior is ensured in \( \tilde{D} \) by the characterization of \( r^*_L \) in (5.4). [This can be checked because Eq. (4.10) with (5.1) will be satisfied when \( r^* \in (r_+, r_0) \).]

\( (3^A III) \) The component \( \theta(s) \) of any geodesic \( \gamma \) in \( \tilde{D} \) will satisfy \( \theta'(s) \neq 0 \) out of the \( z \)-axis (see (4.1), (5.6)); in particular, from (1), \( \theta'(s) > 0 \) initially.

Only geodesics with \( (r^*, S) \in \tilde{D} \) will be used for connectedness.

\( (4^A) \) Now, for each \( (r^*, S) \in \tilde{D} \) we have a reparametrized geodesic \( \gamma(r) \) with the parameter \( r \) going from \( r_0 \) to \( r^* \) and, finally, to \( r_1 \). When this parameter arrives at \( r_1 \), the coordinates \( t, \theta, \varphi \) will have increments \( \Delta t, \Delta \theta, \Delta \varphi \) on \( \gamma \). The increment \( \Delta \varphi \) will not be relevant in this case, because \( p_1 \) lies in the \( z \) axis. But, in order to connect \( p_0 \) and \( p_1 \), we have to find one of such geodesics with \( \Delta t = t_1 - t_0 \) and \( \Delta \theta = \theta_1 - \theta_0 \). Moreover, for the increment of \( \theta \) the following trick will be useful. Let \( \Delta \theta \) be the increment in the coordinate \( \theta \) where it is regarded as a “circular coordinate,” that is, as if \( \theta \) increased even after crossing the \( z \) axis. Then, it is enough for \( \theta \): \( \Delta \theta = \theta_1 - \theta_0 + 2n \pi \), for some integer \( n \geq 0 \).

Topological arguments will be relevant now. If we study how \( \Delta t \) varies with parameter \( S \), we will find that (5.7) holds (see Fig. 1). Thus, applications of Brouwer’s topological degree yield connected sets \( C_m, m \in \mathbb{N} \) of values of the parameters \( (r^*, S) \) such that:

(i) for all \( (r^*, S) \in C_m \), \( \Delta t = t_1 - t_0 \), and

(ii) the projection of the points in \( C_m \) on the \( r^* \) axis [that is, the image of the map \( (r^*, S) \)
FIG. 1. When the behavior of $\Delta t(r^*, S)$ is as in formulas (5.7) then a connected set $C_m$ of zeros of $\Delta t-t_1+t_0$ which connect $r^*=r_1+1/m$ and $r^*=r_L^*$ can be found.

$-r^*$ for $(r^*, S) \in C_m$ is equal to the whole interval $[r_1+1/m, r_L^*]$. Now, the behavior of $\Delta |\theta|$ on $C_m$ can be studied, finding: (a) the value of $\Delta |\theta|$ on the points of $C_m$ in the line $r^*=r_L^*$ admit an upper bound independent of $m$, and (b) the value of $\Delta |\theta|$ on the points of $C_m$ in the line $r^*=r_1+1/m$ becomes arbitrarily big. Thus, obviously, there exists at least one point in some $C_m$ such that $\Delta |\theta|=\theta_1-\theta_0+2n\pi$, for some integer $n \geq 0$ [in addition to $\Delta t=t_1-t_0$, satisfied by (i)]. So, the required geodesic is found.

Let us see what happens if neither $p_0$ nor $p_1$ belongs to the $z$ axis. Now, we will consider always geodesics with $L=1$ (in particular, they do not cross the $z$ axis).

(3\text{B}) Analogously to case (3\text{A}), parameters $(q, K, E)$ will be changed by parameters $(r^*, S, E) \in \mathcal{D}(r_+ r_0) \times (0, \infty)^2$, more closely related to the "returning point" of the geodesic. Analogously, the domain $\mathcal{D}$ of $(r^*, S, E)$ will be restricted to a subset $\mathcal{D}$ to solve the following similar problems to (3\text{A}I), (3\text{A}II), (3\text{A}III):

(3\text{B}I) The values of $q, K, E$ have some restrictions to characterize $\gamma'(s_0)$. Given $(r^*, S, E)$, parameters $q, K, E$ are directly computed [formula (6.1)], but perhaps the restrictions are not satisfied and, thus, they are not associated with any geodesic.

(3\text{B}II) Again, the behavior of $r(s)$ must be as in (3\text{A}II). First, $r(s_0)=r_0$ then $r(s)$ decrease until $r=r^*$ and finally $r(s)$ increase until $r_1$. To achieve this, Eq. (4.10) [with (4.5)] must be satisfied.

(3\text{B}III) In (3\text{A}III) the found connecting geodesic might cross the $z$ axis many times [because of (5.6)]. But now, we are considering geodesics which do not cross the $z$ axis. In fact, for any geodesic with $L=1$, an angle $\theta_L \in (0, \pi/2]$ will exist such that if $\theta_L \leq \theta_0 \leq \pi - \theta_L$ then the component $\theta(s)$ of the geodesic varies between $\theta_L$ and $\pi - \theta_L$. Thus, $\theta_L$ can be regarded as a limit angle for $\theta(s)$ (see Definition 4.1 and Remark 4.2). Recall that, if also $\theta_1 \leq \theta_L$, $\theta(s)$ will find no obstruction to cross $\theta_1$ and $\theta_L$.

(3\text{B}IV) Technical conditions [Eq. (6.4), Remark 6.4] will be required for $\mathcal{D}$ in order to apply topological arguments.

In case $3\text{A}$ a value of $r_L^*$ is found such that the corresponding problems $3\text{A}I-3\text{A}III$ for the parameters $(r^*, S)$ are solved if $r^* \in (r_+, r_L^*)$. Unfortunately, this is not sufficient now. The best
we can find is some domain $D$, with restrictions not only for $r^*$ but also for $S$, such that restrictions $3B I^{3B IV}$ are satisfied when $(r^*, S, E) \in D$ (see Lemma 6.2 and Remark 6.4). More precisely, fixed $\bar{\theta} > 0$ let $S(r^*, E) = 0$ be the minimum non-negative number such that $\theta_2(r^*, S, E) < \bar{\theta}$ if $S > S(r^*, E)$ (see Definition 4.1 and Lemma 6.3). $S(r^*, E)$ is a continuous function with finite supremum $S$, but it will vanish for some $(r^*, E)$ (this must be taken into account noticing that $S > 0$ for the elements of $\bar{D}$). Then there exist $r^*_n \in (r_1, r_0)$ and $\bar{\theta} \in (0, \pi/2)$ such that $D = \{ (r^*, S, E) \in \bar{D}, r^*_n \in (r_1, r^*_n) \} \in (0, \infty), S \in [S(r^*, E), \infty)$ is the required domain.

(4B) Now, for each $(r^*, S, E) \in D$ we have a reparametrized geodesic $\gamma(r)$ with the parameter $r$ going from $r_0$ to $r^*$ and, finally, to $r_1$. As in case (4A), when this parameter arrives at $r_1$, the coordinates $t, \theta$, and $\varphi$ will have increments $\Delta t, \Delta \theta$, and $\Delta \varphi$ on $\gamma$, but now the increment $\Delta \varphi$ is also relevant. In order to connect $p_0$ and $p_1$, it is sufficient to find one of such geodesics with $\Delta t = t_1 - t_0$, $\Delta \theta = \theta_1 - \theta_0$, and $\Delta \varphi = \varphi_1 - \varphi_0$. Moreover, we will denote by $|\Delta \theta|$ the increment in the coordinate $\theta$ computed as if it were positive beyond any rebound of $\theta(s)$ at $\theta_0$ or $\pi - \theta_1$; for example, if $\theta(s)$ increases from $\theta(s_0)$ to $\pi - \theta_1$ and then decreases to $\theta(s_1)$ we define $|\Delta \theta| = |\pi - \theta_1 - \theta(s_0)| + |(\pi - \theta_1) - \theta(s_1)|$. Thus, it is sufficient for $\theta$: $|\Delta \theta| = \theta_1 - \theta_0 + 2n(\pi - 2\theta_1)$, for some integer $n \geq 0$. Finally, as $\varphi$ will be regarded as a circular coordinate it is enough $\Delta \varphi = \varphi_1 - \varphi_0 + 2n' \pi$, for some integer $n' \geq 0$. The topological arguments are now subtler.

First we will find connected sets $C_n$ of parameters such that the associated geodesics have exactly $\Delta t = t_1 - t_0$, $\Delta |\theta| = \theta_1 - \theta_0 + 2n(\pi - 2\theta_1)$.

(4B) Consider the functions $T = \Delta t - (t_1 - t_0)$ and $\Theta_n = |\Delta |\theta| - (\theta_1 - \theta_0) - 2n(\pi - 2\theta_1)$ with domain $D$, for some integer $n \geq 0$. Given $\{S_n\}_n$ diverging, there exist $\{\epsilon_n\}_n$, $\epsilon_n > 0$, $\epsilon_n' > 0$, and $\delta_n > 0, \delta_n' > 0$ such that functions $T$ and $\Theta_n$ satisfy the boundary conditions (6.34) in a domain $D_n$ (Fig. 2), for $n$ big enough (Proposition 6.7).

(4B II) Taking into account that in some points $S(r^*, E)$ vanishes, a natural compact subset $D_n \subset D$ can be defined [see (6.35)] where connected sets $C_{n}$ of simultaneous zeroes of $T$ and $\Theta_n$ will be found. In order to obtain $C_{n}^m$ recall that a homeomorphism $z^m_{n}$ maps $D_{n}$ to a cube as in Fig. 3. Thus, topological arguments imply the existence of a connected set $z^m_{n}(C_{n}^m)$ of zeroes of the two functions $T(z^m_{n})^{-1}, \Theta_n(z^m_{n})^{-1}$ connecting the upper and the lower faces of the cube (Lemma 6.9).

(4B III) Using whyburn arguments on the sets $C_{n}^m$ we construct a new connected set $C_n \subset D_n$ (depicted in Fig. 4). We will prove:

(i) $C_0 \subset D_n$ and thus, points in $C_0$ are zeroes of $T$ and $\Theta_n$, and
(ii) each $C_n$ has a point with $S = S_n$ and a point with $S = S(r^*, E) > 0$ (and, thus $\theta_2 = \bar{\theta}$) for all $n$ (Lemma 6.10 and comments above).

FIG. 2. Domain $D_n$ in (6.32): $A = (r_1 + \epsilon_n, S_n, \delta_n), C = (r_1 + \epsilon_n, S_n, |1/\delta_n|), D = (r_1, S_n, |1/\delta_n|), G = (r_1 + \epsilon_n, S_1(r_1 + \epsilon_n, S_n, \delta_n))$. Behavior of $T, \Theta_n$ from Lemma 6.3 and (6.34) in Proposition 6.7: $T < 0$ on $(A F J) \cup (F E J')$; $T > 0$ on $(C H I D) \cup (D' D')$; $\Theta_n > 0$ on $(A H G)$; $\Theta_n < 0$ on $(F' D' J')$. Lower face: graph of $S = S(r^*, E) (\geq 0)$. 
Now, geodesics associated to each \((r^*, S, E)\) have the required value of \(\Delta t, \Delta |\theta|\), but we have not controlled the value of \(\Delta \varphi\) yet. Our aim will be to find two points in a connected set \(C_n\) for some \((\text{big})\, n\), whose difference in \(\Delta \varphi\) is greater than \(2\pi\). Thus, some point in \(C_n\) will satisfy \(\Delta \varphi = \varphi_1 - \varphi_0 + 2\pi n', \, n' \geq 0\), which concludes the proof. In fact, we prove the existence of a

FIG. 3. (1) Homeomorphism \(z_n^0\) between \(D_n^0\) in (6.35) and the cube \([0,1]^3\). The dashed region \((F'D'I_m')\) corresponds by the homeomorphism. (2) Brouwer's topological degree applied to \(T(z_n^0)^{-1}\) and \(\Theta(z_n^0)^{-1}\) yields a simultaneous connected set of zeroes \(z_n = z_n^0(C_n^0)\) joining the upper and lower faces of the cube \([0,1]^3\).

Take \(\limsup_{m} C_n\) a connected set \(C_n\) is obtained, satisfying: (a) all the points in \(D_n^0\cap C_n\) are zeroes of \(T\) and \(\Theta\), (b) \(C_n\) touches the upper and the lower faces, the latter in a point with \(S > 0\) (Lemma 6.10; in particular, \(C_n \subset D_n\)), and (c) for big \(n\), function \(\Delta \varphi|_{C_n}\) covers an interval of length greater than \(2\pi\) (Lemmas 6.11 and 6.12).
choice of \( \{S_n\}_n \) such that points \((r_n^*, S_n^*, E_n) \in \mathcal{C}_n \) obtained in (ii) satisfy:

\[
2n \pi - (\Delta \varphi)(r_n^*, S_n^*, E_n) < n \cdot \epsilon
\]

for every \( \epsilon > 0 \) and \( n \) big enough (Lemma 6.11). On the other hand, again from (ii) we can take a sequence of points \((r_n^*, S_n^*, E_n) \in \mathcal{C}_n \) with \( \theta_t(r_n^*, S_n^*, E_n^t) = \tilde{\theta} > 0 \) for all \( n \) which, from Lemma 6.12, implies

\[
2n \pi - (\Delta \varphi)(r_n^*, S_n^*, E_n^t) > n \cdot \epsilon_0
\]

for some \( \epsilon_0 > 0 \). The result follows then from (3.2) and (3.3).

**IV. GENERAL BEHAVIOR OF THE GEODESICS**

Let \( \gamma: \mathcal{J} \rightarrow \mathbb{R}^4 \), \( \gamma(s) = (t(s), r(s), \theta(s), \varphi(s)) \) be a (smooth) curve on the interval \( \mathcal{J} \). The first integrals of the geodesics equations of Kerr space–time are

\[
\lambda(r, \theta)t' = aD(\theta) + (r^2 + a^2) \frac{P(r)}{\Delta(r)},
\]

\[
\lambda(r, \theta)^2 r'^2 = \Delta(r)(qr^2 - K) + P^2(r),
\]

\[
\lambda(r, \theta)^2 \theta'^2 = K + qa^2 \cos^2 \theta - \frac{D^2(\theta)}{\sin^2 \theta},
\]

\[
\lambda(r, \theta) \varphi' = \frac{D(\theta)}{\sin^2 \theta} + a \frac{P(r)}{\Delta(r)},
\]

where

\[
D(\theta) = L - Ea \sin^2 \theta,
\]

\[
P(r) = (r^2 + a^2)E - La
\]

and \( q \) (normalization of the geodesic; rest mass), \( K \) (Carter constant), \( L \) (angular momentum), and \( E \) (energy measured by observers in \( \partial_n \)) are constants; we follow essentially the notation in Ref. 31, Chap. 4. So, if we assume \( r'(s_0) \leq 0 \), \( \theta'(s_0) = 0 \) every geodesic is fixed by an initial point \( p_0 \) [i.e., \( \gamma(s_0) = p_0 \)] and the constants \( q, K, L, \) and \( E \). Moreover, from the third equation in (4.1), if \( r \) reaches the \( z \) axis then, necessarily \( L = 0 \); recall that when \( L \neq 0 \) we can normalize \( L = 1 \).

Let \( s(r) \) be the inverse function (where it exists) of \( r(s) \) in (4.1); using \( r \) as the parameter in the other three equations in (4.1):

\[
\frac{dr}{dr} = \epsilon \frac{aD(\theta) + (r^2 + a^2) \frac{P(r)}{\Delta(r)}}{\sqrt{\Delta(r)(qr^2 - K) + P^2(r)}},
\]

\[
\frac{d\theta}{dr} = \epsilon' \frac{\sqrt{K + qa^2 \cos^2 \theta - \frac{D^2(\theta)}{\sin^2 \theta}}}{\sqrt{\Delta(r)(qr^2 - K) + P^2(r)}},
\]

\[
\frac{d\varphi}{dr} = \epsilon \frac{\frac{D(\theta)}{\sin^2 \theta} + a \frac{P(r)}{\Delta(r)}}{\sqrt{\Delta(r)(qr^2 - K) + P^2(r)}}.
\]
on a certain domain, being \( \epsilon, \epsilon' \in \{ \pm 1 \} \). Equation (4.3) will be used to prove the geodesic connectedness of outer Kerr space–time \( \mathbb{K} \). But, previously, we need the following definition related to the expression of \( \theta' \).

**Definition 4.1:** For any \((q, K, L, E) \in \mathbb{R}^4\) we define \( \theta_L(\theta) = \theta(q, K, L, E) \) as the smaller angle in \([0, \pi/2] \) satisfying

\[
K + qa^2 \cos^2 \theta - \frac{1}{\sin^2 \theta} > 0, \quad \forall \theta \in \left( \theta_L, \frac{\pi}{2} \right), \tag{4.4}
\]

with \( D(\theta) = D(\theta, L, E) \) given in (4.2).

[Recall that if for some \((q, K, L, E) \) the left-hand side of inequality (4.4) is nonpositive at \( \theta = \pi/2 \) then \( \theta_L = \pi/2 \).]

**Remark 4.2:** Assume that \((q, K, L, E) \) corresponds to a geodesic with \( \gamma(s_0) = p_0 = (t_0, r_0, \theta_0, \varphi_0) \) and \( \theta_L \leq \theta_0 \leq \pi - \theta_L \) if \( L = 1 \) then \( \theta_L \) can be seen as a limit angle for \( \theta(s) \) as explained in \((3^9)^{III}\); in fact, it is clear from (4.1) that \( \theta(s) \) takes values in \([\theta_L, \pi - \theta_L] \) being \( \theta'(s) = 0 \) only when \( \theta(s) \in \{ \theta_L, \pi - \theta_L \} \).

If one considers geodesics with \( r' \neq 0 \) at any point, then the geodesic can be reparametrized by \( \varphi \) and the increments \( \Delta t, \Delta \theta, \) and \( \Delta \varphi \) can be calculated integrating in (4.3). Nevertheless, in this article we are going to see that two arbitrary points \( p_0 = (t_0, r_0, \theta_0, \varphi_0), \) \( p_1 = (t_1, r_1, \theta_1, \varphi_1) \) in \( \mathbb{K} \) with \( r_0 < r_1 \) can be always joined with a geodesic such that \( r'(s) \) vanishes exactly at one point \( s^* \), and \( r^* = r(s^*) \) satisfies \( s^* < r^* < r_0 \). Recall that, for these geodesics the denominator in formula (4.3), that is

\[
h(r) = \Delta(r)(q r^2 - K) + \Delta^2(r), \tag{4.5}
\]

will vanish at \( r^* \). As \( r(s) \) will go from \( r_0 \) to \( r^* \) then necessarily \( h'(r^*) > 0 \); in fact, this implies \( |s(r^*) - s(r_0)| < \infty \) (see Ref. 29). As later on \( r(s) \) will go from \( r^* \) to \( r_1 \), then \( h(r) > 0 \) if \( r^* < r < r_1 \); we will consider geodesics with \( h(r_1) > 0 \) too. Thus, the increment in the variable \( t \) will be

\[
\Delta t = \int_{r^*}^{r_0} \frac{aD(\theta) + (r^2 + a^2)}{\sqrt{h(r)}} \frac{1}{\Delta(r)} dr + \int_{r^*}^{r_1} \frac{aD(\theta) + (r^2 + a^2)}{\sqrt{h(r)}} \frac{1}{\Delta(r)} dr. \tag{4.6}
\]

For the coordinate \( \theta \), if \( L = 1 \) we will assume that \( \theta(s) \) will go from \( \theta_0 \) to \( \theta_1 \) with \( \theta'(s) \) only vanishing at an even number of points \( s_i, i = 1, \ldots, 2n, \) such that \( \theta(s_i) \) is equal either to the limit angle \( \theta_L \) or to \( \pi - \theta_L \). Remark 4.2. If \( L = 0 \), we will choose always geodesics such that \( q, K, E \) make the left-hand side of (4.4) positive for all \( \theta \) and thus, \( \theta_L = 0 \); that is, \( \theta'(s) \neq 0 \) always except at the points \( s_i, i = 1, \ldots, 2n, \) where \( \gamma \) reaches the \( z \) axis [at these points \( \theta(s) \) is not differentiable and the sign of \( \theta'(s) \) changes]. In particular,

\[
\theta_L \leq \theta_0, \quad \theta_1 \leq \pi - \theta_L \quad \text{if} \quad L = 1,
\]

\[
\theta_L = 0 \quad \text{and} \quad (4.4) \text{ holds strictly at} \quad \theta_L \quad \text{if} \quad L = 0. \tag{4.7}
\]

As explained in (4.5) and (4.6) we will define \( \Delta |\theta| \) considering the increments as positive beyond the points \( s_i \). That is, we define

\[
\Delta |\theta| = \int_{r^*}^{r_0} \frac{\sqrt{K + qa^2 \cos^2 \theta - \frac{1}{\sin^2 \theta}}}{\sqrt{h(r)}} ds + \int_{r^*}^{r_1} \frac{\sqrt{K + qa^2 \cos^2 \theta - \frac{1}{\sin^2 \theta}}}{\sqrt{h(r)}} ds. \tag{4.8}
\]

Finally, the circular coordinate \( \varphi \) will have an increment.
Recall that this increment also represents a certain number \( n' \) of turns around the \( z \) axis (if \( 2n' \pi \leq \Delta \varphi < 2(n' + 1) \pi \)). Summing up:

**Lemma 4.3:** Let \( p_0 = (t_0, r_0, \theta_0, \varphi_0) \), \( p_1 = (t_1, r_1, \theta_1, \varphi_1) \) be two points in \( K \) with \( r_0 \leq r_1 \), such that \( \sin \theta_0 \neq \sin \theta_1 \) (respectively, one or both points in the \( z \) axis). Assume that there exist \( q \), \( K \), and \( E \) as well as \( r^* \in (r_+, r_0) \) such that (4.7) and the following relations (4.10) and (4.11) hold with \( L = 1 \) (respectively, \( L = 0 \)):

\[
\begin{align*}
\Delta t &= t_1 - t_0, \\
\Delta |\theta| &= \theta_1 - \theta_0 + 2n(\pi - 2\theta) \sin \theta, \\
\Delta \varphi &= \varphi_1 - \varphi_0 + 2\pi n' \\
\end{align*}
\]

for some integers \( n,n' \geq 0 \) [for the case \( L = 0 \), without the restriction on \( \Delta \varphi \) in (4.11)]. Then, there exists \( s_1 > s_0 \) such that the geodesic \( \gamma:[s_0,s_1] \to K \), \( \gamma(s_0) = p_0 \) with constants \( L = 1 \) (respectively, \( L = 0 \)), \( q \), \( K \) and \( E \), satisfies \( \gamma(s_1) = p_1 \).

**Remark 4.4:** Recall that the restriction on \( \Delta \varphi \) in (4.11) is not imposed when \( L = 0 \) because if \( \sin \theta_1 = 0 \) (as we assumed in the overview) then the coordinate \( \varphi \) becomes irrelevant. If \( \sin \theta_0 = 0 \neq \sin \theta_1 \) then Lemma 4.3 yields a geodesic \( \gamma(s) = (t(s), r(s), \theta(s), \varphi(s)) \) with the appropriate values of \( \Delta t, \Delta |\theta| \). Then \( \bar{\gamma}(s) = (t(s), r(s), \theta(s), \varphi(s) + \bar{\varphi}) \) with \( \bar{\varphi} = \varphi_1 - \varphi_0 + 2\pi n' \) is the required geodesic.

**Remark 4.5:** In order to prove the connectedness of two arbitrary points \( p_0 = (t_0, r_0, \theta_0, \varphi_0) \), \( p_1 = (t_1, r_1, \theta_1, \varphi_1) \) we will assume \( t_0 \leq t_1 \) and \( r_0 \leq r_1 \). In fact, if this last inequality does not hold, Lemma 4.3 can be obviously modified taking \( r^* \in (r_+, r_1) \).

The following simple technical result will be useful in the following sections (see Ref. 26 for a detailed proof):

**Lemma 4.6:** Let \( \{f_n(x)\}_n \) be a sequence of continuous functions on \([a_n,b_n] \subset \mathbb{R}, a_n \to a, b_n \to b, a \neq b\) satisfying \( 0 < c \leq f_n(x) \leq C \) for all \( n \), and let \( \{p_n(x)\}_n \) be a sequence of polynomials with degree bounded in \( n \) satisfying for all \( n \): \( p_n(a_n) = 0, p_n'(a_n) = R_n > 0 \) and \( p_n^{(k)}(a_n) \geq 0 \) for \( k \geq 2 \).

(i) If \( R_n \to \infty \), then

\[
\int_{a_n}^{b_n} \frac{f_n(x)}{\sqrt{p_n(x)}} \, dx \to 0.
\]

(ii) If \( R_n \to 0 \) and \( p_n^{(k)}(a_n) \) admits an upper bound for \( k \geq 2 \) and for all \( n \), then

\[
\int_{a_n}^{b_n} \frac{f_n(x)}{\sqrt{p_n(x)}} \, dx \to \infty.
\]

**V. GEODESIC CONNECTEDNESS WITH THE z AXIS**

In this section we prove that there exists a geodesic joining two arbitrary points when, at least, one of them lies in the \( z \) axis. From Remark 4.5, it suffices:

**Theorem 5.1:** Given two points \( p_0 = (t_0, r_0, \theta_0, \varphi_0) \), \( p_1 = (t_1, r_1, \theta_1, \varphi_1) \) in \( K \), at least one of them in the \( z \) axis, with \( r_0 \leq r_1 \), \( t_0 \leq t_1 \), there exists a geodesic \( \gamma:[s_0,s_1] \to K \) such that \( \gamma(s_0) = p_0 \), \( \gamma(s_1) = p_1 \).

**Proof:** Taking into account Lemma 4.3 it is sufficient to prove that there exist constants \( q \), \( K \), and \( E \) as well as \( r^* \in (r_+, r_0) \) satisfying (4.7), (4.10), and (4.11) for \( L = 0 \).
First, consider the case \( t_0 < t_1 \) and choose \( E = 1 \); so (4.5) becomes

\[
h(r) = (r - r_-)(r - r_+)(qr^2 - K) + (r^2 + a^2)^2.
\]

(5.1)

If we look for \( r^* \) such that

\[
h(r^*) = 0, \quad h'(r^*) = S > 0
\]

(5.2)

then the following two relations for the constants \( q \) and \( K \) are obtained:

\[
q = \frac{(r^* - r_-)^2}{2r^*(r^* - r_-)(r^* - r_+)} + \frac{(r^* - r_+)^2}{2r^*(r^* - r_-)(r^* - r_+)} - \frac{2(r^* - a^2)}{(r^* - r_-)(r^* - r_+)} + \frac{S}{2r^*(r^* - r_-)(r^* - r_+)}.
\]

(5.3)

Taking into account the dependencies of \( q \) on \( (r^* - r_+) \) in the second equation in (5.3), there exists \( r^*_L \in (r_+, r_0) \) near enough to \( r_+ \) such that if \( r^* \in (r_+, r^*_L] \) then

\[
r^*_L q(r^*, S) > a^2, \quad \forall S > 0 \quad (\Rightarrow K(r^*, S) > a^2, \quad \forall S > 0).
\]

(5.4)

Moreover, deriving in (5.1) in order to apply Lemma 4.6,

\[
h^{(2)}(r^*) = 4qr^*(r^* - r_+) + 4q^2r^*(r^* - r_-) + 2(qr^* - K) + 2q(r^* - r_-)(r^* - r_+) + 12r^* + 4a^2,
\]

\[
h^{(3)}(r^*) = 6q(r^* - r_+) + 6q(r^* - r_-) + (12q + 24)r^*,
\]

\[
h^{(4)}(r^*) = 24q + 24.
\]

(5.5)

Clearly \( h^{(3)}(r^*), h^{(4)}(r^*) > 0 \). Moreover, \( r^*_L \) can be chosen such that the sum of the second plus third terms in the expression of \( q \) in (5.3) is positive and thus, \( h^{(2)}(r^*) > 0 \). In particular, \( h(r) > 0 \) if \( r > r^* \). On the other hand, from (5.4)

\[
K + qa^2 \cos^2 \theta - \frac{D^2(\theta)}{\sin^2 \theta} = K + (qa^2 + a^2) \cos \theta - a^2 > 0
\]

(5.6)

for all \( \theta \) [in particular, (4.7) is satisfied with \( L = 0 \)]. Summing up, because of Lemma 4.3 it is sufficient to find an element of \( D = \{(r^*, S) : r^* \in (r_+, r^*_L], S \in (0, \infty) \} \) such that the corresponding \( (q, K) \) given from (5.3) and the function \( h(r) = h(r, q, K) \) in (5.1) satisfy the first two equalities (4.11) with \( \Delta t, \Delta |\theta| \) as in (4.6) and (4.8) (and \( L = 0, E = 1 \)).

Fix sequences \( \{S_n \}_n \to \infty, \{r^*_L \}_n \to r^* \in (r_+, r^*_L'] \), and put

\[
f_n(r) = \alpha |\theta_n(r)| + (r^2 + a^2) \frac{F(r)}{A(r)}, \quad p_n(r) = h_n(r)
\]

with \( h_n(r) = h(r, q(r_n^*, S_n), K(r_n^*, S_n)) \) and \( \theta_n(r) \) computed with \( (r_n^*, S_n) \) from the second equation in (4.3) [recall (5.6)]. Using (5.2) and the positiveness of (5.5), hypotheses of Lemma 4.6 (i) clearly hold on the interval \( [a_n, b_n] = [r_n^*, r_1] \). Therefore,

\[
(\Delta t)_n = \int_{r_n^*}^{r_0} \frac{f_n(r)}{\sqrt{p_n(r)}} \, dr + \int_{r_n^*}^{r_1} \frac{f_n(r)}{\sqrt{p_n(r)}} \, dr \to 0.
\]
Analogously, if we consider \( \{ S_n \}_n \to 0 \) then, from (5.5) and (5.3), \( h^{(k)}(r_+^n) \) admits an upper bound for \( k \geq 2 \) and all \( n \) thus, from Lemma 4.6(ii), \( (\Delta t)_n \to \infty \). In conclusion, there exists \( \{ \epsilon_n \}_n \), \( \epsilon_n > 0, \epsilon_n \searrow 0 \), such that

\[
\Delta t(r^*, S) < t_1 - t_0 \quad \text{when} \quad (r^*, S) \in \left[ r_+ + \frac{1}{m}, r_L^T \right] \times \left[ \frac{1}{\epsilon_n}, \infty \right),
\]

(5.7)

\[
\Delta t(r^*, S) > t_1 - t_0 \quad \text{when} \quad (r^*, S) \in \left[ r_+ + \frac{1}{m}, r_L^T \right] \times (0, \epsilon_n]
\]

(see Fig. 1). This is a typical situation where Brouwer’s topological degree ensures, for each \( m \), the existence of a connected set \( C_m \) of zeroes of \( \Delta t - t_1 + t_0 \) connecting the extreme vertical lines. More precisely,

**Lemma 5.2:** There exists a connected subset \( C_m \) of zeroes of \( T(r^*, S) = \Delta t - t_1 + t_0 \) such that

\[
C_m \cap \left( \left\{ \frac{1}{m} \right\} \times \left( [\epsilon_n, 1/m] \times \left[ \frac{1}{\epsilon_n}, \infty \right) \right) \right) \neq \emptyset \quad \text{and} \quad C_m \cap \left( \left\{ \frac{1}{m} \right\} \times \left[ 0, \epsilon_n \right) \right) \neq \emptyset
\]

for every \( m \in \mathbb{N} \).

(For a detailed proof, see Ref. 26 Lemma 2.) Therefore, for these subsets (i) and (ii) in (4.4) holds. So, we only need to prove that among the found zeroes of \( T \) there is a \( (r^*, S) \in D \) such that \( \Delta|\theta| \) satisfies (4.11), for some \( n \).

Let \( (r^*_m, S_m) \in C_m \) be with \( r^*_m = r_+ + 1/m \), and take in Lemma 4.6(ii)

\[
f_m(r) = \frac{\sqrt{K_m + q_m a^2 \cos^2 \theta_m(r) - a^2 \sin^2 \theta_m(r)}}{\sqrt{q_m}} \quad \text{and} \quad p_m(r) = \frac{h_m(r)}{q_m}
\]

with \( K_m, q_m \) obtained from (5.3), \( \theta_m(r) \) from (4.3). From (5.3), \( q_m \) will grow at least as \( m^2 \) and, from (5.4) and (5.5) one checks that the hypotheses of this lemma hold on the intervals \( [a_m, b_m] = [r^*_m, r_1] \), obtaining

\[
(\Delta|\theta|)_m = \int_{r_m}^{r_0} \frac{f_m(r)}{\sqrt{p_m(r)}} \, dr + \int_{r_m}^{r_1} \frac{f_m(r)}{\sqrt{p_m(r)}} \, dr \to \infty.
\]

(5.8)

On the other hand, from (5.7) the points in \( C_m \) with \( r^* = r_L^T \) have \( S \in (\epsilon_1, 1/\epsilon_1) \), thus \( q, K \) is upper bounded for these points and all \( m \) [see (5.3)] and so is \( \Delta|\theta| \). This fact, (5.8) and the connectedness of each \( C_m \) imply for some big \( m \) the existence of \( (r^*, S) \in C_m \) such that the second equality (4.11) also holds, as required.

Finally, consider the case \( t_0 = t_1 \). Choose \( E = 0 \) and, thus, \( T = 0 \) automatically. Now (4.5) becomes

\[
h(r) = (r-r_)(r-r_)(qr^2 - K).
\]

By imposing \( h(r^*) = 0 \) and \( h'(r^*) = 1 \) we obtain the following values for the constants \( q \) and \( K \):
If we take \( r_m \to r_+ \), (5.9) and formulas analogous to (5.5) imply that Lemma 4.6(ii) can be applied to the functions

\[
f_m(r) = \frac{\sqrt{K_m + q_m a^2 \cos^2 \theta_m(r)}}{q_m}, \quad p_m(r) = \frac{h_m(r)}{q_m},
\]

on the intervals \([ a_m, b_m ] = [ r_m^*, r_1 ]\). Thus, we obtain (5.8) and, so, \( (\Delta | \theta )_m^0 = \Delta | \theta (r_m^*) \to \infty \).

Therefore, a required value for \( r^* \) can be found in \([ r_m^*, r_L^* ]\) for some big \( m \).

\[\square\]

VI. GEODESIC CONNECTEDNESS BETWEEN POINTS OUT OF THE z AXIS

In this section we conclude the proof of the geodesic connectedness of \( K \) by proving that there exists a geodesic joining two arbitrary points out of the \( z \) axis. Recalling again Remark 4.5, we will prove:

**Theorem 6.1:** Given two points \( p_0 = (t_0, r_0, \theta_0, \varphi_0) \), \( p_1 = (t_1, r_1, \theta_1, \varphi_1) \) in \( K \), \( \sin \theta_0 \neq 0 \), \( r_0 \leq r_1 \), \( t_0 \leq t_1 \), there exists a geodesic \( \gamma : [s_0, s_1] \to K \) such that \( \gamma(s_0) = p_0 \), \( \gamma(s_1) = p_1 \).

Taking into account Lemma 4.3, it is sufficient to prove that there exist constants \( q \), \( K \), and \( E \) as well as \( r^* \in (r_+, r_0) \) satisfying (4.7), (4.10), and (4.11) with \( L = 1 \).

We will again use \( r^* \) such that (5.2) holds. Then the following two relations for the constants \( q \) and \( K \) are obtained from (4.5):

\[
q r^* - K = \frac{[(r^* + a^2)E - a]^2}{(r^* - r_1)(r^* - r_+)}.
\]

Moreover,

\[
h^{(2)}(r^*) = 4q r^*(r^* - r_+) + 4q r^*(r^* - r_-) + 2(q r^* - K) + 2q(r^* - r_-)(r^* - r_+) + 4E[(r^* + a^2)E - a] + 8r^2 E^2,
\]

\[
h^{(3)}(r^*) = 12q r^* + 6q(r^* - r_+) + 6q(r^* - r_-) + 24E^2 r^*;
\]

\[
h^{(4)}(r^*) = 24q + 24E^2.
\]

From (6.1), given \( (r^*, S, E) \in \mathcal{D} = (r_+, r_0) \times (0, \infty)^2 \) we obtain \( q(r^*, S, E, K(r^*, S, E), E) \). So, we can define \( \theta_{l} = \theta_{l}(r^*, S, E) \equiv \theta_{l}(q, K, E) \), Definition 4.1, and write the following result:

**Lemma 6.2:** There exist \( r_L^* \in (r_+, r_0) \) close to \( r_+ \) and \( \bar{\theta} \in (0, \pi/2) \) close to 0 with \( \bar{\theta} < \theta_0, \theta_1 < \pi - \bar{\theta} \) such that for all \( (r^*, S, E) \) with \( r_- < r^* < r_L^* \) and \( \theta_{l}(r^*, S, E) < \bar{\theta} \) we have:

(i) \[q \geq \mu_1 > 0 \quad \text{for some } \mu_1,\]

(ii) derivatives in (6.2) are greater than 0 [in particular, \( h(r) > 0 \) if \( r > r^* \)], and
(iii)
\[
\int_{\theta_L}^{\pi/2} \frac{1}{\sqrt{\frac{K}{q} + a^2 \cos^2 \theta - \frac{1}{q \sin^2 \theta}}} d\theta
\]

is upper bounded.

Proof of Lemma 6.2: From Definition 4.1 and formula (4.2),
\[
\frac{1}{\sin^2 \theta_L} = K + 2Ea - E^2a^2 + (qa^2 + E^2a^2) \cos^2 \theta_L \leq K + 2Ea + qa^2 \cos^2 \theta_L;
\]

thus taking into account the different terms in (6.1), there exist \(a_i, b_i, c_i > 0, i = 1, \ldots, 4\) and \(\nu > 0\) such that:
\[
\frac{1}{\sin^2 \theta_L} < \left[ a_1 \frac{((r^* + 2a^2)E - a)^2}{(r^* - r_+)^2} - a_2 \frac{|(r^* + 2a^2)E - a|E}{(r^* - r_+)} \right] + a_3 \frac{S}{(r^* - r_+)} + a_4. \tag{6.6}
\]
\[
\nu q > 4 qr^*(r^* - r_-) > 4qr^*(r^* - r_-) + 2(qr^* - K)
\]
\[
> \left[ b_1 \frac{((r^* + 2a^2)E - a)^2}{(r^* - r_+)^2} - b_2 \frac{|(r^* + 2a^2)E - a|E}{(r^* - r_+)} \right] + b_3 \frac{S}{(r^* - r_+)}, \tag{6.7}
\]
\[
\frac{K}{2} - E^2a^2 > \left[ c_1 \frac{((r^* + 2a^2)E - a)^2}{(r^* - r_+)^2} - c_2 \frac{|(r^* + 2a^2)E - a|E}{(r^* - r_+)} \right] + c_3 \frac{S}{(r^* - r_+)} - c_4 \tag{6.8}
\]

for any \((r^*, S, E) \in \mathcal{D}\).

Now, we claim that for some \(r^*_L \in (r_+, r_0)\) and \(\bar{\theta} > 0\) small, if \(r^* \in (r_+, r^*_L)\) and \(\theta_L(r^*, S, E) \leq \bar{\theta}\) then
\[
4qr^*(r^* - r_-) + 2(qr^* - K) > \mu_0, \tag{6.9}
\]
\[
\frac{K}{2} - E^2a^2 > 0 \tag{6.10}
\]

for some \(\mu_0 > 0\); in particular, from (6.7) and (6.9), we obtain (i) with \(\mu_1 = \mu_0 / \nu\). In fact, if \(E > 1/\alpha\) then for some \(r^*_L > r_+\) if \(r^* \in (r_+, r^*_L)\):
\[
\frac{(r^* + 2a^2)E - a}{(r^* - r_+)} > d \cdot E, \quad d = \max \left[ \frac{a_2}{a_1}, \frac{b_2}{b_1}, \frac{c_2}{c_1} \right] + 1. \tag{6.11}
\]

Therefore, the square brackets in (6.6), (6.7), and (6.8) are positive. Thus, from (6.6) if \(\bar{\theta}\) is small, then either \(\left(\frac{(r^* + 2a^2)E - a}{(r^* - r_+)}\right)\) or \(S/((r^* - r_+))\) is big, and (6.9), (6.10) are a consequence of (6.7), (6.8), respectively. If \(E \leq 1/\alpha\), some square brackets in (6.6), (6.7), (6.8) might be negative. But if this happens, then instead of (6.11) one has
\[
\frac{|(r^* + 2a^2)E - a|}{(r^* - r_+)} \leq \frac{d}{a}. \tag{6.12}
\]

As a consequence, if \(\bar{\theta}\) is chosen small enough then, from (6.6), \(S/((r^* - r_+))\) is big. Thus, (6.9) and (6.10) are again a consequence of (6.7) and (6.8). Moreover, we can also prove:
\[ \forall M > 0 \exists \tilde{\theta} > 0: \quad q(r^*, S, E) > M \quad \text{if} \quad \theta_L(r^*, S, E) \leq \tilde{\theta}, \quad r^* \in (r_+ , r_L^*]. \quad (6.13) \]

From (6.3), \( h^{(3)}(r^*) , h^{(4)}(r^*) > 0 \) and, from (6.9), \( h^{(2)}(r^*) > 0 \) too and, thus, (ii) is obtained. In order to prove (iii), put

\[ \Lambda_{q,K,E}(\theta) = \frac{K}{q} + a^2 \cos^2 \theta - \frac{D^2(\theta)}{q \sin^2 \theta}, \quad \theta \in \left[ \theta_L , \frac{\pi}{2} \right]. \quad (6.14) \]

When the integral (6.4) is carried out, \( \Lambda_{q,K,E}(\theta) \) vanishes just in \( \theta_L \). It is sufficient to prove that, when \( \Lambda_{q,K,E}(\theta) \) is smaller than, say, \( r^2_+ / 4 \) the contribution to the integral (6.4) is bounded. So, we need just to find \( \tilde{\theta} \) such that, for the corresponding \( q, K, E \) with \( \theta_L \leq \tilde{\theta} \):

\[ \text{if} \quad \Lambda_{q,K,E}(\theta) \leq \frac{r^2_+}{4} \text{ with } \theta \in \left[ \theta_L , \frac{\pi}{2} \right] \quad \text{then} \quad \frac{d}{d\theta} \Lambda_{q,K,E}(\theta) \geq 1. \quad (6.15) \]

Using (6.3) and (6.10) it is easy to check:

\[ \Lambda_{q,K,E}(\theta) > \frac{K}{2q} - \frac{1}{q \sin^2 \theta}. \quad (6.16) \]

Therefore, as \( K/q > r^2_+ \) [from the first Eq. (6.1)], we have

\[ \text{if} \quad \Lambda_{q,K,E}(\theta) \leq \frac{r^2_+}{4} \text{ then} \quad \frac{1}{q \sin^2 \theta} > \frac{K}{4q} \quad (6.17) \]

and, again using (6.10), we obtain

\[ \frac{1}{q \sin^2 \theta} > \mathcal{M}(E, q) := \max \left\{ \frac{r^2_+}{4} , \frac{E^2 a^2}{2q} \right\}. \quad (6.18) \]

On the other hand, from (6.14)

\[ \frac{d}{d\theta} \Lambda_{q,K,E}(\theta) = - \left( 1 + \frac{E^2}{q} \right) a^2 \sin 2 \theta + \frac{2 \cos \theta}{q \sin \theta} \]

which, from (6.18), imply

\[ \frac{d}{d\theta} \Lambda_{q,K,E}(\theta) > - a^2 \sin 2 \theta + 2 \mathcal{M}(E, q) \left( - \sin 2 \theta + \frac{\cos \theta}{\sin \theta} \right). \quad (6.20) \]

In conclusion, from (6.13) if \( \tilde{\theta} \) is small the angles \( \theta \) satisfying (6.18) are small too and, thus, the right-hand side in (6.20) is greater than 1.

From (6.1), \( \lim_{S \rightarrow 0} q(r^*, S, E) = \lim_{S \rightarrow \infty} K(r^*, S, E) = \infty \) and, thus, using (6.5):

\[ \lim_{S \rightarrow 0} \theta_L(r^*, S, E) = \lim_{S \rightarrow \infty} K(r^*, S, E) = \infty \quad \text{and, thus, using (6.5)}: \]

\[ \lim_{S \rightarrow \infty} \theta_L(r^*, S, E) = 0. \]

So, fixed \( \tilde{\theta} \) and \( r_L^* \) in Lemma 6.2 we can define \( S(r^*, E) = \inf \{ S \geq 0: \theta_L(r^*, S, E) \leq \tilde{\theta} , \forall \tilde{S} > S \} \geq 0 \) for all \( r^* \, E \in (r_+ , r_L^*] \times (0, \infty). \) Recall that \( \theta_L(r^*, S, E) \) is differentiable [use \( (d/d\theta) \left( \Lambda_{q,K,E}(\theta) \right) > 0 \) because of (6.15) and, thus \( \theta_L(q, K, E) \) is differentiable]; moreover, \( S(r^*, E) \) is continuous [when \( \theta_L(r^*, S, E) = \tilde{\theta} \) from (6.5) and (6.1) \( d\theta_L/dS > 0 \)]. Recall also that \( S(r^*, E) \) has a finite supremum \( \mathcal{S}. \) In fact, from (6.11) and (6.12) the square brackets in (6.8) are lower bounded. So, for some big value \( S_0 \) of \( \mathcal{S} \), the expression \( K/2 - E^2 a^2 \) can be made arbitrarily big. Thus, from (6.5) and (6.3), \( \theta_L(r^*, S, E) \) will be less than \( \tilde{\theta} \), independently of \( (r^*, E) \), and \( S(r^*, E) < S_0 \) as required. Define now:

\[ D = \{ (r^*, S, E) \in \hat{D} : r^* \in (r_+ , r_L^*], S \in [S(r^*, E), \infty), E \in (0, \infty) \}. \quad (6.21) \]
Lemma 6.3: Consider the continuous function \( T(r^*,S,E) = \Delta t - (t_1 - t_0) \) defined on \( D \). There exist positive constants \( M_1(r^*) \), \( M_2(r^*) \) and \( m(r^*) \) such that
\[
E \leq m(r^*) \Rightarrow T(r^*,S,E) < 0,
\]
\[
M_1(r^*) \cdot \sqrt{S} + M_2(r^*) \leq E \Rightarrow T(r^*,S,E) > 0. \tag{6.22}
\]
Moreover, these constants can be chosen such that if either \( m(n^r) \to 0 \) or \( M_1(n^r) \to \infty \) or \( M_2(n^r) \to \infty \) for some sequence \( \{n^r\}_n \), then, necessarily, \( n^r \to r^+ \).

Proof of Lemma 6.3: Fix \( (r^*,S,E) \in D \), recalling the expression of \( \Delta t \) in (4.6) put:
\[
a [\theta(r)] + (r^2 + a^2) \frac{P(r)}{\Delta(r)} = \left[ \frac{(r^2 + a^2)^2}{(r - r_+)(r - r_-)} - a^2 \sin^2 \theta(r) \right] E - \frac{a(r^2 + a^2)}{(r - r_-)(r - r_+)} - a \tag{6.23}
\]
being \( \theta(r) \) computed from the second equation in (4.3) with constants \( (q(r^*,S,E),K(r^*,S,E),E) \). But each term on the right-hand side of (6.23) will be bounded, say:
\[
0 < r_1^2 < \frac{(r^2 + a^2)^2}{(r - r_-)(r - r_+)} - a^2 \sin^2 \theta(r) < \frac{(r_1^2 + a^2)^2}{(r^*-r_+)^2}, \quad r \in [r^*,r_1], \tag{6.24}
\]
and
\[
0 < \frac{a^3}{r_1^2} < \frac{a(r^2 + a^2)}{(r - r_-)(r - r_+)} - a < \frac{a(r_1^2 + a^2)}{(r^*-r_+)^2}, \quad r \in [r^*,r_1]. \tag{6.25}
\]
Now, put:
\[
m(r^*) = \frac{1}{2} \frac{a^3(r^* - r_+)^2}{r_1^2(r_1^2 + a^2)^2}. \tag{6.26}
\]
Thus, if \( E = m(r^*) \) then (6.23) is less than 0 and \( (\Delta t)(r^*,S,E) \leq 0 \leq t_1 - t_0 \), thus \( T(r^*,S,E) \leq 0 \) [see (4.6)]. Let
\[
M_1(r^*) = \sup_D \left\{ \frac{E}{\sqrt{S}} : T(r^*,S,E) \leq 0, S \geq 1 \right\},
\]
\[
M_2(r^*) = \sup_D \{ E : T(r^*,S,E) \leq 0, S(r^*,E) \leq S < 1 \} + 1.
\]
To prove (6.22) it is sufficient to prove \( M_1(r^*) < \infty \), \( M_2(r^*) < \infty \). But, if some of these inequalities do not hold then there exists a sequence of points \( \{(r^*_n,S_n,E_n)\}_n \subset D \) satisfying hypotheses of Lemma 4.6(ii) for the functions
\[
f_n(r) = \frac{1}{E_n} \left[ a \Delta_n(\theta_n(r)) + (r^2 + a^2) \frac{P_n(r)}{\Delta_n(r)} \right], \quad p_n(r) = \frac{h_n(r)}{E_n^n} \tag{6.27}
\]
with \( h_n(r) = h(r,q(r^*_n,S_n,E_n),K(r^*_n,S_n,E_n),E_n) \) in \( [r^*_n,r_1] \) and thus \( R_n = S_n/E_n^2 \) [see (6.24), (6.25), (6.1), and (6.2)]. Thus, the conclusion of Lemma 4.6(ii) contradicts \( T(r^*_n,S_n,E_n) \leq 0 \) for all \( n \) [see (4.6)].

For the last assertion, it is clear that \( m(r^*_n) \to 0 \) implies \( r^*_n \to r^+ \) [see (6.26)]. Even more, if either \( M_1(r^*_n) \) or \( M_2(r^*_n) \) go to \( \infty \) and, up to a subsequence, \( r^*_n \geq r^+ + \epsilon_0 \), \( \epsilon_0 > 0 \) then there exists a sequence of points \( \{(r^*_n,S_n,E_n)\}_n \subset D \) with \( T(r^*_n,S_n,E_n) \leq 0 \) and either \( S_n/E_n^2 \to 0 \) and \( S \geq 1 \) or \( 1/E_n^2 \to 0 \) and \( S(r^*_n,E_n) \leq S < 1 \). Then, by applying Lemma 4.6(ii) to (6.27) as before but, now, with the sequence \( \{(r^*_n,S_n,E_n)\}_n \subset D \) we obtain a contradiction again. \( \square \)
Remark 6.4: Notice that we can choose $\bar{\theta}$ in Lemma 6.2 small enough such that $\mathcal{D}$ in (6.21) satisfies, additionally:

(i) the component $S$ of the points $(r^*, S, E) \in \mathcal{D}$ with $r^* = r_L^*$ and $m(r_L^*) \leq E \leq M_1(r_L^*) + M_2(r_L^*)$ admits a positive lower bound [from (6.5) if $\theta_i$ is small then either $q$ or $K$ is big and, from (6.1) $S$ will be big too], and

(ii) if $(r^*, S, E) \in \mathcal{D}$ and $S/(r^* - r_+) \leq 1$ then

$$\frac{(r^* + a^2)^E - a}{(r^* - r_+)(r^* - r_+)} \geq \frac{2a}{r_+^2 + a^2}$$

[use (6.6)].

From now on we will assume that $\mathcal{D}$ is the domain fixed by this new value of $\bar{\theta}$, and Lemma 6.2 with Remark 6.4 will ensure that none of the problems 3BII - 3BIV explained in Sec. III. Summing up, it is sufficient to find an element of $\mathcal{D}$ such that the corresponding $(q, K, E)$ given from (6.1) and the function $h(r) = h(r, q, K, E)$ in (4.5) satisfy the equalities (4.11) with $\Delta t$, $\Delta [\theta]$, and $\Delta \varphi$ as in (4.6), (4.8), and (4.9), and with $L = 1$. For this aim, we establish the next two technical Lemmas. First, taking into account Lemma 6.3 define the subset

$$B = \{(r_L^*, S, E) \in \mathcal{D}: 0 < m(r_L^*) \leq E \leq M_1(r_L^*) \cdot \sqrt{S} + M_2(r_L^*)\}.$$

Lemma 6.5: (i) For $(r^*, S, E) \in \mathcal{D}$ then $((r^* + a^2)^E - a)^2/q$ and $E^2/q$ are upper bounded.

(ii) If also $(r_L^*, S, E) \in B$ then

$$(\Delta [\theta])(r_L^*, S, E) = \int_{r_L^*}^{r_0} \frac{\sqrt{\Lambda_{q, K, E}(\theta(r))}}{\sqrt{h(r)}} \frac{dr}{q} + \int_{r_L^*}^{r_1} \frac{\sqrt{\Lambda_{q, K, E}(\theta(r))}}{\sqrt{h(r)}} \frac{dr}{q}$$

is upper bounded too.

Proof of Lemma 6.5: (i) If $E \leq 1/a$ these terms are obviously bounded from (6.3); otherwise, first use (6.7) and (6.11) to prove

$$\nu q \geq b_1 \frac{E((r^* + a^2)^E - a)}{(r^* - r_+)}.$$

(ii) From Remark 6.4(i) the component $S$ of the points in $B$ admits a positive lower bound (see points $J'$, $I'$ in Fig. 2). Then, there exists $L_0 > 0$ such that

$$\frac{h'(r_L^*)}{q} = \frac{S}{q} > L_0 \quad (6.29)$$

for all $(r_L^*, S, E) \in B$ [see (6.1)]. But taking into account (6.1), (6.14) and (i) there exists $C_0 > 0$ such that

$$\sqrt{\Lambda_{q, K, E}(\theta(s))} < C_0 \quad (6.30)$$

along the geodesic $\gamma(s)$ corresponding to each $(r_L^*, S, E) \in B$. Summing up, inequalities (6.29), (6.30) joined with $h^{(k)}(r_L^*)/q \geq 0$ for $k \geq 2$ (see Lemma 6.2) imply that
(\Delta|\theta|)(r_n^*, S, E) = \int_{r_0}^{r_1} \sqrt{\frac{\Lambda_{\alpha, E}(\theta(r))}{q(r)}} dr + \int_{r_1}^{\infty} \sqrt{\frac{\Lambda_{\alpha, E}(\theta(r))}{q(r)}} dr

is upper bounded on B.

\textbf{Lemma 6.6:} For any sequence \( \{(r_n^*, S_n, E_n)\}_{n \in \mathbb{N}} \subset \mathcal{D} \) satisfying either \( r_n^* \searrow r_+ \) or \( S_n \searrow 0 \) then \( (\Delta|\theta|)(r_n^*, S_n, E_n) \to \infty \).

\textbf{Proof of Lemma 6.6:} This result is a consequence of the following two steps:

Step 1. For any such sequence \( \{(r_n^*, S_n, E_n)\}_{n \in \mathbb{N}} \subset \mathcal{D} \), the polynomials \( p_n(r) = h_n(r)/q_n \) satisfy hypotheses in Lemma 4.6(ii) in \( [r_n^*, r_1] \). In fact, \( p_n'(r_n^*) = S_n/q_n \to 0 \) [use that the square brackets in (6.7) is lower bounded and (6.3)] and, from (6.2) and Lemma 6.5(i), derivatives \( p_n^{(k)}(r_n^*) \) (\( \geq 0 \)) admit an upper bound for \( k \geq 2 \).

Step 2. For any sequence \( \{(r_n^*, S_n, E_n)\}_{n \in \mathbb{N}} \subset \mathcal{D} \) such that the polynomials \( p_n(r) = h_n(r)/q_n \) satisfy hypotheses in Lemma 4.6(ii) in \( [r_n^*, r_1] \) then \( (\Delta|\theta|)(r_n^*, S_n, E_n) \to \infty \). Otherwise, \( (\Delta|\theta|)(r_n^*, S_n, E_n) \) is bounded up to a subsequence. Using (6.14) and (4.5), the second equation in (4.3) is rewritten as

\[ e^{-} \sqrt{q_n} dr = \frac{d \theta}{\Lambda_{\alpha, k, E_n}(\theta)} \]

and using that \( (\Delta|\theta|)(r_n^*, S_n, E_n) \) is bounded:

\[ \int_{r_0}^{r_1} \sqrt{q_n} dr + \int_{r_1}^{\infty} \sqrt{q_n} dr = \int_{\theta_0}^{\theta_1} \frac{1}{\sqrt{\Lambda_{\alpha, k, E_n}(\theta)}} d \theta + 2n_0 \int_{\theta_1}^{\infty} \frac{1}{\sqrt{\Lambda_{\alpha, k, E_n}(\theta)}} d \theta \]

for some integer \( n_0 > 0 \). As \( \{(r_n^*, S_n, E_n)\}_{n \in \mathbb{N}} \subset \mathcal{D} \) for all \( n \), Lemma 6.2 implies that the second member of (6.31) is upper bounded. This contradicts Lemma 4.6(ii) applied to the functions \( f_n(r) = 1 \) and \( p_n(r) = h_n(r)/q_n \) in \( [r_n^*, r_1] \).

Next, consider the continuous functions on \( \mathcal{D} \), \( \Theta_n = \Delta|\theta| - (\theta_1 - \theta_0) - 2n(\pi - 2\theta_2) \), \( n \geq 0 \). Then, the following result on boundary conditions of \( T \) and \( \Theta_n \) on \( \mathcal{D} \) holds.

\textbf{Proposition 6.7:} Let \( \{\tilde{S}_n\}_{n \in \mathbb{N}} \) be a sequence with \( \tilde{S}_n > S = \text{Sup}\{S(r^*, E) : (r^*, E) \in (r_+, r_0) \times (0, \infty)\} \) for all \( n \). There exists \( \{\epsilon_n\}_{n \in \mathbb{N}} > 0, \epsilon_n \searrow 0 \) and \( \{\delta_n\}_{n \in \mathbb{N}}, \delta_n \nearrow 0, \delta_\infty \searrow 0, \) such that if

\[ \mathcal{D}_n = \left\{ (r^*, S, E) \in \mathcal{D} : r^* \in [r_+ + \epsilon_n, r_n^*], S \in [S(r^*, E), \tilde{S}_n], E \in \left[ \delta_n, \frac{1}{\delta_n} \right] \right\} \]

then

\[ B \cap \{(r^*, S, E) \in \mathcal{D}_n : r = r_n^*\} \subseteq \left\{ (r^*, S, E) \in \mathcal{D}_n : r = r_n^*, E \in \left[ \delta_n, \frac{1}{\delta_n} \right] \right\} \]

[for \( B \) in (6.28)], and

\[ T > 0 \quad \text{on} \quad \left( r^*, S, \frac{1}{\delta_n} \right) \in \mathcal{D}_n, \]

\[ T < 0 \quad \text{on} \quad (r^*, S, \delta_n) \in \mathcal{D}_n, \]

\[ \Theta_n > 0 \quad \text{on} \quad (r_+ + \epsilon_n, S, E) \in \mathcal{D}_n, \]

\[ \Theta_n > 0 \quad \text{on} \quad (r_+ + \epsilon_n, S, E) \in \mathcal{D}_n, \]
\[ \Theta_n < 0 \text{ on } (r^*, S, E) \in B \cap \{(r^*, S, E) \in D_n : r^* = r_L^* \} \]

for \( n \) big enough (see Fig. 2).

**Proof of Proposition 6.7:** From Lemma 6.5(ii), \( \Delta |\theta| \) is upper bounded on \( B \) and, thus, \( \Theta_n < 0 \) on \( B \) for \( n \) big enough.

On the other hand, there exists a sequence \( \{e_n\}_n \), \( e_n > 0 \), \( e_n \downarrow 0 \) such that \( \Theta_n > 0 \) when \( r^* = r_0 + e_n \). In fact, otherwise we obtain a sequence of points \( \{(r^*_i, S_j, E_i)\} \subseteq D \), \( r_i^* < r_0 + 1/11 \) with \( \Theta_n(r^*_i, S_j, E_i) \leq 0 \) for all \( i \). But these last inequalities contradict the conclusion of Lemma 6.6.

Finally, the sequence \( \{\delta_n\}_n \) is obtained from Lemma 6.3 applied to \( r^* \in [r_0 + e_n, r_L^*] \). \( \square \)

**Remark 6.8:** The choice of \( \{e_n\}_n \) in Proposition 6.7 does not depend on the sequence \( \{S_n\}_n \). Nevertheless, \( \{e_n\}_n \) does depend on \( \{\delta_n\}_n \).

Following step (4) (Sec. III), if we define the sequence of homeomorphisms \( \{z_n^{(m)}\}_m \), \( z_n^{(m)} : D_n^m \to [0,1]^3 \) with

\[ D_n^m = \left\{ (r^*, S, E) \in D_n : S \geq S(r^*, E) + \frac{1}{m} \right\} \]

as depicted in Fig. 3 then, from (6.34) and Lemma 6.3, the functions \( T \circ (z_n^{(m)})^{-1} \), \( \Theta_n \circ (z_n^{(m)})^{-1} \) satisfy

\[ T \circ (z_n^{(m)})^{-1} > 0 \text{ on } (r^*, S, E) \in [0,1]^2 \times \{1\}, \]

\[ T \circ (z_n^{(m)})^{-1} < 0 \text{ on } (r^*, S, E) \in [0,1]^2 \times \{0\}, \]

\[ \Theta_n \circ (z_n^{(m)})^{-1} > 0 \text{ on } (r^*, S, E) \in \{0\} \times [0,1]^2, \]

\[ \Theta_n \circ (z_n^{(m)})^{-1} < 0 \text{ on } (r^*, S, E) \in \{1\} \times [0,1]^2. \]

Now, we are in conditions to apply topological arguments based on Brouwer’s degree. In fact,

**Lemma 6.9:** There exists a connected subset \( z_n^{(m)}(C_n^m) \) of zeroes of \( T \circ (z_n^{(m)})^{-1} \) and \( \Theta_n \circ (z_n^{(m)})^{-1} \)

such that

\[ z_n^{(m)}(C_n^m) \cap ([0,1] \times \{0\} \times [0,1]) \neq \emptyset \text{ and } z_n^{(m)}(C_n^m) \cap ([0,1] \times \{1\} \times [0,1]) \neq \emptyset \]

for every \( m \in \mathbb{N} \).

**Proof of Lemma 6.9:** Consider the function

\[ \mathcal{F}^m_n : (0,1) \times [0,1] \times (0,1) \to X = \mathbb{R}^2, \]

\[ (r^*, S, E) \to (\Theta_n \circ (z_n^{(m)})^{-1}(r^*, S, E), T \circ (z_n^{(m)})^{-1}(r^*, S, E)) + (r^*, E) \]

and put \( \mathcal{F}^m_{n,0}(r^*, E) = \mathcal{F}^m_n(r^*, 0, E), \ G = (0,1)^2. \) By Ref. 32, Lemma 3.4 it is sufficient to prove that, because of (6.36):

\[ \deg(\text{Id} - \mathcal{F}^m_{n,0}, G, 0) \neq 0, \]

where \( \deg(\text{Id} - \mathcal{F}^m_{n,0}, G, 0) \) is the degree of the function \( \text{Id} - \mathcal{F}^m_{n,0} \) in the open subset \( G \) with respect to the value \( 0.30 \). But the affine map

\[ \hat{\mathcal{F}}^m : (0,1)^2 \to \mathbb{R}^2, \]

\[ (r^*, E) \to (1 - 2r^*, -1 + 2E) + (r^*, E) \]
has obviously \( \text{deg}(\text{Id} - \mathcal{F}^m_n, G, 0) \neq 0 \), and \( \text{deg}(\text{Id} - \mathcal{F}^m_n, G, 0) = \text{deg}(\text{Id} - \mathcal{F}^m_n, G, 0) \) [the map \( \lambda: \text{Id} - \mathcal{F}^m_n \mapsto \lambda(\mathcal{F}^m_n - \mathcal{F}^m_n) \), \( \lambda \in [0, 1] \) is a homotopy from \( \text{Id} - \mathcal{F}^m_n \) to \( \text{Id} - \mathcal{F}^m_n \) without zeroes on the boundary from (6.36)] which concludes the proof. \( \square \)

Up to a subsequence, \( \lim_{n \to \infty} f_n \mathcal{C}^m_n \neq \emptyset \) and thus, \( C_n = \limsup_{n \to \infty} \mathcal{C}^m_n \subset \mathcal{D} \), is connected (see Ref. 33 Chap. I, Theorem 9.1). By continuity \( T = \Theta_n = 0 \) in \( \mathcal{C} \) and \( \mathcal{D} \) and

\[
\mathcal{C} \cap \{(r^*, S, E) \in \mathcal{D} : S = S(r^*, E)\} \neq \emptyset \quad \text{and} \quad \mathcal{C} \cap \{(r^*, S, E) \in \mathcal{D} : S = \hat{S}_n\} \neq \emptyset.
\]

(see Fig. 4). This joined with the following result implies (i) and (ii) in (4.8 III):

**Lemma 6.10:** If \( (r^*, S, E) \in \mathcal{C}_n \) then \( S > 0 \) (in particular, \( \mathcal{C}_n \subset \mathcal{D}_n \)).

**Proof of Lemma 6.10:** Otherwise, there exists a sequence \( \{(r^*_i, S_i, E_i)\}_{i} \subset \mathcal{D}_n \subset \mathcal{D} \) in \( \mathcal{C}_n \) such that \( S_i \to 0 \). Then, from Lemma 6.6, we obtain a contradiction with \( \Theta_n = 0 \) in \( \mathcal{C}_n \cap \mathcal{D}_n \). \( \square \)

Finally, as \( \Delta \) and \( \Delta | \theta \) satisfy (4.11) for geodesics in each \( \mathcal{C}_n \) we only have to prove that one of such geodesics satisfies the required value for \( \Delta \varphi \). This will be straightforward from the following two lemmas:

**Lemma 6.11:** There exists a choice of the sequence \( \{\hat{S}_n\}_n \) in Proposition 6.7 such that for any \( \varepsilon > 0 \) points \( (r^*_n, \hat{S}_n, E_n) \in \mathcal{C}_n \) satisfy:

\[
2n \pi - (\Delta \varphi)(r^*_n, \hat{S}_n, E_n) < n \cdot \varepsilon 
\]  

(6.37) for \( n \) big enough.

**Proof of Lemma 6.11:** First, we will prove that for any sequence \( \{\hat{S}_n\}_n \) diverging fast enough there exists a constant \( \delta > 0 \) such that

\[
(\Delta \varphi)(r^*_n, \hat{S}_n, E_n) + \delta > \int_{r_0}^{r_1} \frac{1}{\sqrt{h_n(r)}} \sin^2 \theta_n(r) \, dr + \int_{r_0}^{r_1} \frac{1}{\sqrt{h_n(r)}} \, dr.
\]  

(6.38)

Thus, the right-hand side in (6.38) can replace \( (\Delta \varphi)(r^*_n, \hat{S}_n, E_n) \) in order to prove (6.37). To prove (6.38) note, taking into account (4.9):

\[
P_n(r) = \frac{\Gamma_n(\theta_n(r))}{\sin^2 \theta_n(r)} + \frac{\Gamma_n(r)}{\Delta(r)} = \frac{1}{\sin^2 \theta_n(r)} \left( \frac{a(r^2 + a^2)}{(r - r_+)(r - r_+)} - a \right) - \frac{a^2}{(r - r_+)(r - r_+)}. \]

(6.39)

If \( E_n > 1/a \) then (6.39) is positive. In the case \( E_n \leq 1/a \) note first that, from Remark 6.8, points \( (r^*_n, \hat{S}_n, E_n) \in \mathcal{C}_n \) satisfy \( r^*_n > r_+ + \varepsilon_n \), independently of \( \hat{S}_n \). Thus, for every \( n \) there exists \( \hat{S}_n \), big enough such that

\[
\int_{r_0}^{r_1} \frac{P_n(r)}{\sqrt{h_n(r)}} \, dr + \int_{r_0}^{r_1} \frac{P_n(r)}{\sqrt{h_n(r)}} \, dr > - \delta,
\]

and (6.38) is obtained.

Using the second equation in (4.3), the variable of integration \( r \) in (6.38) can be substituted by \( \theta \). Then, as \( (\Delta | \theta |)_{n} = \Theta_1 - \Theta_0 + 2n(\pi - \Theta_{L_{\infty}}) \), (6.38) can be written

\[
(\Delta \varphi) + \delta > \int_{\Theta_0}^{\Theta_1} \Omega_n(\theta) \, d\theta + 2n \int_{\Theta_{L_{\infty}}}^{\pi - \Theta_{L_{\infty}}} \Omega_n(\theta) \, d\theta,
\]

(6.40)

where
\[ \Omega_n(\theta) = \frac{1}{\sin^2 \theta} \sqrt{K_n + q_n a^2 \cos^2 \theta - \frac{\tan^2 \theta}{\sin^2 \theta}} \] (6.41)

Therefore, from (6.40) and the symmetry of the integrals

\[ \int_{\theta_{L,n}}^{\pi/2} \Omega_n(\theta)\,d\theta, \quad \int_{\pi/2}^{\pi - \theta_{L,n}} \Omega_n(\theta)\,d\theta \]

it is sufficient to prove

\[ \liminf_{n \to \infty} \int_{\theta_{L,n}}^{\pi/2} \Omega_n(\theta)\,d\theta \geq \frac{\pi}{2}. \] (6.42)

In fact, from an elemental integration:

\[ \int_{\theta_{L,n}}^{\pi/2} \frac{1}{\sin^2 \theta} \sqrt{\frac{1}{\sin^2 \theta} - \frac{1}{\sin^2 \theta}}\,d\theta = \frac{\pi}{2}, \] (6.43)

for all \( n \). But from the equality in (6.5)

\[ \frac{1}{\sin^2 \theta_{L,n}} = K_n + 2E_n a - E_n^2 a^2 + (q_n a^2 + E_n^2 a^2) \cos^2 \theta_{L,n} \]

\[ \geq K_n + 2E_n a - E_n^2 a^2 + (q_n a^2 + E_n^2 a^2) \cos^2 \theta \]

\[ = \frac{1}{\sin^2 \theta} + K_n + q_n a^2 \cos^2 \theta - \frac{\tan^2 \theta}{\sin^2 \theta} \] (6.44)

if \( \theta \in \left[ \theta_{L,n}, \pi/2 \right] \). Thus (6.42) is a consequence of (6.43) and (6.44).

As explained in (4BIII), each \( C_n \) contains a point with \( \theta_L = \bar{\theta} \). Thus, the following result will be applicable:

**Lemma 6.12**: Let \( \{(r_n^*, S_n^*, E_n^*)\}_n \) be a sequence of points, each one in the corresponding \( C_n \) with \( \theta_{L,n} = \theta_L(r_n^*, S_n^*, E_n^*) = \bar{\theta} > 0 \). Then, there exists \( \epsilon_0 > 0 \) such that

\[ 2n\pi - (\Delta \varphi)(r_n^*, S_n^*, E_n^*) > n \cdot \epsilon_0 \] (6.45)

for \( n \) big enough.

**Proof of Lemma 6.12**: First, we will prove

\[ h_n'(r_n^*) = S_n^* - 0 \] (6.46)

and then the limits:

\[ r_n^* \to r_+ , \quad E_n^* \to \frac{a}{r_+^2 + a^2}, \quad q_n^* \to q', \quad K_n^* \to q' r_+^2 , \] (6.47)
In order to prove (6.46) and taking into account that \( \Theta_n = 0 \) on \( C_n \) we have that

\[
(\Delta |\theta|)(r_n^* S_n^* E_n^*) = \theta_1 - \theta_0 + 2n(\pi - 2 \bar{\theta}) \to \infty \quad \text{when} \quad n \to \infty.
\]

(6.48)

By using (6.44) with \( \theta_{L,n} = \bar{\theta} \) the numerator in (4.8) is upper bounded by \( 1/\sin \bar{\theta} \) in our sequence, thus from (6.48)

\[
\int_{r_n^*}^{r_0} \frac{1}{\sqrt{h_n(r)}} dr + \int_{r_0}^{r_1} \frac{1}{\sqrt{h_n(r)}} dr \to \infty,
\]

(6.49)

which implies (6.46) (recall that from Lemma 6.2 the remainder of the derivatives of \( h_n \) are positive). Notice that we also obtain \( E_n^* \leq 1/a \) for big \( n \). In fact, otherwise computing directly from (4.2):

\[
a\int \theta_n(r) + (r^2 + a^2) \frac{\Delta r}{\Delta (r)} \geq a + \frac{r^2}{a} > 0,
\]

thus, from (4.6) and (6.49) we have \( (\Delta t)_n \to t_1 - t_0 \), in contradiction with \( (r_n^*, S_n^*, E_n^*) \in C_n \). In order to prove the first limit in (6.47) assume, by contradiction, \( r_n^* \geq r_+ + \delta_0 \) for some \( \delta_0 > 0 \) up to a subsequence. Then, from (6.46) we have

\[
\frac{S_n^*}{(r_n^* - r_+)} \to 0.
\]

This joined with Remark 6.4(ii) allows us to assume

\[
\left| \frac{(r_n^* + a^2)E_n^* - a}{(r_n^* - r_+)(r_n^* - r_-)} \right| > \frac{2a}{r_n^* + a^2}
\]

(6.50)

for \( n \) big enough. Even more, we can extend the lower bound in (6.50) through \( r_n^* \). More precisely, for \( E_n^* \in (0,1/a) \) and \( r \in [r_+ + \delta_0, r_0] \) function

\[
r \to \frac{(r^2 + a^2)E_n'-a}{(r-r_-)(r-r_+)}
\]

has a derivative which admits a bound independent of \( E_n' \). Thus, there exists \( \delta_0 > 0 \) such that

\[
\left| \frac{(r^2 + a^2)E_n'-a}{(r-r_-)(r-r_+)} \right| > \frac{3a}{2(r^2 + a^2)}, \quad \forall r \in [r_n^*, r_n^* + \delta_0];
\]

therefore

\[
a \int \theta_n(r) + (r^2 + a^2) \frac{\Delta r}{\Delta (r)} \geq \frac{a}{2}, \quad \forall r \in [r_n^*, r_n^* + \delta_0].
\]

(6.51)

We will prove that (6.51) implies the contradiction \( (\Delta t)_n \to |t_1 - t_0| \) and, thus, \( r_n^* \to r_+ \). In fact, assuming \( \delta_0 < r_0 - r_L^* \) Lemma 4.6(ii) is applicable to the expression of \( |\Delta t(r_n^*, S_n^*, E_n^*)| \) in (4.6) when the integrals are carried out in \( [r_n^*, r_n^* + \delta_0] \) [use (6.51), (6.46), and the facts that...
$E'_n = 1/a$ and $r^{*'}_n \geq r_r + \delta_0$, with (6.2), (6.1). As the integrals of the expression of $\Delta t(r^{*'}_n, S^{*'}_n, E'_n)$ in $[r^{*'}_n + \delta_0, r_r]$ and $[r^{*'}_n + \delta_0, r_+]$ are bounded [the numerator is bounded and the derivatives $h^{(k)}(r^{*'}_n)$ are positive with $h^{(4)}(r^{*'}_n) > 24 \mu_4 > 0$], the contradiction is obtained, and $r^{*'}_n \to r_+$.

For the convergence of $E'_n$ in (6.47), recall that

$$\left| \frac{(r^{*'}_n)^2 + a^2) E'_n - a}{(r^{*'}_n - r_+)} \right| \leq 6.52$$

is bounded. In fact, otherwise $q'_n$ and $K'_n + E'^2_n a^2$ go to $\infty$, up to a subsequence [see (6.7) and (6.8)]; this and (6.5) contradict that $\theta_{L,n} = \tilde{\theta}$ for all $n$. Therefore, as $r^{*'}_n \to r_+$, the second limit in (6.47) is obtained. Finally, for the other two limits, the boundedness of (6.52) and the convergence of $E'_n$ imply that $q'_n (r^{*'}_n)^2 - K'_n \to 0$ [see the first equation in (6.1)] which, joined to the fact that $\theta_{L,n} = \tilde{\theta}$ for all $n$, implies the two required limits [use the equality in (6.5)].

In order to prove (6.45) notice that the second limit in (6.47) implies

$$\int_{r^{*'}_n}^{r_+} 1 - E'_n a \sin^2 \theta_n(r) \sqrt{h_n(r)} \ dr + \int_{r^{*'}_n}^{r_+} 1 - E'_n a \sin^2 \theta_n(r) \sqrt{h_n(r)} \ dr \leq \frac{r^{*'}_n^2}{2(r_+ + a^2)} \left( \int_{r^{*'}_n}^{r_+} 1 \frac{1}{\sqrt{h_n(r)}} \ dr + \int_{r^{*'}_n}^{r_+} \frac{1}{\sqrt{h_n(r)}} \ dr \right),$$

which goes to $\infty$ by (6.49). But $(\Delta t)_n = t_1 - t_0$ for all $n$, thus from (6.53), (4.6) and the expression of $D(\theta)$ [formula (4.2) with $L = 1$]

$$\int_{r^{*'}_n}^{r_+} \frac{(r^2 + a^2) P_n(r)}{\Delta(r)} \frac{1}{\sqrt{h_n(r)}} \ dr + \int_{r^{*'}_n}^{r_+} \frac{(r^2 + a^2) P_n(r)}{\Delta(r)} \frac{1}{\sqrt{h_n(r)}} \ dr \to -\infty.$$ (6.54)

Obviously, from (6.54) and (4.9)

$$(\Delta \varphi)_n < \int_{r^{*'}_n}^{r_+} \frac{1 - E'_n a \sin^2 \theta_n(r)}{\sin^2 \theta_n(r)} \sqrt{h_n(r)} \ dr + \int_{r^{*'}_n}^{r_+} \frac{1 - E'_n a \sin^2 \theta_n(r)}{\sin^2 \theta_n(r)} \sqrt{h_n(r)} \ dr.$$ (6.55)

If the variable of integration $r$ in (6.55) is changed by $\theta$ (as in Lemma 6.11) and taking into account that now $(\Delta \theta)_n = \theta_1 - \theta_0 + 2n(\pi - 2 \tilde{\theta})$, (6.55) can be written

$$(\Delta \varphi)_n < \int_{\theta_0}^{\theta_1} W_n(\theta) d\theta + \int_{\theta_0}^{\theta_1} W_n(\theta) d\theta,$$ (6.56)

where

$$W_n(\theta) = \frac{1}{\sin^2 \theta} \sqrt{K'_n + q'_n a^2 \cos^2 \theta \left( 1 - E'_n a \sin^2 \theta \right)^2 - \frac{1}{\sin^2 \theta}}.$$ (6.57)

Then, taking the limit $W(\theta)$ of $\{W_n(\theta)\}_n$, we have from (6.47):
\[ W(\theta) = \lim_{n \to \infty} W_n(\theta) = \frac{1}{\sin^2 \theta} \sqrt{\frac{q'}{r_+^2 + a^2 \cos^2 \theta} - \frac{1}{\sin^2 \theta}}. \] (6.57)

Notice that
\[ \frac{d}{d\theta} \frac{(r_+^2 + a^2)^2}{r_+^2 + a^2 \cos^2 \theta} = \frac{a^2(r_+^2 + a^2)^2 \sin 2\theta}{(r_+^2 + a^2 \cos^2 \theta)^2} \begin{cases} 
\geq 0, & \theta \in \left[0, \frac{\pi}{2}\right] \\
\leq 0, & \theta \in \left[\frac{\pi}{2}, \pi\right].
\end{cases} \] (6.58)

and the denominator of (6.57) vanishes just in \( \theta, \pi - \theta \). As \( \{W_n\}_n \) is a sequence of dominated functions, \( \lim_n \int_{\theta}^{\pi - \theta} W_n(\theta) d\theta = \int_{\theta}^{\pi - \theta} W(\theta) d\theta \), and from (6.56):
\[ (\Delta \varphi)_n < (2n + 1) \int_{\theta}^{\pi - \theta} W(\theta) d\theta, \] (6.59)

for \( n \) big enough. But from (6.58), if \( \cos \theta \) is replaced by \( \cos \bar{\theta} \) in (6.57):
\[ \int_{\theta}^{\pi - \theta} W(\theta) d\theta < \int_{\theta}^{\pi - \theta} \frac{1}{\sin^2 \theta} \sqrt{\frac{q'}{r_+^2 + a^2 \cos^2 \theta} - \frac{1}{\sin^2 \theta}} d\theta = \pi. \]

Thus, the proof concludes putting \( \int_{\theta}^{\pi - \theta} W(\theta) d\theta = \pi - \epsilon_0 \) in (6.59).

From Lemmas 6.11 and 6.12, there exist two points in a \( C_n \) for some (big) \( n \) whose difference in \( \Delta \varphi \) is greater than \( 2\pi \), which implies the existence of \( (r^\pm, S, E) \in C_n \) such that (4.11) holds, as required.

**VII. CONCLUSION**

We have proven a relevant geometric property of outer Kerr space–time \( K \) (\( a^2 < m^2 \)), the geodesic connectedness. The proof uses essentially topological arguments. These arguments are applicable to most of the classical relativistic space–times (Schwarzschild, Robertson–Walker, Reissner–Nordström, etc.).

The difficulty to study \( K \) seems to arise from the following facts:

1. No Killing vector field of \( K \) is timelike; otherwise, the problem would be reducible to a “Riemannian” problem, where variational methods yields very precise results.

2. \( K \) can be seen as a splitting type manifold such as those studied by using variational methods and Rabinowitz’s saddle point theorem in Ref. 34. These results are especially appropriate to study globally hyperbolic space–times under a splitting with complete Cauchy hypersurfaces \( t = \text{constant} \). But no clear choice of the time function \( t \) seems to be natural to apply such techniques for \( K \).

3. At any case, the role of the event horizon \( r = r_+ \) seems to be unavoidable (recall that, under our approach, geodesics approaching the event horizon play an essential role). It is known that the convexity of the boundary of a semi-Riemannian manifold sometimes yields the geodesic connectedness of the manifold, especially in the Riemannian case.\(^{18}\) The boundary \( r = r_+ \) of \( K \) is singular, and the approaching hypersurfaces \( r = r_+ + \nu, \nu > 0 \), are not convex (regions \( r > r_+ + \nu \)
are not geodesically connected. In the Riemannian case, there are techniques to measure if the lack of convexity goes to zero, when \( n \to 0 \), yielding geodesic connectedness. In the static case, some of these techniques are translatable, and geodesic connectedness of some space–times with singular boundary, including outer Schwarzschild, have been proven. But none of these techniques seem applicable to a nonstationary situation.

Thus, our method circumvents previous difficulties.

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27We point out that there is a recent book (Ref. 31) devoted systematically to the geometric study of Kerr space–time, which we refer as a basic reference.
28If \( M \) is not included in \( \mathbb{R}^n \), assume that suitable coordinates can be chosen. Even though Kerr space–time is an open subset of \( \mathbb{R}^4 \), arbitrary values for coordinates \( \theta, \phi \) will be permitted for convenience; thus, an extension of the idea underlying in (4.1) is used.
29Recall that \( x(r) - x(r') \leq 0 \) for all \( r, r' \) and thus \( r \) is a valid new parameter.
30Recall that when \( f \in C^1(a,b) \cap C[a,b] \), \( f(a) \neq 0 \neq f(b) \) and \( f'(x) \neq 0 \) if \( f(x) = 0 \) then \( \deg(f,a,b,0) \)
\[= \sum_{s=1}^{\infty} \int_{a_-}^{a_+} \text{sign}(f)(x) \quad \text{In the standard notation of Ref}. \ 32, \deg(Id - \mathcal{F}_{\mathbb{R}^n}^0, G, 0) \text{ is written } i_{\mathcal{F}_{\mathbb{R}^n}^0}(G). \]