Global attractivity in Goodwin’s oscillator with finite delay

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Abstract

In the paper, several delay dependent sufficient conditions are established for the global stability of the positive steady state of a system of delay differential equations which generalizes the Goodwin oscillator system. We improve some previous results.

\textit{Key words:} Functional differential equations, global attractor, one-dimensional maps, Goodwin oscillator

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1 Introduction

In this paper we will be concerned with the following n-dimensional system of delay differential equations

\begin{equation}
\begin{align*}
x'_1(t) + a_1 x_1(t) &= b_1 [1 + x_n(t - h)]^{-1} \\
x'_i(t) + a_i x_i(t) &= b_i \int_{-h}^{0} x_{i-1}(t + s)dk(s), \quad i = 2, \ldots, n
\end{align*}
\end{equation}

where \( a_i, b_i > 0 \) for \( i = 1, \ldots, n \), \( \mu \in \mathbb{N} \) and \( k \) is a nonconstant nondecreasing function with \( k(-h) = 0 \). For simplicity we assume that \( \int_{-h}^{0} dk(s) = 1 \).

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It is easy to see that System (1) has a unique positive equilibrium \((z_1, \ldots, z_n)\), determined by the equations \(g(z_n) = z_n, z_{i-1} = a_i b_i^{-1} z_i, i = 2, \ldots, n\), being

\[
g(x) = (b_1 \ldots b_n)(a_1 \ldots a_n)^{-1} f(x),
\]

where \(f(x) = 1/(1 + x^\mu)\) is strictly decreasing on \((0, +\infty)\).

This system is a generalization of the celebrated system proposed by Goodwin [3] to model certain regulatory mechanisms in cellular physiology. The original model by Goodwin was a three-dimensional system of ordinary (without delay) differential equations. However, the number of linked reactions can be greater than three and this leads to an \(n\)-dimensional system

\[
x_1'(t) + a_1 x_1(t) = b_1 [1 + x_n(t)^\mu]^{-1}, \quad x_i'(t) + a_i x_i(t) = b_i x_{i-1}(t), \quad i = 2, \ldots, n.
\]

For \(n = 2\), this system does not admit any periodic solution and the unique positive equilibrium is globally asymptotically stable. However, the introduction of delays, as it was suggested by Goodwin, makes possible that, even for \(n = 2\), periodic solutions can appear. MacDonald [9] proposed a system with delay in the form

\[
x_1'(t) + a_1 x_1(t) = b_1 [1 + x_n(t - h)^\mu]^{-1}
\]

\[
x_i'(t) + a_i x_i(t) = b_i x_{i-1}(t - h), \quad i = 2, \ldots, n.
\]

Banks and Mahaffy [1] prove that for \(\mu = 1\) the introduction of delays does not produce periodic solutions since in that case the equilibrium is globally asymptotically stable. Later, Mahaffy [11] found a range of values of the parameters \(a_i, b_i, \mu\) for which sufficiently large delays induce periodic solutions in (3).

We restrict our attention to the problem of finding sufficient conditions for the global attractivity of the unique positive equilibrium in (3); (with more generality, in (1)), improving in this direction condition given in [1]. In relation to this problem, Kerscher and Nagel [7] proved that condition

\[
\frac{a_1 \ldots a_n}{b_1 \ldots b_n} > \frac{\mu - 1}{\mu} (\mu - 1)^{1/\mu}
\]

is sufficient to assure that the equilibrium in (3) is locally exponentially stable (in fact, they consider a more general system, replacing \(x_i(t - h)\) by \(\int_0^h x_i(t + h(s)) d\eta_i(s)\), being \(\eta_i\) a normalized positive measure and \(h : [-1, 0] \rightarrow \mathbb{R}^+\)). Hence condition (4) is a delay-independent condition for the local asymptotic stability of \((z_1, \ldots, z_n)\). In [8] we show that condition (4) in fact ensures the global
asymptotical stability in (1), also allowing the case of infinite delay. Let us note that in particular (4) is always true for \( \mu = 1 \). Comparing this condition with the results in [11] we can conclude, roughly speaking, that for small values of \((a_1 \ldots a_n)(b_1 \ldots b_n)^{-1}\) nonconstant periodic solutions of (3) cannot exist, whereas for \((a_1 \ldots a_n)(b_1 \ldots b_n)^{-1}\) large there exist values of \( h \) for which (3) has a nontrivial periodic solution [11, Corollary 5.2].

On the other hand, we emphasize that the analogue one-dimensional of (1)

\[
x'(t) + ax(t) = b [1 + x(t - h)]^{-1}
\]

(5)

is the Mackey-Glass equation [10]. Gopalsamy et al. [4] found delay-dependent sufficient conditions for the global attractivity in Eq. (5), improving condition (4) for \( n = 1 \). Furthermore, their condition matches (4) when \( h \to +\infty \).

The aim of this paper is to derive a similar result for System (1), using some arguments based on the negativity of the Schwarz derivative of function \( f \). In fact, we establish three different methods to obtain an interval of values of the delay parameter \( h \) for which the equilibrium in (1) is a global attractor. These methods are illustrated in the final part of our paper for a particular case of system (1).

We recall that the Schwarz derivative \((Sf)(x)\) of a real smooth function \( f \) at the point \( x \) is defined as \((Sf)(x) = f''(x)(f'(x))^{-1} - 3/2 \left( f''(x)(f'(x))^{-1}\right)^2\). The next proposition, which can be easily deduced from the Singer's results [2], will be very important in the sequel.

**Proposition 1** Assume that the function \( f : [a, b] \to [a, b], f \in C^3[a, b] \), is strictly decreasing. If \( f \) has a unique fixed point \( x^* \) in \([a, b]\) which is locally asymptotically stable and \((Sf)(x) < 0\) for all \( x \in [a, b] \), then \( x^* \) is the global attractor of \( f \).

Lemma 1 in [4] proves that \((Sg)(x) < 0\) for all \( x > 0 \) if \( \mu > 1 \), being \( g \) the function defined in (2). Since for \( \mu \leq 1 \) we have the global attractivity of \((z_1, \ldots, z_n)\) independently of the delay, we shall only consider here the case \( \mu > 1 \).

First of all, we indicate some important differences existing between Goodwin's system and scalar equation (5) and which can complicate our studies in the multidimensional case. Set, for example, \( n = 2 \) and \( h = 1 \) in (3) and consider the variational equations along positive steady states to Eq. (5) and System (3) respectively. Evidently, the first variational equation has the characteristic equation of the form \( z + a + c \exp(-z) = 0 \), \( a, c > 0 \), which has the pure imaginary root \( i\omega, \omega > 0 \) only if \( \omega > \pi/2 \). This means that the linearized equation could have a trigonometric periodic solution with 4 as the maximum period only. As a consequence, the distance between every point of local maximum of this trigonometric solution and its nearest zero always is less than \( h = 1 \), the well-known and very
immediate fact. Unfortunately, this is not the case for the multi-dimensional system: we can observe this fact investigating the following characteristic equation for linearized Eq. (3)

$$z^2 + (a_1 + a_2)z + a_1a_2 + d\exp(-2z) = 0, \quad d > 0.$$  

For example, if $$a_1 = a_2 = a$$, this equation has an imaginary root $$z = i\omega$$ if $$\cos 2\omega = 1 - 2a^2d^{-1}$$ and $$\omega^2 = d - a^2$$. Since the equation $$\cos 2\omega = 1 - 2a^2(\omega^2 + a^2)^{-1}$$ has arbitrary small zeros for small values of $$a$$, the equation linearized along steady state of (3) generally can have trigonometric solutions with arbitrary large period and this period depends on the coefficients $$a_i, b_i$$. This means that the distance $$\Delta$$ between a critical point and the nearest zero of the periodic solutions to Eq. (3) can be arbitrarily large for a fixed $$h$$ and depends, in general, on the coefficients of the system. On the other hand, to obtain an analog of results from [4], we need to estimate this $$\Delta$$. Below we will show how this estimation could be realized.

### 2 Main Results

For the sake of simplicity, we shall prove our results in this section for System (1) with $$n = 2$$. The generalization to the $$n$$-dimensional case, $$n > 2$$, could be obtained following similar arguments.

The unique bounded solution of System (1) with initial conditions $$x_i(s) = \phi_i(s), \quad s \in [-h, 0], \quad \phi_i \in \mathbb{C}^+ = \mathbb{C}([-h, 0], \mathbb{R}^+), \quad i = 1, 2$$, is attracted by a compact invariant $$\omega$$-limit set $$\omega(\phi) \subset \mathbb{C}^+ \times \mathbb{C}^+$$ (see [8]). Let $$(z_1, z_2)$$ denote the unique steady state of Eq. (1) and $$p_i : \mathbb{C}^+ \times \mathbb{C}^+ \rightarrow \mathbb{C}^+$$ denote the projection on the $$i$$-th copy of $$\mathbb{C}^+$$. It is a simple exercise to verify that either $$\omega(\phi) = (z_1, z_2)$$ or $$p_i(\omega(\phi)) \neq z_i$$ for $$i = 1, 2$$. Moreover, in the latter case, $$M_i = \max_{\psi \in \omega_i} \max_{s \in [-h, 0]} p_i(\psi)(s) > z_i$$ and

$$m_i = \min_{\psi \in \omega_i} \min_{s \in [-h, 0]} p_i(\psi)(s) < z_i.$$  

Define $$q = \max\{M_1 - z_1, z_1 - m_1\}$$ and let $$(x_1(t), x_2(t))$$ be a solution of (1) such that $$x_1(0) = M_1$$. Assuming below $$q = M_1 - z_1 > 0$$ (the case $$q = z_1 - m_1 > 0$$ is analogous) we get $$q \in (0, b_1a_1^{-1} - z_1]$$ because of $$M_1 - z_1 \leq b_1a_1^{-1}f(0) - z_1 = b_1a_1^{-1} - z_1$$. Next, $$a_2(z_2 - m_2) \leq b_2(z_1 - m_1) \leq b_2q$$, from where

$$m_2 \geq z_2 - b_2a_2^{-1}q.$$  

Set

$$\lambda = 2h - \frac{1}{a_2}\ln \left( \min \left\{ 1, \frac{a_1a_2}{b_1b_2} \inf_{u > 0} \frac{u - z_2}{f(z_2) - f(u)} \right\} \right)$$  

(7)
and \( \theta = b_2 a_2^{-1} \exp\{-a_2(\lambda - 2h)\} \). Evidently \( \lambda \geq 2h \). Next, from the definition of \( \lambda \) and \( \theta \) it is easy to prove that \( f^{-1}(a_1 b_1^{-1}(z_1 + q)) \leq z_2 - \theta q \) for every \( q \in (0, b_1 a_1^{-1} - z_1) \). Using this fact, we will prove that \( x_1(s) = z_1 \) for some \( s \in (-\lambda, 0) \).

Indeed, in the opposite case \( x_1(t) > z_1, t \in (-\lambda, 0) \). Since \( x'_1(0) = 0 \), we have that

\[
x_2(-h) = f^{-1}(a_1 b_1^{-1}(z_1 + q)).
\]

On the other hand, since \( x_1(t) > z_1 \) for all \( t \in (-\lambda, 0) \), we obtain that \( x'_2(t) + a_2 x_2(t) > b_2 z_1 = a_2 z_2 \) for all \( t \in (-\lambda + h, 0) \). Thus \( x_2(t) > z_2 + (x_2(-\lambda + h) - z_2) \exp(-a_2(t + \lambda - h)) \) for all \( t \in (-\lambda + h, 0) \). From (6), we have \( x_2(-\lambda + h) \geq m_2 \geq z_2 - b_2 a_2^{-1} q \), that implies

\[
x_2(t) > z_2 - b_2 a_2^{-1} q \exp(-a_2(t + \lambda - h)), \quad t \in (-\lambda + h, 0).
\]

Finally, setting \( t = -h \) in (9) and combining this inequality with (8), we get

\[
f^{-1}(a_1 b_1^{-1}(z_1 + q)) = x_2(-h) > z_2 - b_2 a_2^{-1} q \exp(-a_2(\lambda - 2h)) = z_2 - \theta q,
\]

a contradiction with the choice of \( \lambda \).

To end this part, we notice that for \( q = z_1 - m_1 \), we get a similar contradiction, taking into account that the definitions of \( \lambda \) and \( \theta \) also ensure that \( f^{-1}(a_1 b_1^{-1}(z_1 - q)) \geq z_2 + \theta q \) for every \( q \in (0, z_1) \).

Next we use the previous considerations to prove the following result:

**Theorem 2** Define \( \lambda \) as in (7). The positive equilibrium \((z_1, z_2)\) of (1) is the global attractor if

\[
(1 - e^{-a_1 \lambda}) g'(z_2) > -1,
\]

**PROOF.** We shall use the notations of previous reasoning and assume that \( q = M_1 - z_1 \). Since \( x_1(0) = M_1 \) and \( x_1(-\sigma) = z_1 \) for some \( \sigma \in [0, \lambda] \), we have

\[
M_1 = e^{-a_1 \sigma} z_1 + b_1 \int_{-\sigma}^{0} e^{a_1 u} f(x_2(u - h)) du
\]

\[
\leq e^{-a_1 \lambda} z_1 + (1 - e^{-a_1 \lambda}) b_1 a_1^{-1} \max_{t \in [m_2, M_2]} f(t).
\]
Now, since \( x(t) = (x_1(t), x_2(t)) \) belongs to an \( \omega \)-limit set of system (1), we can obtain the following integral representation:

\[
x_2(t) = b_2 \int_{-\infty}^{t} e^{-a_2(t-s)} \left( \int_{-h}^{0} x_1(s + u) du \right) ds.
\]

Thus we have \( M_2 \leq b_2 a_2^{-1} M_1 \) and \( m_2 \geq b_2 a_2^{-1} m_1 \).

In consequence, \([a_2 b_2^{-1} m_2, a_2 b_2^{-1} M_2] \subset [m_1, M_1]\), and therefore, setting \( u = a_2 b_2^{-1} t \) in (11), we have \( M_1 \leq e^{-a_1 \lambda} z_1 + (1 - e^{-a_1 \lambda}) b_1 a_1^{-1} \max_{u \in [m_1, M_1]} f(b_2 a_2^{-1} u) \). Analogously, \( m_1 \geq e^{-a_1 \lambda} z_1 + (1 - e^{-a_1 \lambda}) b_1 a_1^{-1} \min_{u \in [m_1, M_1]} f(b_2 a_2^{-1} u) \), and, consequently,

\[
[m_1, M_1] \subset \zeta([m_1, M_1]), \quad \text{where} \quad \zeta(x) = e^{-a_1 \lambda} z_1 + (1 - e^{-a_1 \lambda}) b_1 a_1^{-1} f(b_2 a_2^{-1} x). \tag{12}
\]

It is obvious that \( \zeta(z_1) = z_1 \) iff \( g(z_2) = z_2 \). Then the attractivity properties of the equilibrium of (1) can be reduced to the attractivity properties of the equilibrium \( z_1 \) of the discrete dynamical system defined by \( \zeta \).

Since Schwarzian of \( \zeta \) is negative, we can assure that the condition (10) is sufficient for the global attractivity of the fixed point \( z_1 \) of equation \( x = \zeta(x) \). Therefore, it follows from (12) that \( m_1 = M_1 = z_1 \) and hence \( m_2 = M_2 = z_2 \), showing that \( (z_1, z_2) \) is the global attractor in (1).

As immediate consequences of Theorem 2, we have the following corollaries:

**Corollary 3** The positive equilibrium \((z_1, z_2)\) of (1) is the global attractor if one of the following conditions is verified

\[\begin{align*}
& (i) \quad \mu \leq (1 - e^{-a_1 \lambda})^{-1} \\
& (ii) \quad \mu > (1 - e^{-a_1 \lambda})^{-1} \text{ and the following inequality holds:} \\
& a_1 a_2 (b_1 b_2)^{-1} > \left( 1 - \frac{1}{\mu(1 - e^{-a_1 \lambda})} \right) (\mu(1 - e^{-a_1 \lambda}) - 1)^{1/\mu}, \tag{13}
\end{align*}\]

where \( \lambda \) is given by (7).

**PROOF.** We shall prove that condition (10) is satisfied if (i) or (ii) hold.

Let us denote \( B = b_1 b_2 (a_1 a_2)^{-1} \). We observe that \( g(z_2) = z_2 \) if and only if \( z_2^{\mu+1} + z_2 = B \). Hence (10) is equivalent to the relation

\[
(1 - e^{-a_1 \lambda}) B \mu z_2^{\mu-1} (1 + z_2^2)^{-2} < 1 \tag{14}
\]
Since \( 1 + z_2^\mu = Bz_2^{-1} \) and \( z_2^{\mu - 1} = (B - z_2)z_2^{-2} \), condition (14) becomes \( B > (1 - e^{-a_1\lambda})\mu(B - z_2) \) or, equivalently,
\[
(1 - e^{-a_1\lambda})\mu z_2 > B((1 - e^{-a_1\lambda})\mu - 1).
\]
Clearly (15) is true whenever \((1 - e^{-a_1\lambda})\mu - 1 \leq 0\), that is, if condition (i) in the statement of the corollary holds. In the following, let us assume that \((1 - e^{-a_1\lambda})\mu > 1\). In this case, condition (15) reads
\[
z_2 > B \left(1 - ((1 - e^{-a_1\lambda})\mu)^{-1}\right) = z^* > 0.
\]
Denote \( F(z) = z^{\mu+1} + z - B \). Since \( F(0) = -B < 0 \) and \( z_2 \) is the unique positive solution of equation \( F(z) = 0 \), we have that \( z_2 > z^* \) if and only if \( F(z^*) < 0 \). Now, using the expression (16) it is easy to see that condition \( F(z^*) < 0 \) is equivalent to condition (13).

**Remark 4** We note that condition (13) is better than conditions obtained in previous works ([7,8,12]). In fact, when \( \lambda \to +\infty \), this condition is precisely the same obtained in [7] for the local exponential stability of (1) and in [8] when delay is infinite. (Compare with (4) when \( n = 2 \)).

The following corollary is a straightforward consequence of Theorem 2:

**Corollary 5** The positive equilibrium \((z_1, z_2)\) of (1) is the global attractor for all \( h \in [0, h^0) \), where
\[
h^0 = \frac{-1}{2a_1} \ln \left( \beta^{-(a_1/a_2)} \left[ 1 + \frac{1}{g'(z_2)} \right] \right)
\text{ and } \beta = \min \left\{ 1, \frac{a_1a_2}{b_1b_2} \inf_{u > 0} \frac{u - z_2}{f(z_2) - f(u)} \right\}.
\]

In the following proposition, which gives other sufficient conditions for the global attractivity in Eq. (1), we will use the auxiliary functions \( \sigma_1(x, \rho) = (\zeta \circ w)(x, \rho) \) and \( \sigma_2(x, \rho) = (w \circ \zeta)(x, \rho) \), where \( \rho \in (0, 1) \), \( \zeta(x, \rho) = (1 - \rho)z_1 + \rho b_1a_1^{-1}f(b_2a_2^{-1}x) \), and \( w(x, \rho) = b_1a_1^{-1}f(\delta x + \delta_1) \), being \( \delta = \rho b_2a_2^{-1} \), \( \delta_1 = (1 - \rho)z_2 \).

**Theorem 6** Assume that, for some \( \rho^* \in (0, 1) \),
\[
\rho^*g'(z_2) > -1, \text{ and } 0 < h \leq h^* = (1/2) \left[ a_2^{-1} \ln(\rho^*) - a_1^{-1} \ln(1 - \rho^*) \right] .
\]
Then the steady state solution of Eq. (1) is globally attracting if one of the following conditions is satisfied:
(a) \( f''(z_2) \leq 0 \) and \( \sigma_1(x, \rho^*) \neq x \) for all \( x \in [0, z_1) \).
(b) \( f''(z_2) \geq 0 \) and \( \sigma_2(x, \rho^*) \neq x \) for all \( x \in [0, z_1) \).

PROOF. First we prove that the result is true for \( h = h^* \). For the above indicated value of \( h^* = h(\rho^*) \), the equation \( \exp(-a_2(\epsilon - 2h^*)) = (1 - \exp(-a_1\epsilon)) = \rho^* \) has a unique positive solution \( \epsilon = -a_1^{-1}\ln(1 - \rho^*) = 2h^* - a_2^{-1}\ln \rho^* > 2h^* \). Let us fix this value of \( \epsilon \). For short, we will write in this case \( \zeta(x) = \zeta(x, \rho^*), w(x) = w(x, \rho^*), \sigma_i(x) = \sigma_i(x, \rho^*), i = 1, 2 \).

Next, suppose again that

\[
M_i = \max_{\psi \in \omega(\psi_i)} \max_{s \in [-h, 0]} p_i(\psi)(s) > z_i \quad \text{and} \quad m_i = \min_{\psi \in \omega(\psi_i)} \min_{s \in [-h, 0]} p_i(\psi)(s) < z_i.
\]

Thus we can indicate a solution \((x_1(t), x_2(t))\) to Eq. (1) such that \( x_1(0) = M_1 > z_1 \) and \( x_1(s) > z_1 \) for all \( s \in (-\lambda_1, 0) = \Lambda_1 \). Analogously, there exists a solution \((y_1(t), y_2(t))\) to Eq. (1) such that \( y_1(0) = m_1 < z_1 \) and \( y_1(s) < z_1 \) for all \( s \in (-\lambda_2, 0) = \Lambda_2 \). Now, we can meet only the following four possibilities for \( \lambda_i \): i) \( \lambda_i \leq \epsilon \), ii) \( \lambda_i \geq \epsilon > 2h \), iii) \( \lambda_i \leq \epsilon \leq \lambda_2 \) and iv) \( \lambda_2 \leq \epsilon \leq \lambda_1 \).

In the first case, proceeding analogously to the proof of Theorem 2, we get again the chain of inclusions (12), where \( \lambda = \epsilon \). Since \( \zeta'(z_1) = \rho^* g'(z_2) > -1 \), the global stability in Eq. (1) follows from Proposition 1.

Assume now ii). Since \( x_1'(0) = 0 \), we have \( x_2(-h) = f^{-1}(a_1 b_1^{-1} M_1) \). On the other hand, since \( x_1(t) > z_1 \) for all \( t \in \Lambda_1 \), we obtain that \( x_2(t) + a_2 x_2(t) > b_2 z_1 = a_2 z_2 \) for all \( t \in (-\epsilon + h, -h) \). Hence, the inequality \( x_2(-\epsilon + h) \geq m_2 \geq b_2 a_2^{-1} m_1 \) implies that

\[
x_2(-h) \geq (b_2 a_2^{-1} m_1 - z_2) \exp(-a_2(\epsilon - 2h)) + z_2.
\]

Finally, we have that \( f^{-1}(a_1 b_1^{-1} M_1) \geq (b_2 a_2^{-1} m_1 - z_2) \exp(-a_2(\epsilon - 2h)) + z_2 \), that is, \( M_1 \leq b_1 a_1^{-1} f(\delta m_1 + \delta_1) \). Analogously, considering the solution \((y_1(t), y_2(t))\), we obtain that \( m_1 \geq b_1 a_1^{-1} f(\delta M_1 + \delta_1) \), and therefore \( [m_1, M_1] \subset w([m_1, M_1]) \). Since \( w \) is a monotonically decreasing function with the negative Schwarz derivative and \( w(z_1) = z_1 \), the inequality \( w'(z_1) = \rho^* g(z_2) > -1 \) implies that \( m_1 = M_1 = z_1 \).

Now, to consider iii) or iv), we have to combine estimations from i) and ii).

First we prove that in the case iii) we have the global stability of Eq. (1) if either \( f''(z_2) \leq 0 \) or \( \sigma_2 \) does not have fixed points in \([0, z_1)\). Indeed, in this case we have \( m_1 \geq w(M_1) \) together with

\[
M_1 \leq z_1 \exp(-a_1 \lambda_1) + b_1 a_1^{-1}(1 - \exp(-\lambda_1 a_1)) f(b_2 a_2^{-1} m_1) \leq
\]

\[
\leq z_1 \exp(-a_1 \epsilon) + b_1 a_1^{-1}(1 - \exp(-\epsilon a_1)) f(b_2 a_2^{-1} m_1) = \zeta(m_1).
\]

Since \( \zeta, w \) are decreasing, the compositions \( \sigma_1 = \zeta \circ w, \sigma_2 = w \circ \zeta \) are increasing. Moreover, \( M_1 \leq \sigma_1(M_1), m_1 \geq \sigma_2(m_1), \sigma_i(z_1) = z_1 \). When \( \sigma_i'(z_1) = (\rho^* g(z_2))^2 < \)
1, and when \( \sigma_2(z) = z \) has a unique positive solution \( z = z_1 \in [0, z_1] \), this implies that \( m_1 = z_1 \). Similarly, when \( \sigma_1(z) = z \) has a unique positive solution \( z = z_1 \in [z_1, b_1a_1^{-1}] \), this implies that \( M_1 = z_1 \). It is easy to check that the above conditions \( (\sigma_2(z) \neq z \text{ for all } z \in [0, z_1]) \text{ and } (\sigma_1(z) \neq z \text{ for all } z \in (z_1, b_1a_1^{-1}]) \) are equivalent. As a consequence, we have \( m_1 = M_1 = z_1 \) if \( \sigma_2 \) does not have fixed points in \([0, z_1]\).

Next we prove that if \( f''(z_2) \leq 0 \) then this last condition is always true. Indeed, consider the decreasing function \( \beta : \mathbb{R}^+ \to \mathbb{R}^+ \) defined by \( \beta(x) = \zeta(x) \) if \( x \in [0, z_1] \) and \( \beta(x) = w(x) \) if \( x \geq z_1 \). It is obvious that the nonexistence of fixed points of \( \sigma_2 \) in \([0, z_1]\) is equivalent to affirm that \( z_1 \) is the unique fixed point in \([0, z_1]\) of the second iteration of \( \beta \).

Suppose that there exists another fixed point \( q < z_1 \) of \( \alpha = \beta^2 \). Since \( \alpha \) is increasing and \( \alpha'(z_1) < 1 \) we may assume, without restricting the generality, that \( \alpha'(q) \geq 1 \). Now, it is easy to prove that \( \alpha \) is continuously differentiable and

\[
\alpha''(z_1^-) = w'(z_1)\zeta''(z_1)(1 + \rho^*\zeta'(z_1)).
\]

Therefore \( \alpha''(z_1^-) \geq 0 \) if \( \zeta''(z_1) \leq 0 \), or, which is the same, if \( f''(z_2) \leq 0 \). Finally we note that in this case function \( \alpha' \) should have one positive minimum in \((q, z_1)\), but this is impossible since \((S\alpha)(x) < 0 \) for all \( x \in (q, z_1) \) (see, for instance, [2, Lemma 11.5]).

To end the first part of the proof, we note that the case iv) can be treated analogously to iii). In this case one can prove that the global stability of Eq. (1) holds if either \( f''(z_2) \geq 0 \) or \( \sigma_1 \) does not have fixed points in \([0, z_1]\).

In the second part of the proof we show that the result is also true for \( h \in (0, h^*) \). Notice that \( h'(\rho) = (2a_2\rho)^{-1} + (2a_1(1 - \rho))^{-1} > 0 \) for \( \rho \in (0, 1) \), so that the strictly increasing map \( h : [\rho_*, \rho^*] \to [0, h^*] > 0 \) is one-to-one if \( \rho_*> 0 \) solves \( a_2 \ln(\rho) = a_2 \ln(1 - \rho) \). This means that for every \( h^* \in (0, h^*) \) we can find \( \rho^* = \rho(h^*) < \rho^* \) such that (17) holds if we replace there \( \rho^*, h^* \) by \( \rho^*, h^* \). In this way, taking into account the first part of the proof, we have to prove only that either (a) or (b) holds for \( \rho^* \). Consider, for example, \( \sigma_2 \) in (b) (the proof for \( \sigma_1 \) in (a) is completely analogous). We have \( \sigma_2(0, \rho^*) > 0 \) and consequently

\[
\sigma_2(x, \rho^*) > x \text{ for all } x < z_1.
\]

Hence, if we prove that function \( \sigma_2(x, \rho) \) is decreasing as a function of \( x \) for each \( x \in (0, z_1) \), then (18) also holds for \( \rho \in (0, \rho^*) \) and we can conclude that \( \sigma_2(x, \rho) \neq x \) for all \( x \in [0, z_1] \) and \( \rho^* < \rho^* \). Indeed, the partial derivative \( D_\rho \sigma_2(x, \rho) \) of \( \sigma_2(x, \rho) \) with respect to \( \rho \) is equal to

\[
b_1a_1^{-1}f'(\delta(z(x, \rho) + \delta_1) \left[ \rho(g(b_2a_2^{-1}x) - z_2) + b_2a_2^{-1}(\zeta(x, \rho) - z_1) \right].
\]
For \( x < z_1 \) we have that \( \zeta(x, \rho) > \zeta(z_1, \rho) = z_1 \). Analogously, \( x < z_1 \) implies that \( b_2 a_2^{-1} x < b_2 a_2^{-1} z_1 = z_2 \). Since \( g \) is decreasing, it follows that \( g(b_2 a_2^{-1} x) > g(z_2) = z_2 \). Hence, the two factors in (19) have opposite signs and \( D_{\rho \sigma_2}(x, \rho) < 0 \) for each fixed \( x < z_1 \). This ends the proof.

It should be noted that verification of the condition (a) (or (b)) requires additional efforts and this test can be simplified considerably if we admit the use of computers. Of course, the optimal value of \( \rho^* \) in Theorem 6, which is completely determined by (17) and (a) (or (b)), depends essentially on the form of nonlinearity \( g(x) \) of our system. This means that taking another \( g_1(x) \) with \( g_1(z_2) = g(z_2) = z_2 \) and with \( g_1'(z_2) = g'(z_2), \ g_1''(z_2) = g''(z_2) \), we can obtain other value of \( \rho^* \) while checking out one of the assumptions (a), (b). Below, we will establish a corollary of Theorem 6 that overcomes the indicated peculiarities giving sufficient and easily verifiable conditions for the assumption (b) be true. However, since these conditions will be formulated in terms of \( z_2 = g(z_2), \ g'(z_2), \ g''(z_2) \) only, it is natural to expect that they will impose more restrictions on the parameters of Eq. (1) than (a) or (b).

Setting
\[
\lambda(x, a, b) = \frac{az + (a^2 - bz)(x - z_1)}{a - b(x - z_1)},
\]
we start with the following comparison result:

**Lemma 7** Suppose that \( g''(z_2) > 0 \) and \( 2g'(z_2) + z_2 g''(z_2) < 0 \). Then, for all \( x \in [0, z_1] \), we have that \( \zeta(x, \rho) < \lambda_-(x) = \lambda(x, \zeta'(z_1), \zeta''(z_1)/2) \). Moreover, \( w(x, \rho) > \lambda_+(x) = \lambda(x, w'(z_1), w''(z_1)/2) = \lambda(x, \zeta'(z_1), \rho \zeta''(z_1)/2) \) for all \( x > z_1 \).

**PROOF.** Notice that \( \lambda(z_1, a, b) = z_1, \ \lambda'(z_1, a, b) = a \) and \( \lambda''(z_1, a, b) = 2b \). Moreover, \( \lambda_-(x) : [0, z_1] \rightarrow \mathbb{R} \) and \( \lambda_+(x) : [z_1, +\infty) \rightarrow \mathbb{R} \) are well defined and strictly decreasing. Next, it should be noted that \( Sg(x) = G'(x) - (1/2)G^2(x) \), where \( G(x) = g''(x)/g'(x) \) and therefore the function \( g \) with negative S-derivative satisfies the differential Riccati inequality \( G'(x) - (1/2)G^2(x) < 0 \). Furthermore, every \( w \) with \( Sw < 0 \) can have at most one point \( c \) of inflection on \( \mathbb{R} \) (e.g. see [2]).

Moreover, taking into account that \( w''(z_1) > 0 \) and \( w(x) > 0 \) for all \( x \geq 0 \), we obtain that \( w''(x) > 0 \) for all \( x > z_1 \). Now, the second part of Lemma 7 follows from standard comparison results (see, e.g. [5, p.26]) if we observe that \( \Lambda_+ = \lambda''_+/\lambda'_+ \) and \( W = w''/w' \) satisfy \( S\Lambda_+(x) = \lambda''_+(x) - (1/2)\lambda'^2_+(x) = 0 \) and \( Sw(x) = W'(x) - (1/2)W^2(x) < 0 \) with \( \Lambda_+(z_1) = W(z_1) \). Indeed, the above relations imply that \( \Lambda_+(x) > W(x) \) for all \( x > z_1 \). Now, integrating \( \Lambda_+ \) and \( W \) over the interval \( (z_1, x) \), we get \( \lambda'_+(x) < w'(x) \). Integrating \( \lambda'_+ \) and \( w' \) from \( z_1 \) to \( x \) again, we obtain \( \lambda_+(x) < w(x) \) for all \( x > z_1 \).

Now, the proof of the first part of lemma (when \( x < z_1 \)) is slightly different from the given above since \( \zeta \) can have an inflexion point \( c \in (0, z_1) \). In this case, we can use the same arguments only to show that \( \lambda_+(c) > \zeta(x) \) and \( \lambda'_+(c) < \zeta'(x) \).
for \( x \in [c, z_1) \). Next, by convexity arguments, \( \lambda_-(x) \geq \lambda_-(c) + \lambda'_-(c)(x - c) > \zeta(c) + \zeta'(c)(x - c) \geq \zeta(x) \) for all \( x \in [0, c) \).

**Corollary 8** Assume that \( g'(z_2) < -1 \), \( g''(z_2) > 0 \) and \( 2g'(z_2) + z_2g''(z_2) < 0 \). Then the steady state solution of Eq. (1) is globally attracting if \( 0 < h < h^* \), where

\[
h^* = \frac{1}{2} \left[ \frac{\ln(\rho^*)}{a_2} - \frac{\ln(1 - \rho^*)}{a_1} \right] \quad \text{and} \quad \rho^* = \sqrt{\frac{2g'(z_2) + z_2g''(z_2)}{2(g'(z_2))^3 - z_2g''(z_2)g'(z_2)}} \in (0, 1).
\]

**PROOF.** Let us apply Theorem 6. Evidently, \( \rho^*|g'(z_2)| < 1 \) and therefore it suffices to prove that the condition (b) of Theorem 6 is also satisfied. Define the piecewise rational function \( r : \mathbb{R}^+ \to \mathbb{R}^+ \) as \( r(x) = \lambda_-(x) \) if \( x \in [0, z_1) \), \( r(x) = \lambda_+(x) \) if \( x \in [z_1, \alpha) \) and \( r(x) = 0 \) if \( x \geq \alpha \), where \( \alpha = z_1 + z_1\zeta'(z_1)[\rho(z_1/2)\zeta''(z_1) - (\zeta'(z_1))^2]^{-1} \) is such that \( \lambda_+(\alpha) = 0 \). Now, it could be easily seen that (b) holds if the one dimensional map \( r^2 = r \circ r \) has only one fixed point \( x = z_1 \). Finally, this is always true if \( |r'(z_1)| = \rho|g'(z_2)| < 1 \) and \( r(0) < \alpha \). It is easy to check that this last inequality holds whenever \( \rho < \rho^* \).

**Example 9** Assume that \( (x, y) \in \mathbb{R}^2_+ \), \( \int_{-h}^0 dk(s) = 1 \) and consider the system

\[
x'(t) + x(t) = \frac{1}{1 + y^2(t - h)} , \quad y'(t) + y(t) = 3 \int_{-h}^0 x(t + s)dk(s) . \quad (20)
\]

It is easy to see that the conditions given in [7,8,12] do not work here to prove the global attractivity of the unique positive equilibrium \( (z_1, z_2) = (0.40447, 1.12341) \). First, let us apply Corollary 8. In this case, \( g(x) = 3(1 + x^2)^{-1} \) (so that \( z_2 = 1.12341 \ldots \) is the unique real root of equation \( x^3 + x - 3 = 0 \)) and \( g'(z_2) = -1.19 \ldots \), \( g''(z_2) = 1.35 \ldots \) Finally, \( \rho^* = 0.72 \ldots \) and \( h^* = 0.47 \ldots \) so that we will have global asymptotical stability in Eq. (20) for all \( h \in [0, 0.47] \) even if we replace \( f(x) = (1 + x^2)^{-1} \) with another function \( f_1(x) \) such that \( Sf_1 < 0 \) and \( f_1^{(j)}(z_2) = f^{(j)}(z_2) , j = 0, 1, 2. \)

On the other hand, it is easy to apply Corollary 5. Indeed, function \( v : (0, +\infty) \to \mathbb{R} \) defined by

\[
v(u) = \frac{a_1a_2(u - z_2)}{b_1b_2(f(z_2) - f(u))} = \frac{u - 1.21341}{3(0.40447 - f(u))}
\]

reaches its global minimum at \( x_0 = 0.358965 \), so \( \beta = v(x_0) = 0.59166 \), which finally gives \( h^0 = 0.65259 \), improving the interval given by Corollary 8.

Finally, the direct application of Theorem 6 to Eq. (20) gives better result for our particular \( f(x) \). Here, the first inequality in (17) is satisfied for \( \rho = 0.839589 \), and because of \( f''(z_2) > 0 \), we have to check only the option (b) of the theorem. One
can verify directly that the map $\sigma_2(x, \rho^*)$ does not have fixed points $x \in (0, z_1)$ for $\rho^* = 0.836$. Thus the steady state $(z_1, z_2)$ attracts all solutions of Eq. (20) with positive initial values at least for $h \leq h^* = (1/2)[\ln(\rho^*) - \ln(1 - \rho^*)] = 0.814381$. For comparison: in the particular case $\int_0^h x(t+s)dk(s) = x(t-h)$, the equilibrium is asymptotically stable for $h \in [0, 2.65144\ldots)$ and there is at least one nontrivial positive periodic solution for all $h > 2.65144\ldots$ (see [6] for details).

References


