Negation and BCK-algebras

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In this paper we consider twelve classical laws of negation and study their relations in the context of BCK-algebras. A classification of the laws of negation is established and some characterizations are obtained. For example, using the concept of translation we obtain some characterizations of Hilbert algebras and commutative BCK-algebras with minimum. As a consequence we obtain a theorem relating those algebras to Boolean algebras.

Introduction

At the time of the dispute concerning the foundations of mathematics, one of the most controversial subjects was the role attributed to negation in reasoning. G. F. C. Griss, a representative of L. J. E. Brouwer's school, even went so far as to reject negation as a mathematical concept (see [7] and [8]).

According to the intuitionistic view of negation, the *principle of the excluded middle*, for example, is not valid. Therefore the formula $\neg\neg p \to p$ is not considered a logical law, unlike the formula $p \to \neg\neg p$. Likewise, in the intuitionist conception the formula $(p \to q) \to (\neg q \to \neg p)$ can be upheld, but $(\neg q \to \neg p) \to (p \to q)$ can is not.

For intuitionism $\neg\neg p$ and p cannot be considered equivalent; nor can one accept a reasoning such as that schematized in the formula $(\neg p \to q) \to \big((\neg p \to \neg q) \to p\big)$. The alternative is the following formula that A. Heyting included in the calculus in [6]: $\big((p \to q) \wedge (p \to \neg q)\big) \to \neg p$ which is equivalent in that system to $(p \to q) \to \big((p \to \neg q) \to \neg p\big)$. These are the most representative differences between the intuitionistic and the classical conceptions of negation.

However, there are certain points of contact between the intuitionist and the classical conceptions involving negation. In the formal system proposed by A. Heyting in [6] the law $\neg p \to (p \to q)$ appears as an axiom. This is what J. Łukasiewicz called the *Duns Scotus law*, and A. Heyting explains in [6] that it is a property that helps to give the definition of implication. However, I. Johansson presented a "minimal calculus", qualified by A. Heyting as intuitionist, in which he rejects the Duns Scotus law (see [6] and [3]), and thus gives a restricted interpretation of $\to$.

There are also certain classical observations on negation. For example, in Proposition 30 of the seventh book of "The Elements", Euclid proves that if a prime number p divides a product ab, then p divides a or b. The particular case of this proof when b is chosen equal to a provides the scheme for reasoning $(\neg p \to p) \to p$ to which J. Łukasiewicz refers in [9]; he names it the *Clavius law*. However, a reading of Book II of the Prior Analytics suggests that Aristotle had intuited the Clavius law; as did the precursor of non-Euclidean geometry, Girolamo Saccheri, in his book "Logica Demonstrativa" of 1697 (see [10]).

Also in Book II, Aristotle refers to another statement of propositional logic and applies it to the proofs of imperfect syllogisms. It is one of the laws of transposition: $(p \to q) \to (\neg q \to \neg p)$ (see [9]), which is present in some way in Stoic logic.

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To axiomatize the logical system named theory of deduction, which J. Łukasiewicz considers the foundation of logical systems, some time before 1929, he proposed an axiomatization taking as primitive functors $\to$ and $\neg$, following Frege. This axiomatization included the *law of the strong syllogism*, the *Clavius law*, and what he called the *Duns Scotus law*.

In this paper we consider the main ideas on negation that can be deduced from the above observations and give equational expressions which permit its algebraic study. This research was started by J. Pla i Carrera [11, 12, 13] and A. J. Rodríguez Salas [14] who studied the relationships between negations in the context of Hilbert algebras and commutative BCK-algebras, respectively. The pioneering work of K. Iseki and S. Tanaka includes some considerations about negation in [7]. The present study considers the much less restrictive context of BCK-algebras, applying a similar approach. We believe that this ambit is sufficiently permissive to show the value of the properties of negation. Some of our results are generalizations of those given by the above researchers; in some cases we use the same demonstrations, but in the most cases we need new ones. After a brief chapter on basic notions, we produce a map of implications between the properties of negation, providing counter-examples for reciprocal implications. When one property of negation appears "stronger" than others, we aim to find what "weak" properties must be added to obtain the "strong" one. This is the aim of Section 3 on characterizations. Section 2 is summarized in Figure 1, and the third in the table in Figure 2. Section 4 analyses the variations in the diagram and table when we restrict ourselves to the case of Hilbert algebras and commutative BCK-algebras, respectively. Section 4 also studies the relation between Boolean algebras and the properties of negation.

1 Basic definitions and results

Here a BCK-algebra is an algebra $\boldsymbol{A} = \langle A, \to, 1\rangle$ of type $\langle 2, 0\rangle$ such that for all $a, b, c \in A$:

DO1 $(a \to b) \to \big((b \to c) \to (a \to c)\big) = 1$,

DO2 $a \to 1 = 1$,

DO3 $1 \to a = a$,

DO4 $a = b$, whenever $a \to b = 1 = b \to a$.

There is an important binary relation in A, namely $\leq$. It is defined by $a \leq b$ iff $a \to b = 1$. In every BCK-algebra $\boldsymbol{A}$ the equalities $a \to a = 1$ (*identity law*) and $a \to (b \to c) = b \to (a \to c)$ (*commutation of premises law*) hold for all $a, b, c \in A$. From the identity law, quasi-identity D04, and transitivity – deduced from D01 and D03 – it is clear that $\leq$ is an order in A. Throughout the paper the only order we use is $\leq$, hence $\leq$ underlies each statement related to order. Moreover, for all $a, b, c \in A$, if $a \leq b$, then $b \to c \leq a \to c$ (resp. $c \to a \leq c \to b$). The above property is known as *antitonicity of right multiplication* (resp. *isotonicity of left multiplication*).

A BCK-algebra satisfying the equation $(a \to b) \to \big((b \to a) \to a\big) = (a \to b) \to \big((b \to a) \to b\big)$ is called a *Hilbert algebra*. Nevertheless there is another well-known definition of Hilbert algebras, namely the algebras $\langle A, \to, 1\rangle$ of type $\langle 2, 0\rangle$ satisfying the quasi-identity D04 and equations:

(A) $a \to (b \to a) = 1$,

(F) $\big(a \to (b \to c)\big) \to \big((a \to b) \to (a \to c)\big) = 1$.

Property (A) is called *a fortiori* and is sometimes written as $a \leq b \to a$. Identity (F) is equivalent in BCK-algebras to the equality $a \to (b \to c) = (a \to b) \to (a \to c)$, called *Frege identity* (or *Frege law*). Moreover, if a BCK-algebra satisfies the equation:

(T) $(b \to a) \to a = (a \to b) \to b$,

then it is called a *commutative* BCK-*algebra* or a *Tanaka algebra*.

In [4] we introduced the class V_3 of the BCK-algebras $\boldsymbol{A}$ satisfying the following equations:

A4 $(a \to b)^2 \to \big((b \to a)^2 \to a\big) = (a \to b)^2 \to \big((b \to a)^2 \to b\big)$,

A5 $\big((a \to b)^2 \to c\big) \to \big(((b \to a)^2 \to c) \to c\big) = 1$,

A6 $\big((a \to c) \to b\big) \to \big(((b \to a) \to b) \to b\big) = 1$.

In fact V_3 is the variety of BCK-algebras generated by the Hilbert algebra of three elements $\boldsymbol{H}_3$ and the commutative BCK-algebra $\boldsymbol{T}_3$ with three elements.

A BCK-algebra $\boldsymbol{A}$ is an *Abbott algebra* if it is a Hilbert algebra and if for all $a,\, b \in A$ the following equality holds:

(P) $(a \to b) \to a = a.$

There is a very important example of Abbott algebra in V_3, namely the Abbott algebra $\boldsymbol{A}_2$ with two elements.

A *negation* inside a BCK-algebra $\boldsymbol{A}$, in the broad sense of the word, is a monary operation in A. Nevertheless, any interesting operation in the algebra must be linked to the other operations, so we will assume that a monary operation $\neg$ in a BCK-algebra $\boldsymbol{A}$ is a negation if it satisfies some of the equations listed below with a brief reference to a system including the respective law. Since our considerations are within the logical framework, instead of the word "equation" we will sometimes use the word "law" or "property"; and so we will say, for example, "Duns Scotus law" instead of "Duns Scotus equation". The above mentioned list of equations is the following:

1. $(a \to b) \to (\neg b \to \neg a) = 1$, the *contrast law* "tollendo tollens" (Frege 1879; Hilbert-Bernays 1934), denoted by the abbreviation TTC.
2. $(a \to \neg b) \to (b \to \neg a) = 1$, the *contrast law* "ponendo tollens" (Russell 1906), denoted by PTC.
3. $(\neg a \to b) \to (\neg b \to a) = 1$, the *contrast law* "tollendo ponens", denoted by TPC.
4. $(\neg a \to \neg b) \to (b \to a) = 1$, the *contrast law* "ponendo ponens" (Łukasiewicz 1930), denoted by PPC.
5. $(a \to b) \to \big((a \to \neg b) \to \neg a\big) = 1$, the *intuitionistic law of* "reductio ad absurdum" (Heyting 1930), denoted by WRA.
6. $(\neg a \to \neg b) \to \big((\neg a \to b) \to a\big) = 1$, the *classical law of* "reductio ad absurdum" (Mendelson 1979), denoted by SRA.
7. $a \to (\neg a \to \neg b) = 1$, the *weak Duns Scotus law* denoted by WDS.
8. $a \to (\neg a \to b) = 1$, the *strong Duns Scotus law* (Heyting 1930), denoted by SDS.
9. $(a \to \neg a) \to \neg a = 1$, the *weak Clavius law* (Russell 1906), denoted by WCL.
10. $(\neg a \to a) \to a = 1$, the *strong Clavius law* (Łukasiewicz 1929), denoted by SCL.
11. $\neg\neg a \to a = 1$, the *strong double-negation law* (Frege 1879), denoted by SDN.
12. $a \to \neg\neg a = 1$, the *weak double-negation law* (Frege 1879), denoted by WDN.

Remember that if $\langle A, \to, 1\rangle$ is an Abbott algebra and there exists $z = \min A$, then $\langle A, \wedge, \vee, \neg,\, 1\rangle$ is a Boolean algebra, where $\wedge$, $\vee$, and $\neg$ are defined for all $a,\, b \in A$ as follows: $a \vee b = (a \to b) \to b$, $\neg a = a \to z$, and $a \wedge b = \neg(\neg a \vee \neg b)$.

In what follows we will abbreviate the sentence "let $\langle A, \to, \neg,\, 1\rangle$ be an algebra of type $\langle 2, 1, 0\rangle$, where $\langle A, \to,\, 1\rangle$ is a BCK-algebra and $\neg$ is a negation in $\langle A, \to,\, 1\rangle$" to the phrase: "let $\langle A, \to, \neg,\, 1\rangle$ be a BCK-algebra with negation". Moreover, the sentence "the negation $\neg$ satisfies the equation X" will be abbreviated to "the negation $\neg$ is X"; for example we will say "the negation $\neg$ is WRA" instead of "the negation $\neg$ satisfies the equation WRA". Finally, for all $B \subseteq A$, $\neg B$ will represent the set $\{\neg b : b \in B\}$.

2 The scheme of relationship between properties of negation

The main goal of this section is to find the relationships between properties of negation in order to establish what properties imply anothers. Of course, for this we need an underlying logical system. As system we choose the BCK-logic; but since the quasivariety of BCK-algebras is definitionally equivalent to the equivalent quasivariety semantics for the BCK-logic, then we will focus our study on the class of BCK-algebras. We will see that if the property P_1 is stronger than P_2, then there exists a BCK-algebra with negation such that the negation is P_2 but not P_1. So we can say that "BCK-logic distinguishes between properties of negation". Understanding the arrow as a "proper implication", Figure 1 summarizes the results of the section.

Given a BCK-algebra $\langle A, \to, \neg,\, 1\rangle$ with negation, for all $a,\, b \in A$ the equality $a \to (\neg a \to \neg b) = 1$ is straightforwardly obtained from SDS as a particular case – thence the label "weak"– or from TTC via a fortiori. From this we obtain:

Proposition 2.1 *Let $\langle A, \to, \neg,\, 1\rangle$ be a BCK-algebra with negation. If $\neg$ is* TTC *or* SDS*, then it is* WDS.

Lemma 2.2 *Let $\langle A, \to, \neg, 1\rangle$ be a BCK-algebra with negation. The following statements are equivalent:*

1. $\neg$ *is* PTC.
2. *For all* $a, b \in A$, $a \to \neg b = b \to \neg a$.
3. *For all* $a \in A$, $\neg a = a \to \neg 1$.

Proof. Let us assume that $\neg$ is PTC. Then for all $a, b \in A$, $a \to \neg b \leq b \to \neg a$ and $b \to \neg a \leq a \to \neg b$; therefore $a \to \neg b = b \to \neg a$. Now if we assume 2., then for all $a \in A$, $a \to \neg 1 = 1 \to \neg a = \neg a$. Finally, let us assume 3. and take $a, b \in A$; hence $a \to \neg b = a \to (b \to \neg 1) = b \to (a \to \neg 1) = b \to \neg a$. ☐

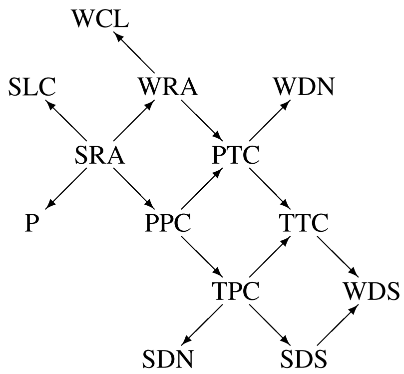

Figure 1. Relations between laws of negation in BCK-algebras

The proof of Lemma 2.3 follows that of Lemma 2.2.

Lemma 2.3 *Let $\langle A, \to, \neg, 1\rangle$ be a BCK-algebra with negation. $\neg$ is* TPC *if and only if for all $a, b \in A$, $\neg a \to b = \neg b \to a$.*

Proposition 2.4 *Let $\langle A, \to, \neg, 1\rangle$ be a BCK-algebra with negation.*

1. *If* $\neg$ *is* PTC*, then it is* WDN *and* TTC.
2. *If* $\neg$ *is* TPC*, then it is* SDN, TTC, *and* SDS.

Proof. Let us suppose that $\neg$ is PTC. Then for all $a \in A$, $a \to \neg\neg a = \neg a \to \neg a = 1$; and so WDN holds. Moreover, by isotonicity of left multiplication in BCK-algebras and WDN we have for all $a, b \in A$, $a \to b \leq a \to \neg\neg b$; but it follows from Lemma 2.2 that $a \to \neg\neg b = \neg b \to \neg a$, therefore $a \to b \leq \neg b \to \neg a$; that is, $\neg$ is TTC. Moreover, let us assume that $\neg$ is TPC. For all $a \in A$, $(\neg a \to \neg a) \to (\neg\neg a \to a) = 1$, but $\neg a \to \neg a = 1$, hence $\neg\neg a \to a = 1 \to (\neg\neg a \to a) = 1$, that is, $\neg$ is SDN. Since for all $b \in A$, $\neg\neg a \to a = 1$, it follows by antitonicity of right multiplication that for all $a, b \in A$, $a \to b \leq \neg\neg a \to b$. Using TPC, $\neg\neg a \to b \leq \neg b \to \neg a$ and so, by transitivity we have $a \to b \leq \neg b \to \neg a$, that is, $\neg$ is TTC. By Lemma 2.3 and *a fortiori*, if $\neg$ is TPC, we have for all $a, b \in A$ that $a \leq \neg b \to a = \neg a \to b$; hence $\neg$ is SDS. ☐

Lemma 2.5 *Let $\langle A, \to, \neg, 1\rangle$ be a BCK-algebra with negation. If $\neg$ is* PPC*, then we have:*

1. *For all* $a \in A$, $\neg a \leq a \to \neg 1$ *and* $\neg 1 \to a = 1$.
2. $\neg$ *is* SDS.
3. $\neg$ *is* SDN.

Proof. By *a fortiori* we have for all $a \in A$, $\neg a \leq \neg\neg 1 \to \neg a$; but by PPC, $\neg\neg 1 \to \neg a \leq a \to \neg 1$, therefore by transitivity we obtain $\neg a \leq a \to \neg 1$. By PPC, $\neg a \to \neg 1 \leq 1 \to a = a$; but by isotonicity and *a fortiori* we have $1 = \neg 1 \to (\neg a \to \neg 1) \leq \neg 1 \to a$, hence $\neg 1 \to a = 1$. Moreover, in virtue of statement 1. we have, for all $a, b \in A$, $\neg 1 \leq b$ and by isotonicity, $a \to \neg 1 \leq a \to b$; but by 1. and transitivity, $\neg a \leq a \to b$. Commutation of premises leads us to $a \leq \neg a \to b$ and so SDS holds. Taking $b = \neg 1$ we have for all $a \in A$, $a \leq \neg a \to \neg 1$ and by PPC, $\neg a \to \neg 1 \leq 1 \to a$; therefore $a = \neg a \to \neg 1$. Since SDS holds, $\neg\neg a \leq \neg a \to \neg 1$ and so $\neg\neg a \leq a$. ☐

Proposition 2.6 *Let* $\langle A, \to, \neg, 1\rangle$ *be a BCK-algebra with negation. If* $\neg$ *is* PPC*, then it is* PTC.

Proof. From Lemma 2.5 we know that $\neg$ is SDN. Therefore, by antitonicity we have, for all $a \in A$, $a \to \neg 1 \le \neg\neg a \to \neg 1$, but since $\neg$ is PPC, we have $\neg\neg a \to \neg 1 \le 1 \to \neg a = \neg a$. Hence, it follows that $a \to \neg 1 \le \neg a$. From this and Lemma 2.5 it follows, for all $a \in A$, that $\neg a = a \to \neg 1$, which implies – as states Lemma 2.2 – that $\neg$ is PTC. □

Given a BCK-algebra with negation, $\boldsymbol{A}$, the equality $\neg\neg a = a$ holds for all $a \in A$ if and only if $\neg$ is SDN and WDN, in particular if $\neg$ is SDN and PTC (see Lemma 2.4); hence we have:

Lemma 2.7 *Let* $\langle A, \to, \neg, 1\rangle$ *be a BCK-algebra with negation. If* $\neg$ *is* SDN *and* PTC*, then for all* $a \in A$, $\neg\neg a = a$.

Proposition 2.8 *Let* $\langle A, \to, \neg, 1\rangle$ *be a BCK-algebra with negation.*

1. *If* $\neg$ *is* PPC*, then it is* TPC.
2. *If* $\neg$ *is* WRA*, then it is* PTC.

Proof. If $\neg$ is PPC, then by Proposition 2.6 $\neg$ is PTC and so (see Lemma 2.2) $\neg a \to \neg 1 = \neg\neg a$ for all $a \in A$, and by Lemma 2.7 $a = \neg\neg a$; therefore $\neg a \to \neg 1 = a$. We have, for all $a, b \in A$, $\neg a \to b = \neg a \to \neg\neg b$; hence $\neg a \to b = \neg a \to (\neg b \to \neg 1) = \neg b \to (\neg a \to \neg 1) = \neg b \to a$. It follows that $\neg$ is TPC. Moreover, let us assume that $\neg$ is WRA. By antitonicity, for all $a, b \in A$, $(a \to b) \to \neg a \le b \to \neg a$ and by isotonicity, $(a \to \neg b) \to \big((a \to b) \to \neg a\big) \le (a \to \neg b) \to (b \to \neg a)$. If $\neg$ is WRA, as we assumed, the latter inequality is $1 \le (a \to \neg b) \to (b \to \neg a)$, and so PTC holds. □

Proposition 2.9 *Let* $\langle A, \to, \neg, 1\rangle$ *be a BCK-algebra with negation.*

1. *If* $\neg$ *is* SRA*, then it is* PPC*,* WRA*, and* SCL.
2. *If* $\neg$ *is* WRA*, then it is* WCL.

Proof. Let us assume that $\neg$ is SRA. By *a fortiori* and antitonicity we have $(\neg a \to b) \to a \le b \to a$ for all $a, b \in A$; but by SRA, $\neg a \to \neg b \le (\neg a \to b) \to a$; therefore $\neg a \to \neg b \le b \to a$ and so $\neg$ is PPC. Since each PPC negation is PTC and SDN (see Proposition 2.6 and Lemma 2.5), then, by Lemma 2.7, our SRA negation satisfies for all $a \in A$, $a = \neg\neg a$. From this, since $(\neg\neg a \to \neg\neg b) \to \big((\neg\neg a \to \neg b) \to \neg a\big) = 1$, it follows that for all $a, b \in A$, $(a \to b) \to \big((a \to \neg b) \to \neg a\big) = 1$, that is, $\neg$ is WRA. Moreover, by the identity law, $\neg a \to \neg a = 1$ holds for all $a \in A$, and so $(\neg a \to a) \to a = (\neg a \to \neg a) \to \big((\neg a \to a) \to a\big) = 1$; therefore $(\neg a \to a) \to a = 1$ and so SCL holds. Finally, let us assume that $\neg$ is WRA and take $a \in A$. Then $1 = (a \to \neg a) \to \big((a \to a) \to \neg a\big) = (a \to \neg a) \to \neg a$, and so WCL holds. □

To complete our scheme we establish an interesting relationship between equation P and negations. Note that Proposition 2.10 holds when $\neg$ is SRA.

Proposition 2.10 *Let* $\langle A, \to, \neg, 1\rangle$ *be a BCK-algebra with negation. If* $\neg$ *is* SDS *and* SCL*, then* P *holds in* $\langle A, \to, 1\rangle$.

Proof. Let us assume that $\neg$ is SDS, therefore $a \to (\neg a \to b) = 1$ for all $a, b \in A$, and so $\neg a \le a \to b$. By antitonicity applied twice, $(\neg a \to a) \to a \le \big((a \to b) \to a\big) \to a$. If $\neg$ is also SCL, then we conclude from the above inequality that $\big((a \to b) \to a\big) \to a = 1$ for all $a, b \in A$. □

Therefore, the scheme in Figure 1 is established. In the following, for the sake of brevity, we need to assume some notational agreements. Any natural number $0 \le n \le 26$ represents only one negation. In effect, given $0 \le n \le 26$, if $(uvw)_3$ is the base 3 expression of n and ξ is the mapping from $\{0, 1, 2\} \subseteq \omega$ into $A = \{0, a, 1\}$ given by $2 \longmapsto 0$, $1 \longmapsto a$, and $0 \longmapsto 1$; then we agree that n will represent the negation: $0 \longmapsto \xi(u)$, $a \longmapsto \xi(v)$, and $1 \longmapsto \xi(w)$, abbreviated to the symbol $(\xi(u), \xi(v), \xi(w))$. For example, $7 = (021)_3$, and so 7 represents the negation $(1, 0, a)$.

The symbol $[law_1, law_2, n_1, n_2]$, where $0 \le n_1,\ n_2 \le 26$, will represent the sentence "a given negation $\neg$ can be law_1 without to be law_2, for example the negation n_1 in $\boldsymbol{H}_3$ and n_2 in $\boldsymbol{T}_3$". The symbol $[law_1, law_2, \_, n_2]$

(resp. $[law_1, law_2, n_1, \_]$) represents the same sentence unless there is no example in $\boldsymbol{H}_3$ (resp. $\boldsymbol{T}_3$). It is advisable for us to give examples in $\boldsymbol{H}_3$ and $\boldsymbol{S}_3$ simultaneously since we will consider negations in Hilbert algebras and commutative BCK-algebras later.

Now we begin the stage of obtaining examples and counterexamples:

1. Relating to the axis WRA, PTC, TTC, and WDS ("the intuitionistic spine"): $[\mathrm{WDS}, \mathrm{TTC}, 10, 10]$, $[\mathrm{TTC}, \mathrm{PTC}, 4, 4]$, and $[\mathrm{PTC}, \mathrm{WRA}, \_, 5]$.
2. Relating to the axis SRA, PPC, TPC, SDS ("the classic spine"): $[\mathrm{SDS}, \mathrm{TPC}, 8, 8]$, $[\mathrm{TPC}, \mathrm{PPC}, 17, 17]$, and $[\mathrm{PPC}, \mathrm{SRA}, \_, 5]$.
3. The transverse axis: $[\mathrm{WCL}, \mathrm{WRA}, 2, 2]$, $[\mathrm{WRA}, \mathrm{SRA}, 0, 0]$, $[\mathrm{WDN}, \mathrm{PTC}, 3, 3]$, $[\mathrm{PTC}, \mathrm{PPC}, 1, 1]$, $[\mathrm{TTC}, \mathrm{TPC}, 13, 13]$, $[\mathrm{WDS}, \mathrm{SDS}, 4, 4]$, $[\mathrm{SCL}, \mathrm{SRA}, 2, 2]$, and $[\mathrm{SDN}, \mathrm{TPC}, 15, 15]$.
4. Relating to WCL: $[\mathrm{PPC}, \mathrm{WCL}, \_, 5]$ (5 is the only PPC negation in $\boldsymbol{T}_3$; in $\boldsymbol{H}_3$ there is none) and $[\mathrm{SCL}, \mathrm{WCL}, 11, \_]$.
5. Relating to SCL: $[\mathrm{PPC}, \mathrm{SCL}, \_, 5]$ and $[\mathrm{WRA}, \mathrm{SCL}, 8, \_]$.
6. Relating to SDN: $[\mathrm{SDS}, \mathrm{SDN}, 8, 8]$ and $[\mathrm{WRA}, \mathrm{SDN}, 1, 1]$.
7. Relating to WDN: $[\mathrm{WCL}, \mathrm{WDN}, 2, 2]$, $[\mathrm{TPC}, \mathrm{WDN}, 17, 17]$, and $[\mathrm{SCL}, \mathrm{WDN}, 2, 2]$.
8. Downward diagonals: $[\mathrm{WRA}, \mathrm{SDS}, 0, 0]$.
9. Upward diagonals: $[\mathrm{SDS}, \mathrm{TTC}, \_, 23]$.
10. Others: $[\mathrm{SDN}, \mathrm{WDS}, 2, 2]$, $[\mathrm{SCL}, \mathrm{SDN}, 1, 1]$.

Of course there are other impossible implications between properties of negation, but each one of them can be straightforwardly deduced from the preceding results and counterexamples. We will show later that it is impossible to find an example of SRA negation in $\boldsymbol{H}_3$ or $\boldsymbol{T}_3$; nevertheless to complete this stage it is necessary to give an example of SRA negation in a BCK-algebra $\boldsymbol{A}$. For this let us take $\boldsymbol{A} = \boldsymbol{A}_2$ and $\neg$ the mapping: $0 \longmapsto 1$ and $1 \longmapsto 0$, that is, $\neg = (1, 0)$. Contrary to the above example, there are BCK-algebras $\boldsymbol{A}$ and monary operations $\neg$ in A such that $\neg$ does not satisfy any of the properties enumerated above. This is the case of $\boldsymbol{T}_3$ and the mapping: $0 \longmapsto a$, $a \longmapsto 1$, and $1 \longmapsto 0$. Nevertheless, this is not the frequent case; usually monary operations in BCK-algebras satisfy one or more properties.

3 Characterizations

We begin this section by a characterization of the key law PTC based on the concept of *translation*. Essentially this is the concept of negation of K. Iseki and S. Tanaka in [7]. Lemma 3.1 gives some consequences of translations; we avoid its straightforward proof.

Definition 3.1 Let $\langle A, \rightarrow, \tau, 1\rangle$ be an algebra of type $\langle 2, 1, 0\rangle$, where $\langle A, \rightarrow, 1\rangle$ is a BCK-algebra, and let $a \in A$. τ is a *translation by* a if for all $x \in A$, $\tau x = x \rightarrow a$.

Lemma 3.1 *Let $\langle A, \rightarrow, \neg, 1\rangle$ be a BCK-algebra with negation, $z \in A$, and let $\neg$ be a translation by z. Then for all x, $y \in A$ we have:*

1. *If $x \leq y$, then $\neg y \leq \neg x$.*
2. *$\neg z = 1$ and $\neg 1 = z$.*
3. *$z = \min \neg A$.*
4. *$x \leq z$ if and only if $\neg x = 1$.*
5. *$\neg\neg\neg x = \neg x$ for all $x \in A$.*

Theorem 3.2 *Let $\langle A, \rightarrow, \neg, 1\rangle$ be a BCK-algebra with negation. The following statements are equivalent:*

1. *$\neg$ is a translation.*
2. *$\neg$ is* PTC.
3. *$\neg$ is* TTC *and* WDN.
4. *$\neg$ is* TTC *and $\neg\neg 1 = 1$.*

Proof. Let us assume that $\neg$ is a translation, that is, there is $z \in A$ such that $\neg a = a \to z$, for all $a \in A$. From Lemma 3.1 we have that $z = \neg 1$, therefore $\neg a = a \to \neg 1$ for all $a \in A$ and so $\neg$ is PTC (see Lemma 2.2). Now let us assume that $\neg$ is TTC and that $\neg\neg 1 = 1$. In virtue of TTC we have for all $a \in A$, $a = 1 \to a \leq \neg a \to \neg 1$ and so $\neg a \leq a \to \neg 1$. Moreover, $a \to \neg 1 \leq \neg\neg 1 \to \neg a = 1 \to \neg a = \neg a$; hence $\neg a = a \to \neg 1$ and it follows that $\neg$ is a translation. The rest of the proof is straightforward. □

Remark 3.1 None of the statements in Lemma 3.1 implies that $\neg$ is a translation. In fact, 9 in $\boldsymbol{H}_3$ or $\boldsymbol{T}_3$ satisfies statement 3.1.1., but it is not PTC; taking 19 in $\boldsymbol{H}_3$ or $\boldsymbol{T}_3$ as $\neg$, we have $\neg a = 1$ and $\neg 1 = a$, nevertheless 19 is not PTC in $\boldsymbol{H}_3$ or $\boldsymbol{T}_3$; if we select 13 in $\boldsymbol{H}_3$ or $\boldsymbol{T}_3$ as $\neg$, we have $\neg A = \{a\}$ and so $a = \min \neg A$, but 13 is not PTC in $\boldsymbol{H}_3$ or $\boldsymbol{T}_3$; for the statement 3.1.4 consider 8 in $\boldsymbol{T}_3$ which is not PTC.

Theorem 3.3 *Let $\langle A, \to, \neg, 1\rangle$ be a BCK-algebra with negation. The following statements are equivalent:*

1. $\neg$ *is* TPC.
2. $\neg$ *is* TTC *and* SDN.

Proof. It remains to be shown that if $\neg$ is TTC and SDN, then it is TPC (see Proposition 2.4). By SDN and isotonicity we have, for all $a, b \in A$, $\neg b \to \neg\neg a \leq \neg b \to a$. If $\neg$ is TTC, as we assumed, $\neg a \to b \leq \neg b \to \neg\neg a$, therefore $\neg a \to b \leq \neg b \to a$ and so $\neg$ is TPC. □

Remark 3.2 There are negations which are SDN and SDS but not TPC, for example 23 in $\boldsymbol{T}_3$.

Theorem 3.4 *Let $\langle A, \to, \neg, 1\rangle$ be a BCK-algebra with negation. The following statements are equivalent:*

1. $\neg$ *is* PPC.
2. $\neg$ *is* PTC *and* SDN.
3. *For all $a, b \in A$, $\neg a \to \neg b = b \to a$.*
4. $\neg$ *is* PTC *and* TPC.
5. $\neg$ *is* TPC *and* WDN.

Proof. Let us assume that $\neg$ is PTC and SDN. By PTC, for all $a, b \in A$, $\neg a \to \neg b = b \to \neg\neg a$; but by Lemma 2.7, $\neg\neg a = a$; hence $\neg a \to \neg b = b \to a$. Now let us suppose that $\neg$ is TPC and WDN. By TPC it follows for all $a, b \in A$ that $\neg a \to \neg b \leq \neg\neg b \to a$. Since $b \leq \neg\neg b$, by antitonicity we have $\neg\neg b \to a \leq b \to a$. Therefore, $\neg$ is PPC as a consequence of transitivity. The rest of the proof is straightforward or a direct consequence of Lemma 2.5, Proposition 2.6, Lemma 2.7, and Proposition 2.8. □

Every TTC negation is close to being a translation. In effect, it needs only to satisfy the equality $\neg\neg 1 = 1$, as we deduce from Theorem 3.2. The case of TPC is analogous, as Corollary 3.5 shows.

Corollary 3.5 *Let $\langle A, \to, \neg, 1\rangle$ be a BCK-algebra with negation. If $\neg$ is* TPC*, then the following statements are equivalent:*

1. $\neg$ *is* PPC.
2. $\neg$ *is a translation.*
3. $\neg\neg 1 = 1$.

Proof. It is obvious from Theorems 3.4 and 3.2 that every PPC negation must be a translation and that every translation satisfies $\neg\neg 1 = 1$. Moreover, if $\neg$ is TPC, then it is TTC. If we assume that $\neg\neg 1 = 1$, then $\neg$ is PTC. Therefore, $\neg$ is PTC and TPC; because of Theorem 3.4, $\neg$ must be PPC. □

Proposition 3.6 *Let $\langle A, \cdot, \neg, 1\rangle$ be a BCK-algebra with negation and let us assume that $\neg$ is* PPC. *Then the following hold:*

1. *For all $a \in A$, $a = \neg a \to \neg 1$.*
2. *If $\neg$ is* SCL*, then for all $a \in A$, $\neg a = a \to \neg a$.*

Proof. If $\neg$ is PPC, then it is PTC. Hence for all $a \in A$, $\neg a \leq a \to \neg 1$ (Lemma 2.2) and so $a \leq \neg a \to \neg 1$; but by PPC, $\neg a \to \neg 1 \leq 1 \to a$. It follows, for all $a \in A$, that $a = \neg a \to \neg 1$. Moreover, if $\neg$ is SCL too, then for all $a \in A$, $(\neg\neg a \to \neg a) \to \neg a = 1$. Since $\neg$ is PPC, then it is PTC and SDN, hence $\neg\neg a = a$, and so $(a \to \neg a) \to \neg a = 1$; but $\neg a \to (a \to \neg a) = 1$, therefore $\neg a = a \to \neg a$. □

Lemma 3.7 *Let $\langle A, \to, \neg, 1\rangle$ be a BCK-algebra with negation. The following statements hold:*

1. *If $\neg$ is* WDS *and* SDN*, then it is* SDS.
2. *If $\neg$ is* SDS *and* SCL*, then it is* SDN.

Proof. Let $\neg$ be a WDS and SDN negation. Then for all $b \in A$, $\neg\neg b \le b$; by isotonicity we have for all $a \in A$, $a \to (\neg a \to \neg\neg b) \le a \to (\neg a \to b)$ and this implies, in virtue of WDS, that $\neg$ is SDS. Moreover, let $\neg$ be a SCL negation and $a \in A$, then $\neg a \to a \le a$; hence $\neg\neg a \to (\neg a \to a) \le \neg\neg a \to a$. If $\neg$ is SDS too, we have that $\neg\neg a \to (\neg a \to a) = 1$ and so $\neg\neg a \to a = 1$, that is, $\neg$ is SDN. □

Lemma 3.8 *Let $\langle A, \to, \neg, 1\rangle$ be a BCK-algebra with negation. If $\neg$ is* SDS *(resp.* WDS *), then A (resp. $\neg A$) has minimum and $\neg 1 = \min A$ (resp. $\neg 1 = \min \neg A$).*

Proof. If $\neg$ is SDS, then, in particular, $\neg 1 \to a = 1 \to (\neg 1 \to a) = 1$ for all $a \in A$. Hence, for all $a \in A$, $\neg 1 \le a$. If we assume WDS instead of SDS, the latter reasoning leads to $\neg 1 = \min \neg A$. □

Remark 3.3 Note that any BCK-algebra without minimum has no negation in the classic spine. Moreover, 8 is an SDS negation in $\boldsymbol{T}_3$, but $\neg A \ne A$.

Theorem 3.9 *Let $\langle A, \to, \neg, 1\rangle$ be a BCK-algebra with negation. The following statements are equivalent:*

1. *$\neg$ is* PTC *and* WCL.
2. *$\neg$ is* WRA.
3. *$\neg$ is* TTC *and* WCL.
4. *$\neg$ is* WDS *and* WCL.

Proof. Let us assume that $\neg$ is PTC and WCL. By PTC we have for all $a, b \in A$, $a \to \neg b = b \to \neg a$, but by the syllogism and WCL, $b \to \neg a \le (a \to b) \to (a \to \neg a) = (a \to b) \to \neg a$. Therefore we have $(a \to b) \to \big((a \to \neg b) \to \neg a\big) = 1$. Now suppose that $\neg$ is WDS and WCL. In virtue of Lemma 3.8 we have for all $a \in A$, $\neg 1 \le \neg a$. By isotonicity it follows that $a \to \neg 1 \le a \to \neg a$; but we assumed WCL, hence $a \to \neg 1 \le \neg a$. From this with WDS and Lemma 2.2 we obtain that $\neg$ is PTC. The rest of the proof is obvious or an immediate consequence of the latter. □

Theorem 3.10 *Let $\langle A, \to, \neg, 1\rangle$ be a BCK-algebra with negation. The following statements are equivalent:*

1. *$\neg$ is* WRA *and* SDN.
2. *$\neg$ is* SRA.
3. *$\neg$ is* WRA, SDS, *and* SCL.

Proof. If $\neg$ is WRA, then it is PTC, and for all $a, b \in A$, $(\neg a \to \neg b) \to \big((\neg a \to \neg\neg b) \to \neg\neg a\big) = 1$. Since $\neg$ is PTC and SDN, it follows from Lemma 2.7 that in our algebra the equation $\neg\neg a = a$ holds; hence $\neg$ is SRA. The rest of the proof is immediate from the scheme in Figure 1 and Lemma 3.7. □

Theorem 3.11 *Let $\langle A, \to, \neg, 1\rangle$ be a BCK-algebra with negation. The following statements are equivalent:*

1. *$\neg$ is* PPC *and* SCL.
2. *$\neg$ is* TPC *and* SCL.
3. *$\neg$ is* SRA.
4. *$\neg$ is* WDN, SDS, *and* SCL.

Proof. Let us assume that $\neg$ is TPC. Then for all $a, b \in A$, $\neg b \le (\neg a \to b) \to a$; by isotonicity it follows that $\neg a \to \neg b \le \neg a \to \big((\neg a \to b) \to a\big)$, or equivalently, $\neg a \to \neg b \le (\neg a \to b) \to (\neg a \to a)$. Furthermore, if $\neg$ is SCL, then $\neg a \to a \le a$ and by isotonicity, $(\neg a \to b) \to (\neg a \to a) \le (\neg a \to b) \to a$. Therefore we obtain, for all $a, b \in A$, $\neg a \to \neg b \le (\neg a \to b) \to a$ and so $\neg$ is SRA. Moreover, let us suppose that $\neg$ is WDN, SDS, and SCL. In virtue of Lemma 3.8, for all $a \in A$, $\neg 1 \le a$. By isotonicity and SCL it follows that $\neg a \to \neg 1 \le \neg a \to a \le a$. From SDS, $a \le \neg a \to \neg 1$ and so $a = \neg a \to \neg 1$. If $\neg$ is SDS and SCL, then it is SDN, as established in Lemma 3.7; since, by hypothesis, $\neg$ is WDN, then for all $a \in A$, $\neg\neg a = a$. Therefore, for all $a \in A$, $\neg a = \neg\neg a \to \neg 1 = a \to \neg 1$ and so $\neg$ is PTC. By Theorem 3.4 we conclude that $\neg$ is PPC. The rest of the proof is straightforward. □

Equivalences between negations in BCK-algebras						
PTC	TTC, WDN	$\neg a = a \rightarrow \neg 1$	$\neg$ is translation	TTC, $\neg\neg 1 = 1$		
TPC	TTC, SDN	$\neg a \rightarrow b = \neg b \rightarrow a$				
PPC	PTC, SDN	TPC, $\neg\neg 1 = 1$	TPC, WDN	$\neg a \rightarrow \neg b = b \rightarrow a$	PTC, TPC	
WRA	WDS, WCL	TTC, WCL	PTC, WCL			
SRA	WRA, SDN	WRA, PPC	WRA, SDS, SCL	PPC, SCL	TPC, SCL	SDS, SCL, WDN

Figure 2. Characterization of negation law in BCK-algebras

4 Negation in Hilbert algebras and commutative BCK-algebras

In this section we will study how axioms of Hilbert algebras and commutative BCK-algebras affect Figures 1 and 2. In the variety of Hilbert algebras, Figure 1 takes on the appearance of Figure 3. We observe two important differences: PTC is equivalent to WRA and PPC is equivalent to SRA.

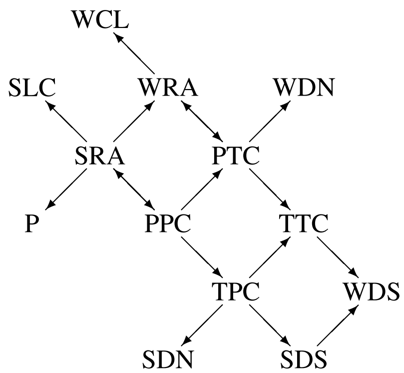

Figure 3. Relations between laws of negation in Hilbert algebras

Theorem 4.1 *Let $\langle A, \rightarrow, \neg, 1\rangle$ be a Hilbert algebra with negation. The following are equivalent:*

1. $\neg$ *is* PTC.
2. *For all* $a, b \in A$, $a \rightarrow \neg b = (a \rightarrow b) \rightarrow \neg a$.
3. $\neg$ *is* WRA.

Proof. Let $\neg$ be a PTC negation. Therefore, for all $a, b \in A$ we have $a \rightarrow \neg b = a \rightarrow (b \rightarrow \neg 1) = (a \rightarrow b) \rightarrow (a \rightarrow \neg 1) = (a \rightarrow b) \rightarrow \neg a$. The rest of the proof is obvious. □

Theorem 4.2 *Let $\langle A, \rightarrow, \neg, 1\rangle$ be a Hilbert algebra with negation. The following are equivalent:*

1. $\neg$ *is* SDS *and* SCL.
2. $\neg$ *is* PPC.
3. $\neg$ *is* SRA.

Proof. Let us assume that $\neg$ is SDS and SCL. By *a fortiori*, we have for all $a, b \in A$ that $\neg b \rightarrow a \leq \neg a \rightarrow (\neg b \rightarrow a)$. Since $\neg a \rightarrow a = a$, it follows that $\neg a \rightarrow (\neg b \rightarrow a) = (\neg a \rightarrow \neg b) \rightarrow a$; therefore $\neg a \rightarrow \neg b \leq (\neg b \rightarrow a) \rightarrow a$. Since $b \leq \neg b \rightarrow a$, antitonicity leads us to $(\neg b \rightarrow a) \rightarrow a \leq b \rightarrow a$ and so $\neg a \rightarrow \neg b \leq b \rightarrow a$, that is, $\neg$ is PPC. Moreover, assuming now that $\neg$ is PPC we have, in virtue of Theorem 3.4, that $\neg$ is PTC and SDN, and so for all $a \in A$, $a = \neg\neg a = \neg a \rightarrow \neg 1$ (see Lemmas 2.7 and 2.2). Therefore, for all $a, b \in A$, $b \rightarrow a = b \rightarrow (\neg a \rightarrow \neg 1) = (\neg a \rightarrow b) \rightarrow (\neg a \rightarrow \neg 1) = (\neg a \rightarrow b) \rightarrow a$. Since $\neg a \rightarrow \neg b \leq b \rightarrow a$, it follows that $\neg a \rightarrow \neg b \leq (\neg a \rightarrow b) \rightarrow a$; therefore $\neg$ is SRA. The rest of the proof is straightforward. □

In the case of commutative BCK-algebras, the obvious fact is that SCL and WCL are equivalent properties of negation. In fact, taking in consideration the examples at the end of Section 2, this is all we can add to the general scheme Figure 1.

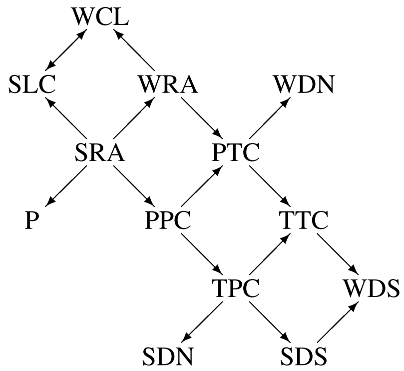

Figure 4. Relations between laws of negation in commutative BCK-algebras

Proposition 4.3 *Let $\langle A, \to, \neg, 1\rangle$ be a BCK-algebra with negation. If $\neg$ is* SDS *and* SCL*, then it is* SDN *and for all $a \in A$, $a = \neg a \to \neg\neg a$.*

P r o o f. Let us assume that $\neg$ is SDS and SCL and let us take $a \in A$. Since $\neg$ is SDS we have, in particular, $1 = \neg a \to (\neg\neg a \to a) = \neg\neg a \to (\neg a \to a)$. In virtue of the fact that $\neg$ is SCL, $(\neg a \to a) \to a = 1$. Hence we conclude $\neg\neg a \leq a$, and so $\neg$ is SDN. Moreover, by isotonicity, $\neg a \to \neg\neg a \leq \neg a \to a$; but $\neg a \to a \leq a$, therefore $\neg a \to \neg\neg a \leq a$. The reciprocal inequality follows from SDS taking as b the element $\neg\neg a$. □

Theorem 4.4 *Let $\langle A, \to, \neg, 1\rangle$ be a commutative BCK-algebra with negation. The following are equivalent:*

1. $\neg$ *is* SDS *and* SCL.
2. $\neg$ *is* SRA.

P r o o f. If $\neg$ is SCL, then, for all $a \in A$, $1 = (\neg a \to a) \to a = (a \to \neg a) \to \neg a$; therefore, by *a fortiori*, $\neg a = a \to \neg a$. Now Proposition 4.3 leads us to the equality $\neg\neg a = \neg a \to \neg\neg a = a$. In virtue of Theorem 3.11 it follows that $\neg$ is SRA. The rest of the proof is straightforward from the results summarized in Figure 1. □

As we will see in Corollary 4.7, the concept of translation provides some characterizations for the existence of minimum in any Hilbert algebra or commutative BCK-algebra. First we need two results.

Lemma 4.5 *Let $\boldsymbol{A} = \langle A, \to, 1\rangle$ be a BCK-algebra. There exists a unique* SDS *translation in $\boldsymbol{A}$, whenever there exists one such translation.*

P r o o f. Let τ and σ be SDS translations in $\boldsymbol{A}$. From Lemma 2.2 it follows that for all $a \in A$, $\tau a = a \to \tau 1$ and $\sigma a = a \to \sigma 1$. But by hypothesis τ, as well as σ, is SDS negation. Therefore in virtue of Lemma 3.3 it follows that $\boldsymbol{A}$ has minimum and $\tau 1 = \min A = \sigma 1$; hence $\tau = \sigma$. □

Theorem 4.6 *Let $\boldsymbol{A} = \langle A, \to, 1\rangle$ be a Hilbert algebra or commutative BCK-algebra. If there exists $0 = \min A$, then $\neg$ defined for $a \in A$ by $\neg a = a \to 0$ is an* SDS *translation.*

P r o o f. Let us assume that A has minimum 0 for $\leq$ and let us consider $\neg$ defined by $\neg a = a \to 0$, for all $a \in A$. If $\boldsymbol{A}$ is a Hilbert algebra, then $\neg$ is SDS. In effect, for all $a,\ b \in A$ we have

$$a \to (\neg a \to b) = a \to \big((a \to 0) \to b\big) = (a \to 0) \to (a \to b) = a \to (0 \to b).$$

Since $0 = \min A$ it follows that $0 \to b = 1$, and so $a \to (\neg a \to b) = 1$. Moreover, if $\boldsymbol{A}$ is a commutative BCK-algebra, then $\neg$ is an SDN translation. In effect, take $a \in A$; then $(a \to 0) \to 0 = (0 \to a) \to a = 1 \to a$; therefore, $\neg\neg a = a$ and so $\neg$ is SDN. It follows from Theorem 3.2, Lemma 3.7, and the above that $\neg$ is SDS. □

Corollary 4.7 *Let $\boldsymbol{A} = \langle A, \to, 1\rangle$ be a Hilbert algebra or commutative BCK-algebra. The following statements are equivalent:*

1. *There is an* SDS *translation in* $\boldsymbol{A}$.
2. *There is a unique* SDS *translation in* $\boldsymbol{A}$.
3. A *has minimum.*

P r o o f. If $\boldsymbol{A}$ is any BCK-algebra, then statement 1. implies statement 2. If there exists a unique SDS translation in $\boldsymbol{A}$, then we are in a particular case of Lemma 3.8. Therefore A has a minimum. Finally, if A has minimum 0, then $\neg$ defined by $\neg a = a \to 0$ is an SDS translation as we proved in Theorem 4.6. □

Remark 4.8 For the sake of brevity, we omit a great deal of information which can be straightforwardly deduced from the results summarized in Figure 1 and the table in Figure 2. For example, in the case of commutative BCK-algebras, the statement 1. in Corollary 4.7 could be replaced by "there is a PPC translation in $\boldsymbol{A}$" or by "there is a TPC translation in $\boldsymbol{A}$".

It is well known that every commutative BCK-algebra which satisfies P is an Abbott algebra (see, for example, [7], [14], or [2]). This result is included in

Lemma 4.9 *Let $\boldsymbol{A} = \langle A, \to, 1\rangle$ be a commutative BCK-algebra. The following statements are equivalent:*

1. *For all* $a, b \in A$, $\big((a \to b) \to a\big) \to a = 1$.
2. *For all* $a, b \in A$, $a \to (a \to b) = a \to b$.
3. *For all* $a, b, c \in A$, $\big(a \to (b \to c)\big) \to \big((a \to b) \to (a \to c)\big) = 1$.

Theorem 4.10 *Let $\boldsymbol{A} = \langle A, \to, \neg, 1\rangle$ be a Hilbert algebra or commutative BCK-algebra with negation. The following statements are equivalent:*

1. $\neg$ *is* SRA.
2. $\langle A, \to, 1\rangle$ *is an Abbott algebra with minimum,* $\neg 1 = \min A$, *and* $\neg a = a \to \neg 1$.

P r o o f. In either case, if $\neg$ is SRA, then it is PTC and SDS (see Figure 1). In virtue of Corollary 4.7, $\boldsymbol{A}$ has minimum 0. As stated in Theorem 4.6 and Lemma 4.5, $\neg$ must be the negation defined by $\neg a = a \to 0$ and so $0 = \neg 1$. Moreover, if $\neg$ is SRA, then the equality P holds in $\langle A, \to, 1\rangle$ for all $a, b \in A$ (see Figure 1). Therefore, if $\langle A, \to, 1\rangle$ is a Hilbert algebra, then it is obviously an Abbott algebra. If $\langle A, \to, 1\rangle$ is a commutative BCK-algebra, then it is an Abbott algebra too in virtue of Lemma 4.9. Conversely, let us assume that statement 2. holds. If $\boldsymbol{A}$ is an Abbott algebra, then $\boldsymbol{A}$ is a Hilbert algebra and $\neg$ is SCL. In effect, $\neg$ is PTC and P holds in $\langle A, \to, 1\rangle$, therefore for all $a \in A$, $\neg a \to a = (a \to \neg 1) \to a = a$. Since $\neg$ is SDS, as shown in Theorem 4.6, then it is SRA in virtue of Theorem 4.2. □

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